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ON SOME PROPERTIES OF CHEBYSHEV POLYNOMIALS

Hacène Belbachir and Farid Bencherif

USTHB, Faculty of Mathematics,
P.O.Box 32, El Alia, 16111, Algiers, Algeria

e-mail: hbelbachir@usthb.dz
or hacenebelbachir@gmail.com

e-mail: fbencherif@usthb.dz
or fbencherif@gmail.com

Abstract

Letting T_n (resp. U_n) be the n-th Chebyshev polynomials of the first (resp. second) kind, we prove that the sequences $\left(X^k T_{n-k}\right)_k$ and $\left(X^k U_{n-k}\right)_k$ for $n - 2\lfloor n/2 \rfloor \leq k \leq n - \lfloor n/2 \rfloor$ are two basis of the $\mathbb{Q}$-vectorial space $\mathbb{E}_n[X]$ formed by the polynomials of $\mathbb{Q}[X]$ having the same parity as n and of degree $\leq n$. Also T_n and U_n admit remarkableness integer coordinates on each of the two basis.

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1. Introduction and main results

For each integer $n \geq 0$, $T_n = T_n(X)$ and $U_n = U_n(X)$ denote the unique polynomials, with integer coefficients, satisfying

$$\cos nx = T_n(\cos x) \text{ and } \sin((n+1)x) = \sin x \, U_n(\cos x), \quad (x \in \mathbb{R}).$$

The well known Simpson's formulae [2], for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$:

$$\cos nx = 2\cos x \cos(n-1)x - \cos(n-2)x$$

$$\sin(n+1)x = 2\cos x \sin nx - \sin(n-1)x$$

give the recurrence relations

$$T_n = 2XT_{n-1} - T_{n-2}, \quad \text{with } T_0 = 1,\ T_1 = X, \tag{1}$$

$$U_n = 2XU_{n-1} - U_{n-2}, \ \text{with } U_0 = 1,\ U_1 = 2X. \tag{2}$$

We deduce (see [1]), for $n \geq 1$, the following relations

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 2^{n-1-2k} \frac{n}{n-k} \binom{n-k}{k} X^{n-2k}, \tag{3}$$

$$U_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 2^{n-2k} \binom{n-k}{k} X^{n-2k}. \tag{4}$$

These relations allows to state that, for any $n \geq 0$, T_n and U_n belong to $\mathbb{E}_n[X]$, (3) and (4) are their decompositions in the canonical basis $\mathfrak{B}_n := \left(X^{n-2k}\right)_{0 \leq k \leq \lfloor n/2 \rfloor}$

For instance, the first fifth values of thus polynomials are

$T_0 = 1$	$U_0 = 1$
$T_1 = X$	$U_1 = 2X$
$T_2 = 2X^2 - 1$	$U_2 = 4X^2 - 1$
$T_3 = 4X^3 - 3X$	$U_3 = 8X^3 - 4X$
$T_4 = 8X^4 - 8X^2 + 1$	$U_4 = 16X^4 - 12X^2 + 1$

The main goal of this paper is to prove that the families $\mathfrak{T}_n := \left(X^k T_{n-k}\right)_k$ and $\mathfrak{U}_n := \left(X^k U_{n-k}\right)_k$ for $n - 2\lfloor n/2 \rfloor \le k \le n - \lfloor n/2 \rfloor$ constitute two other basis of $\mathbb{E}_n[X]$ (Theorem 1) for which T_n and U_n admit remarkableness integer coordinates.

Our first result is the following

Theorem 1. *For any $n \geq 0$, $\mathfrak{T}_n$ and $\mathfrak{U}_n$ are two basis of $\mathbb{E}_n[X]$.*

So we can decompose T_{2n+1} and U_{2n+1} (resp. T_{2n} and U_{2n}) over each of the basis $\mathfrak{T}_{2n+1}$ and $\mathfrak{U}_{2n+1}$ (resp. $\mathfrak{T}_{2n}$ and $\mathfrak{U}_{2n}$). Decompositions of T_{2n} over $\mathfrak{T}_{2n}$ and U_{2n} over $\mathfrak{U}_{2n}$ are trivial, it remains to examine the six decompositions:

1. The decomposition of T_{2n+1} over $\mathfrak{T}_{2n+1}$ and U_{2n+1} over $\mathfrak{U}_{2n+1}$ in Theorem 2.
2. The decomposition of T_{2n} over $\mathfrak{U}_{2n}$ and U_{2n} over $\mathfrak{T}_{2n}$ in Theorem 3.
3. The decomposition of T_{2n+1} over $\mathfrak{U}_{2n+1}$ and U_{2n+1} over $\mathfrak{T}_{2n+1}$ in Theorem 4.

