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# ON SOME PROPERTIES OF CHEBYSHEV POLYNOMIALS

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**Abstract**

Letting $T_n$ (resp. $U_n$) be the $n$-th Chebyshev polynomials of the first (resp. second) kind, we prove that the sequences $\left(X^k T_{n-k}\right)_k$ and $\left(X^k U_{n-k}\right)_k$ for $n - 2\lfloor n/2\rfloor \le k \le n - \lfloor n/2\rfloor$ are two basis of the $\mathbb{Q}$-vectorial space $\mathbb{E}_n[X]$ formed by the polynomials of $\mathbb{Q}[X]$ having the same parity as $n$ and of degree $\le n$. Also $T_n$ and $U_n$ admit remarkableness integer coordinates on each of the two basis.

**Keywords:** Chebyshev polynomials, integer coordinates.

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## 1. Introduction and main results

For each integer $n \ge 0$, $T_n = T_n(X)$ and $U_n = U_n(X)$ denote the unique polynomials, with integer coefficients, satisfying

$$\cos nx = T_n(\cos x) \text{ and } \sin((n+1)x) = \sin x\, U_n(\cos x), \quad (x \in \mathbb{R}).$$

The well known Simpson's formulae [2], for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ :

$$\cos nx = 2\cos x \cos(n-1)x - \cos(n-2)x$$

$$\sin(n+1)x = 2\cos x \sin nx - \sin(n-1)x$$

give the recurrence relations

$$T_n = 2XT_{n-1} - T_{n-2}, \quad \text{with } T_0 = 1,\ T_1 = X, \tag{1}$$

$$U_n = 2XU_{n-1} - U_{n-2}, \ \text{with } U_0 = 1,\ U_1 = 2X. \tag{2}$$

We deduce (see [1]), for $n \geq 1$, the following relations

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 2^{n-1-2k} \frac{n}{n-k} \binom{n-k}{k} X^{n-2k}, \tag{3}$$

$$U_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 2^{n-2k} \binom{n-k}{k} X^{n-2k}. \tag{4}$$

These relations allows to state that, for any $n \geq 0$, $T_n$ and $U_n$ belong to $\mathbb{E}_n[X]$, (3) and (4) are their decompositions in the canonical basis $\mathfrak{B}_n := \left(X^{n-2k}\right)_{0 \leq k \leq \lfloor n/2 \rfloor}$

For instance, the first fifth values of thus polynomials are

$$\begin{array}{ll}
T_0 = 1 & U_0 = 1 \\
T_1 = X & U_1 = 2X \\
T_2 = 2X^2 - 1 & U_2 = 4X^2 - 1 \\
T_3 = 4X^3 - 3X & U_3 = 8X^3 - 4X \\
T_4 = 8X^4 - 8X^2 + 1 & U_4 = 16X^4 - 12X^2 + 1
\end{array}$$

The main goal of this paper is to prove that the families $\mathfrak{T}_n := \left(X^k T_{n-k}\right)_k$ and $\mathfrak{U}_n := \left(X^k U_{n-k}\right)_k$ for $n - 2\lfloor n/2 \rfloor \leq k \leq n - \lfloor n/2 \rfloor$ constitute two other basis of $\mathbb{E}_n[X]$ (Theorem 1) for which $T_n$ and $U_n$ admit remarkableness integer coordinates.

Our first result is the following

**Theorem 1.** *For any $n \geq 0$, $\mathfrak{T}_n$ and $\mathfrak{U}_n$ are two basis of $\mathbb{E}_n[X]$.*

So we can decompose $T_{2n+1}$ and $U_{2n+1}$ (resp. $T_{2n}$ and $U_{2n}$) over each of the basis $\mathfrak{T}_{2n+1}$ and $\mathfrak{U}_{2n+1}$ (resp. $\mathfrak{T}_{2n}$ and $\mathfrak{U}_{2n}$). Decompositions of $T_{2n}$ over $\mathfrak{T}_{2n}$ and $U_{2n}$ over $\mathfrak{U}_{2n}$ are trivial, it remains to examine the six decompositions:

1. The decomposition of $T_{2n+1}$ over $\mathfrak{T}_{2n+1}$ and $U_{2n+1}$ over $\mathfrak{U}_{2n+1}$ in Theorem 2.
2. The decomposition of $T_{2n}$ over $\mathfrak{U}_{2n}$ and $U_{2n}$ over $\mathfrak{T}_{2n}$ in Theorem 3.
3. The decomposition of $T_{2n+1}$ over $\mathfrak{U}_{2n+1}$ and $U_{2n+1}$ over $\mathfrak{T}_{2n+1}$ in Theorem 4.