Let us define the families of integers $(\alpha_{n,k})$, $(\beta_{n,k})$ and $(\gamma_{n,k})$ by the following equalities

$$-(1-2X)^n = \sum_{k\geq 0} \alpha_{n,k} X^k, \tag{5}$$

$$(X-1)(1-2X)^n = \sum_{k\geq 0} \beta_{n,k} X^k, \tag{6}$$

$$(2X-1)^n + 2\left(1 + (2X-1) + \cdots + (2X-1)^{n-1}\right) = \sum_{k\geq 0} \gamma_{n,k} X^k, \tag{7}$$

we deduce then, for $(n,k) \in \mathbb{N}^2$, the following relations

$$\alpha_{n,k} = (-1)^{k+1} 2^k \binom{n}{k}, \tag{8}$$

$$\beta_{n,k} = \alpha_{n,k} - \alpha_{n,k-1}, \tag{9}$$

$$\gamma_{n,k} = (-1)^{n+1}\alpha_{n,k} + 2\sum_{j=0}^{n-1}(-1)^{j+1}\alpha_{j,k}. \tag{10}$$

Theorem 2. *For each $n \geq 0$, we have*

$$T_{2n+1} = \sum_{k=1}^{n+1} \beta_{n,k} X^k T_{2n+1-k}, \tag{11}$$

$$U_{2n+1} = \sum_{k=1}^{n+1} \alpha_{n+1,k} X^k U_{2n+1-k}. \tag{12}$$

Theorem 3. *For each $n \geq 0$, we have*

$$T_{2n} = U_{2n} - XU_{2n-1}, \quad n \geq 1, \tag{13}$$

$$U_{2n} = \sum_{k=0}^{n} \gamma_{n,k} X^k T_{2n-k}. \tag{14}$$

Theorem 4. *For each $n \geq 0$, we have*

$$T_{2n+1} = \sum_{k=1}^{n+1} \left(\alpha_{n+1,k} - \delta_{k,1}\right) X^k U_{2n+1-k}, \tag{15}$$

$$U_{2n+1} = \sum_{k=1}^{n+1} \left(\gamma_{n,k-1} + \beta_{n,k}\right) X^k T_{2n+1-k}, \tag{16}$$

where $\delta_{i,j}$ denotes the Kronecker symbol.

The sequences of integers $(\alpha_{n,k}), (\beta_{n,k})$ and $(\gamma_{n,k})$ satisfy the following recurrence relation

$$\begin{cases} \alpha_{n,0} = -1, \text{ for } n \geq 0, \\ \alpha_{0,k} = 0, \text{ for } k \geq 1, \\ \alpha_{n,k} = \alpha_{n-1,k} - 2\alpha_{n-1,k-1}, \text{ for } n, k \geq 1, \end{cases}$$

$$\begin{cases} \beta_{n,0} = -1, \text{ for } n \geq 0, \\ \beta_{0,1} = 1 \text{ and } \beta_{0,k} = 0, \text{ for } k \geq 2, \\ \beta_{n,k} = \beta_{n-1,k} - 2\beta_{n-1,k-1}, \text{ for } n, k \geq 1, \end{cases}$$

$$\begin{cases} \gamma_{n,0} = 1, \text{ for } n \geq 0, \\ \gamma_{0,k} = 0, \text{ for } k \geq 1, \\ \gamma_{n,k} = -\gamma_{n-1,k} + 2\gamma_{n-1,k-1}, \text{ for } n, k \geq 1. \end{cases}$$

The following tables give the values of $\alpha_{n,k}, \beta_{n,k}$ and $\gamma_{n,k}$ for $0 \leq n \leq 4$

n	$\alpha_{n,0}$	$\alpha_{n,1}$	$\alpha_{n,2}$	$\alpha_{n,3}$	$\alpha_{n,4}$
0	-1				
1	-1	2			
2	-1	4	-4		
3	-1	6	-12	8	
4	-1	8	-24	32	-16

n	$\beta_{n,0}$	$\beta_{n,1}$	$\beta_{n,2}$	$\beta_{n,3}$	$\beta_{n,4}$	$\beta_{n,5}$
0	-1	1				
1	-1	3	-2			
2	-1	5	-8	4		
3	-1	7	-18	20	-8	
4	-1	9	-32	56	-48	16

n	$\gamma_{n,0}$	$\gamma_{n,1}$	$\gamma_{n,2}$	$\gamma_{n,3}$	$\gamma_{n,4}$
0	1				
1	1	2			
2	1	0	4		
3	1	2	-4	8	
4	1	0	8	-16	16

Notice that $\alpha_{n,k} = \gamma_{n,k} = 0$ for $k > n$ and $\beta_{n,k} = 0$ for $k > n+1$.

According to these tables, one obtains

- Using Theorem 2

$$T_1 = XT_0$$
$$T_3 = 3XT_2 - 2X^2T_1$$
$$T_5 = 5XT_4 - 8X^2T_3 + 4X^3T_2$$
$$T_7 = 7XT_6 - 18X^2T_5 + 20X^3T_4 - 8X^4T_3$$
$$T_9 = 9XT_8 - 32X^2T_7 + 56X^3T_6 - 48X^4T_5 + 16X^5T_4$$

$$U_1 = 2XU_0$$
$$U_3 = 4XU_2 - 4X^2U_1$$
$$U_5 = 6XU_4 - 12X^2U_3 + 8X^3U_2$$
$$U_7 = 8XU_6 - 24X^2U_5 + 32X^3U_4 - 16X^4U_3$$

- Using Theorem 3

$$T_0 = U_0$$
$$T_2 = U_2 - XU_1$$
$$T_4 = U_4 - XU_3$$
$$T_6 = U_6 - XU_5$$
$$T_8 = U_8 - XU_7$$

$$U_0 = T_0$$
$$U_2 = T_2 + 2XT_1$$
$$U_4 = T_4 + 0XT_3 + 4X^2T_2$$
$$U_6 = T_6 + 2XT_5 - 4X^2T_4 + 8X^3T_3$$
$$U_8 = T_8 + 0XT_7 + 8X^2T_6 - 16X^3T_5 + 16X^4T_4$$

- Using Theorem 4

$$T_1 = XU_0$$
$$T_3 = 3XU_2 - 4X^2U_1$$
$$T_5 = 5XU_4 - 12X^2U_3 + 8X^3U_2$$
$$T_7 = 7XU_6 - 24X^2U_5 + 32X^3U_4 - 16X^4U_3$$
$$T_9 = 9XU_8 - 40X^2U_7 + 80X^3U_6 - 80X^4U_5 + 32X^5U_4$$

$$U_1 = 2XT_0$$
$$U_3 = 4XT_2$$
$$U_5 = 6XT_4 - 8X^2T_3 + 8X^3T_2$$
$$U_7 = 8XT_6 - 16X^2T_5 + 16X^3T_4$$
$$U_9 = 10XT_8 - 32X^2T_7 + 64X^3T_6 - 64X^4T_5 + 32X^5T_4$$

2. Proofs of Theorems

2.1. Proof of Theorem 1

$\mathfrak{T}_n$ and $\mathfrak{U}_n$ are two families of polynomials of $\mathbb{E}_n[X]$ with $card\mathfrak{T}_n = card\mathfrak{U}_n = \dim \mathbb{E}_n[X] = \lfloor n/2 \rfloor + 1$. Using the following Lemma, we prove that the determinant of $\mathfrak{T}_n$ and $\mathfrak{U}_n$ relatively to the canonical basis $\mathfrak{B}_n$ of $\mathbb{E}_n[X]$ are not zero. Theorem 1 follows.