Let us define the families of integers $(\alpha_{n,k})$, $(\beta_{n,k})$ and $(\gamma_{n,k})$ by the following equalities

$$-(1-2X)^n = \sum_{k\geq 0} \alpha_{n,k} X^k, \tag{5}$$

$$(X-1)(1-2X)^n = \sum_{k\geq 0} \beta_{n,k} X^k, \tag{6}$$

$$(2X-1)^n + 2\Big(1 + (2X-1) + \cdots + (2X-1)^{n-1}\Big) = \sum_{k\geq 0} \gamma_{n,k} X^k, \tag{7}$$

we deduce then, for $(n,k) \in \mathbb{N}^2$, the following relations

$$\alpha_{n,k} = (-1)^{k+1} 2^k \binom{n}{k}, \tag{8}$$

$$\beta_{n,k} = \alpha_{n,k} - \alpha_{n,k-1}, \tag{9}$$

$$\gamma_{n,k} = (-1)^{n+1}\alpha_{n,k} + 2\sum_{j=0}^{n-1}(-1)^{j+1}\alpha_{j,k}. \tag{10}$$

**Theorem 2.** *For each $n \geq 0$, we have*

$$T_{2n+1} = \sum_{k=1}^{n+1} \beta_{n,k} X^k T_{2n+1-k}, \tag{11}$$

$$U_{2n+1} = \sum_{k=1}^{n+1} \alpha_{n+1,k} X^k U_{2n+1-k}. \tag{12}$$

**Theorem 3.** *For each $n \geq 0$, we have*

$$T_{2n} = U_{2n} - XU_{2n-1}, \quad n \geq 1, \tag{13}$$

$$U_{2n} = \sum_{k=0}^{n} \gamma_{n,k} X^k T_{2n-k}. \tag{14}$$

**Theorem 4.** *For each $n \geq 0$, we have*

$$T_{2n+1} = \sum_{k=1}^{n+1} \left(\alpha_{n+1,k} - \delta_{k,1}\right) X^k U_{2n+1-k}, \tag{15}$$

$$U_{2n+1} = \sum_{k=1}^{n+1} \left(\gamma_{n,k-1} + \beta_{n,k}\right) X^k T_{2n+1-k}, \tag{16}$$

*where $\delta_{i,j}$ denotes the Kronecker symbol.*

The sequences of integers $(\alpha_{n,k}), (\beta_{n,k})$ and $(\gamma_{n,k})$ satisfy the following recurrence relation

$$\begin{cases} \alpha_{n,0} = -1, \ \text{for } n \geq 0, \\ \alpha_{0,k} = 0, \ \text{for } k \geq 1, \\ \alpha_{n,k} = \alpha_{n-1,k} - 2\alpha_{n-1,k-1}, \ \text{for } n, k \geq 1, \end{cases}$$

$$\begin{cases} \beta_{n,0} = -1, \ \text{for } n \geq 0, \\ \beta_{0,1} = 1 \text{ and } \beta_{0,k} = 0, \ \text{for } k \geq 2, \\ \beta_{n,k} = \beta_{n-1,k} - 2\beta_{n-1,k-1}, \ \text{for } n, k \geq 1, \end{cases}$$

$$\begin{cases} \gamma_{n,0} = 1, \ \text{for } n \geq 0, \\ \gamma_{0,k} = 0, \ \text{for } k \geq 1, \\ \gamma_{n,k} = -\gamma_{n-1,k} + 2\gamma_{n-1,k-1}, \ \text{for } n, k \geq 1. \end{cases}$$