Lemma 5. *For any integer $n \geq 0$, by setting $m = \lfloor n/2 \rfloor$, we have*

$$\det_{\mathfrak{B}_n}(\mathfrak{T}_n) = 2^{m(m-1)/2} \quad \text{and} \quad \det_{\mathfrak{B}_n}(\mathfrak{U}_n) = 2^{m(m+1)/2}.$$

Proof. For any integer $m \geq 0$ and for $1 \leq k \leq m+1$, set $V_k^{(m)} = (2X)^{k-1} T_{2m+1-k}$ and $W_k^{(m)} = (2X)^{k-1} U_{2m+1-k}$. Notice that $V_k^{(m)}$ and $W_k^{(m)}$ are polynomials of $\mathbb{E}_{2m}[X]$ with dominant coefficient 2^{2m-1} and 2^{2m} respectively. Using the recurrence equations (1) and (2), we obtain for $m \geq 1$

$$V_{k+1}^{(m)} - V_k^{(m)} = V_k^{(m-1)} \quad \text{and} \quad W_{k+1}^{(m)} - W_k^{(m)} = W_k^{(m-1)}.$$

Let $\quad \Delta_m = \det_{\mathfrak{B}_{2m}}\left(V_1^{(m)}, V_2^{(m)}, \ldots, V_{m+1}^{(m)}\right) \quad$ and

$$D_m = \det_{\mathfrak{B}_{2m}}\left(W_1^{(m)}, W_2^{(m)}, \ldots, W_{m+1}^{(m)}\right).$$

We have

$$\begin{aligned}
\Delta_m &= \det_{\mathfrak{B}_{2m}}\left(V_1^{(m)},\ V_2^{(m)} - V_1^{(m)},\ V_3^{(m)} - V_2^{(m)}, \ldots,\ V_{m+1}^{(m)} - V_m^{(m)}\right) \\
&= \det_{\mathfrak{B}_{2m}}\left(V_1^{(m)},\ V_1^{(m-1)},\ V_2^{(m-1)}, \ldots,\ V_m^{(m-1)}\right) \\
&= 2^{2m-1}\Delta_{m-1} \\
&= 2^{(2m-1)+(2m-3)+\cdots+1}\Delta_0 \\
&= 2^{m^2},
\end{aligned}$$

and similarly, we obtain $D_m = 2^{2m} D_{m-1} = 2^{2m+(2m-2)+\cdots+2} D_0 = 2^{m(m+1)}$.

For $n = 2m + r$, with $m = \lfloor n/2 \rfloor$ and $r \in \{0, 1\}$, we have

$$\begin{aligned}
\det_{\mathfrak{B}_n}(\mathfrak{T}_n) &= \det_{\mathfrak{B}_{2m+r}}\left(X^r T_{2m},\ X^{r+1} T_{2m-1}, \ldots,\ X^{r+m} T_m\right) \\
&= \det_{\mathfrak{B}_{2m}}(T_{2m},\ X T_{2m-1}, \ldots,\ X^m T_m) \\
&= 2^{-(1+2+\cdots+m)} \Delta_m \\
&= 2^{m(m-1)/2},
\end{aligned}$$

and similarly $\det_{\mathfrak{B}_n}(\mathfrak{U}_n) = 2^{-(1+2+\cdots+m)} D_m = 2^{m(m+1)/2}$. ■

2.2. Proof of Theorems 2, 3 and 4

Let us denote E denote the shift operator on $\mathbb{Q}[X]^{\mathbb{N}}$ defined by

$$E((W_n)_n) = (W_{n+1})_n,$$

or in a more simple form

$$E W_n = W_{n+1}, \quad (n \geq 0).$$

For any $m \geq 0$, define the operators

$$\begin{aligned}
A_m &= -(E - 2X)^m, \\
B_m &= (X - E)(E - 2X)^m, \\
C_m &= (2X - E)^m + 2 \sum_{k=1}^{m} E^k (2X - E)^{m-k}.
\end{aligned}$$

Using relations (5) , (6) and (7) , we have also

$$A_m = \sum_{k=0}^{m} \alpha_{m,k} X^k E^{m-k},$$

$$B_m = \sum_{k=0}^{m+1} \beta_{m,k} X^k E^{m+1-k},$$

$$C_m = \sum_{k=0}^{m} \gamma_{m,k} X^k E^{m-k}.$$

Lemma 6. *For any integer n, we have*

(a) $(2X-E)^n T_m = T_{m-n}$ *and* $(2X-E)^n U_m = U_{m-n}$, $(m \geq n \geq 0)$.