The following tables give the values of $\alpha_{n,k}, \beta_{n,k}$ and $\gamma_{n,k}$ for $0 \leq n \leq 4$

| $n$ | $\alpha_{n,0}$ | $\alpha_{n,1}$ | $\alpha_{n,2}$ | $\alpha_{n,3}$ | $\alpha_{n,4}$ |
|---|---|---|---|---|---|
| *0* | $-1$ | | | | |
| *1* | $-1$ | 2 | | | |
| *2* | $-1$ | 4 | $-4$ | | |
| *3* | $-1$ | 6 | $-12$ | 8 | |
| *4* | $-1$ | 8 | $-24$ | 32 | $-16$ |

| $n$ | $\beta_{n,0}$ | $\beta_{n,1}$ | $\beta_{n,2}$ | $\beta_{n,3}$ | $\beta_{n,4}$ | $\beta_{n,5}$ |
|---|---|---|---|---|---|---|
| 0 | $-1$ | 1 | | | | |
| 1 | $-1$ | 3 | $-2$ | | | |
| 2 | $-1$ | 5 | $-8$ | 4 | | |
| 3 | $-1$ | 7 | $-18$ | 20 | $-8$ | |
| 4 | $-1$ | 9 | $-32$ | 56 | $-48$ | 16 |

| $n$ | $\gamma_{n,0}$ | $\gamma_{n,1}$ | $\gamma_{n,2}$ | $\gamma_{n,3}$ | $\gamma_{n,4}$ |
|---|---|---|---|---|---|
| 0 | 1 | | | | |
| 1 | 1 | 2 | | | |
| 2 | 1 | 0 | 4 | | |
| 3 | 1 | 2 | $-4$ | 8 | |
| 4 | 1 | 0 | 8 | $-16$ | 16 |

Notice that $\alpha_{n,k} = \gamma_{n,k} = 0$ for $k > n$ and $\beta_{n,k} = 0$ for $k > n+1$.

According to these tables, one obtains

- Using Theorem 2

$$T_1 = XT_0$$

$$T_3 = 3XT_2 - 2X^2T_1$$

$$T_5 = 5XT_4 - 8X^2T_3 + 4X^3T_2$$

$$T_7 = 7XT_6 - 18X^2T_5 + 20X^3T_4 - 8X^4T_3$$

$$T_9 = 9XT_8 - 32X^2T_7 + 56X^3T_6 - 48X^4T_5 + 16X^5T_4$$

$$U_1 = 2XU_0$$

$$U_3 = 4XU_2 - 4X^2U_1$$

$$U_5 = 6XU_4 - 12X^2U_3 + 8X^3U_2$$

$$U_7 = 8XU_6 - 24X^2U_5 + 32X^3U_4 - 16X^4U_3$$

- Using Theorem 3

$$
\begin{aligned}
T_0 &= U_0\\
T_2 &= U_2 - XU_1\\
T_4 &= U_4 - XU_3\\
T_6 &= U_6 - XU_5\\
T_8 &= U_8 - XU_7
\end{aligned}
$$

$$
\begin{aligned}
U_0 &= T_0\\
U_2 &= T_2 + 2XT_1\\
U_4 &= T_4 + 0XT_3 + 4X^2T_2\\
U_6 &= T_6 + 2XT_5 - 4X^2T_4 + 8X^3T_3\\
U_8 &= T_8 + 0XT_7 + 8X^2T_6 - 16X^3T_5 + 16X^4T_4
\end{aligned}
$$

- Using Theorem 4

$$
\begin{aligned}
T_1 &= XU_0\\
T_3 &= 3XU_2 - 4X^2U_1\\
T_5 &= 5XU_4 - 12X^2U_3 + 8X^3U_2\\
T_7 &= 7XU_6 - 24X^2U_5 + 32X^3U_4 - 16X^4U_3\\
T_9 &= 9XU_8 - 40X^2U_7 + 80X^3U_6 - 80X^4U_5 + 32X^5U_4
\end{aligned}
$$

$$
\begin{aligned}
U_1 &= 2XT_0\\
U_3 &= 4XT_2\\
U_5 &= 6XT_4 - 8X^2T_3 + 8X^3T_2\\
U_7 &= 8XT_6 - 16X^2T_5 + 16X^3T_4\\
U_9 &= 10XT_8 - 32X^2T_7 + 64X^3T_6 - 64X^4T_5 + 32X^5T_4
\end{aligned}
$$

## 2. Proofs of Theorems

### 2.1. Proof of Theorem 1

$\mathfrak{T}_n$ and $\mathfrak{U}_n$ are two families of polynomials of $\mathbb{E}_n[X]$ with $card\mathfrak{T}_n = card\mathfrak{U}_n = \dim \mathbb{E}_n[X] = \lfloor n/2 \rfloor + 1$. Using the following Lemma, we prove that the determinant of $\mathfrak{T}_n$ and $\mathfrak{U}_n$ relatively to the canonical basis $\mathfrak{B}_n$ of $\mathbb{E}_n[X]$ are not zero. Theorem 1 follows.

**Lemma 5.** *For any integer $n \geq 0$, by setting $m = \lfloor n/2 \rfloor$, we have*

$$\det_{\mathfrak{B}_n}(\mathfrak{T}_n) = 2^{m(m-1)/2} \quad \text{and} \quad \det_{\mathfrak{B}_n}(\mathfrak{U}_n) = 2^{m(m+1)/2}.$$

***Proof.*** For any integer $m \geq 0$ and for $1 \leq k \leq m+1$, set $V_k^{(m)} = (2X)^{k-1} T_{2m+1-k}$ and $W_k^{(m)} = (2X)^{k-1} U_{2m+1-k}$. Notice that $V_k^{(m)}$ and $W_k^{(m)}$ are polynomials of $\mathbb{E}_{2m}[X]$ with dominant coefficient $2^{2m-1}$ and $2^{2m}$ respectively. Using the recurrence equations (1) and (2), we obtain for $m \geq 1$

$$V_{k+1}^{(m)} - V_k^{(m)} = V_k^{(m-1)} \quad \text{and} \quad W_{k+1}^{(m)} - W_k^{(m)} = W_k^{(m-1)}.$$

Let $\Delta_m = \det_{\mathfrak{B}_{2m}}\left(V_1^{(m)}, V_2^{(m)}, \ldots, V_{m+1}^{(m)}\right)$ and

$$D_m = \det_{\mathfrak{B}_{2m}}\left(W_1^{(m)}, W_2^{(m)}, \ldots, W_{m+1}^{(m)}\right).$$

We have

$$\begin{aligned}
\Delta_m &= \det_{\mathfrak{B}_{2m}}\left(V_1^{(m)},\ V_2^{(m)} - V_1^{(m)},\ V_3^{(m)} - V_2^{(m)}, \ldots,\ V_{m+1}^{(m)} - V_m^{(m)}\right) \\
&= \det_{\mathfrak{B}_{2m}}\left(V_1^{(m)},\ V_1^{(m-1)},\ V_2^{(m-1)}, \ldots,\ V_m^{(m-1)}\right) \\
&= 2^{2m-1}\Delta_{m-1} \\
&= 2^{(2m-1)+(2m-3)+\cdots+1}\Delta_0 \\
&= 2^{m^2},
\end{aligned}$$

and similarly, we obtain $D_m = 2^{2m} D_{m-1} = 2^{2m+(2m-2)+\cdots+2} D_0 = 2^{m(m+1)}$.