(b) $T_n = U_n - XU_{n-1}$ $(n \geq 1)$.

(c) $2T_n = U_n - U_{n-2}$ $(n \geq 2)$.

(d) $U_{2n} = 1 + 2\sum_{k=1}^{n} T_{2k}$ $(n \geq 0)$.

Proof.

(a) For $m \geq 1$, one has $(2X-E) T_m = 2XT_m - T_{m+1} = T_{m-1}$ and $(2X-E) U_m = 2XU_m - U_{m+1} = U_{m-1}$. We conclude by induction.

(b) Letting for $n \geq 1$, $W_n = T_n - U_n + XU_{n-1}$. The sequence $(W_n)_{n \geq 1}$ satisfies the following relation

$$W_n = 2XW_{n-1} - W_{n-2}, \quad (n \geq 3), \quad \text{with } W_1 = W_2 = 0,$$

leading to $W_n = 0$, for $n \geq 1$.

(c) For $n \geq 2$, one has $2T_n = U_n + (U_n - 2XU_{n-1})$ from (b) and thus $2T_n = U_n - U_{n-2}$ from (2) .

(d) For $n \geq 0$, one has $U_{2n} = U_0 + \sum_{k=1}^{n} (U_{2k} - U_{2k-2}) = 1 + 2\sum_{k=1}^{n} T_{2k}$ from (c).

■

***Proof of Theorem* 2.** Relations (11) and (12) are respectively equivalent to

$$B_n T_n = 0 \quad \text{and} \quad A_{n+1} U_n = 0, \text{ for } n \geq 0.$$

These last relations follows from Lemma 6 (a). We have, for any integer $n \geq 0$

$$\begin{aligned} B_n T_n &= (-1)^n (X - E)(2X - E)^n T_n \\ &= (-1)^n (X - E) T_0 \\ &= (-1)^n (XT_0 - T_1) = 0, \end{aligned}$$

and

$$\begin{aligned} A_{n+1} U_n &= (-1)^{n+1} (2X - E)(2X - E)^n U_n \\ &= (-1)^{n+1} (2X - E) U_0 \\ &= (-1)^{n+1} (2XU_0 - U_1) = 0. \end{aligned}$$

***Proof of Theorem* 3.** Relation (13) follows from Lemma 6 (b).

Relation (14) is equivalent to $U_{2n} = C_n T_n$, for $n \geq 0$. One has indeed, for any $n \geq 0$

$$C_nT_n = \left((2X-E)^n + 2\sum_{k=1}^{n} E^k\,(2X-E)^{n-k}\right)T_n$$

$$= T_0 + 2\sum_{k=1}^{n} T_{2k}$$

$$= U_{2n}, \quad \text{(from (d) of Lemma 6).}$$

***Proof of Theorem* 4**. For any $n \geq 0$, we have

$$T_{2n+1} = U_{2n+1} - XU_{2n}, \quad \text{(from (b) of Lemma 6)}$$

$$= \sum_{k=1}^{n+1} \alpha_{n+1,k} X^k U_{2n+1-k} - XU_{2n} \quad \text{(from (12))}$$

$$= \sum_{k=1}^{n+1} \left(\alpha_{n+1,k} - \delta_{k,1}\right) X^k U_{2n+1-k}.$$

We have also, for any $n \geq 0$

$$U_{2n+1} = XU_{2n} + T_{2n+1}, \quad \text{(from (b) of Lemma 6)}$$

$$= \sum_{k=0}^{n} \gamma_{n,k} X^{k+1} T_{2n-k} + \sum_{k=1}^{n+1} \beta_{n,k} X^k T_{2n+1-k}, \quad \text{(from (11) and (14))}$$

$$= \sum_{k=1}^{n+1} \left(\gamma_{n,k-1} + \beta_{n,k}\right) X^k T_{2n+1-k}.$$

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