For $n = 2m + r$, with $m = \lfloor n/2 \rfloor$ and $r \in \{0, 1\}$, we have

$$\begin{aligned}
\det_{\mathfrak{B}_n}(\mathfrak{T}_n) &= \det_{\mathfrak{B}_{2m+r}}\left(X^r T_{2m},\ X^{r+1} T_{2m-1}, \ldots,\ X^{r+m} T_m\right) \\
&= \det_{\mathfrak{B}_{2m}}(T_{2m},\ X T_{2m-1}, \ldots,\ X^m T_m) \\
&= 2^{-(1+2+\cdots+m)} \Delta_m \\
&= 2^{m(m-1)/2},
\end{aligned}$$

and similarly $\det_{\mathfrak{B}_n}(\mathfrak{U}_n) = 2^{-(1+2+\cdots+m)} D_m = 2^{m(m+1)/2}$. ■

### 2.2. Proof of Theorems 2, 3 and 4

Let us denote $E$ denote the shift operator on $\mathbb{Q}[X]^{\mathbb{N}}$ defined by

$$E((W_n)_n) = (W_{n+1})_n,$$

or in a more simple form

$$E W_n = W_{n+1}, \quad (n \geq 0).$$

For any $m \geq 0$, define the operators

$$\begin{aligned}
A_m &= -(E - 2X)^m, \\
B_m &= (X - E)(E - 2X)^m, \\
C_m &= (2X - E)^m + 2 \sum_{k=1}^{m} E^k (2X - E)^{m-k}.
\end{aligned}$$

Using relations (5) , (6) and (7) , we have also

$$A_m = \sum_{k=0}^{m} \alpha_{m,k} X^k E^{m-k},$$

$$B_m = \sum_{k=0}^{m+1} \beta_{m,k} X^k E^{m+1-k},$$

$$C_m = \sum_{k=0}^{m} \gamma_{m,k} X^k E^{m-k}.$$

**Lemma 6.** *For any integer $n$, we have*

(a) $(2X-E)^n T_m = T_{m-n}$ *and* $(2X-E)^n U_m = U_{m-n}$, $(m \geq n \geq 0)$.

(b) $T_n = U_n - XU_{n-1}$ $(n \geq 1)$.

(c) $2T_n = U_n - U_{n-2}$ $(n \geq 2)$.

(d) $U_{2n} = 1 + 2\sum_{k=1}^{n} T_{2k}$ $(n \geq 0)$.

***Proof.***

(a) For $m \geq 1$, one has $(2X-E)\,T_m = 2XT_m - T_{m+1} = T_{m-1}$ and $(2X-E)\,U_m = 2XU_m - U_{m+1} = U_{m-1}$. We conclude by induction.

(b) Letting for $n \geq 1$, $W_n = T_n - U_n + XU_{n-1}$. The sequence $(W_n)_{n\geq 1}$ satisfies the following relation

$$W_n = 2XW_{n-1} - W_{n-2}, \ (n \geq 3), \quad \text{with } W_1 = W_2 = 0,$$

leading to $W_n = 0$, for $n \geq 1$.

(c) For $n \geq 2$, one has $2T_n = U_n + (U_n - 2XU_{n-1})$ from (b) and thus $2T_n = U_n - U_{n-2}$ from (2) .

(d) For $n \geq 0$, one has $U_{2n} = U_0 + \sum_{k=1}^{n} (U_{2k} - U_{2k-2}) = 1 + 2\sum_{k=1}^{n} T_{2k}$ from (c).

■

***Proof of Theorem* 2.** Relations (11) and (12) are respectively equivalent to

$$B_n T_n = 0 \quad \text{and} \quad A_{n+1} U_n = 0, \text{ for } n \geq 0.$$

These last relations follows from Lemma 6 (a). We have, for any integer $n \geq 0$

$$\begin{aligned} B_n T_n &= (-1)^n (X - E)(2X - E)^n T_n \\ &= (-1)^n (X - E) T_0 \\ &= (-1)^n (XT_0 - T_1) = 0, \end{aligned}$$

and

$$\begin{aligned} A_{n+1} U_n &= (-1)^{n+1} (2X - E)(2X - E)^n U_n \\ &= (-1)^{n+1} (2X - E) U_0 \\ &= (-1)^{n+1} (2XU_0 - U_1) = 0. \end{aligned}$$

***Proof of Theorem* 3.** Relation (13) follows from Lemma 6 (b).

Relation (14) is equivalent to $U_{2n} = C_n T_n$, for $n \geq 0$. One has indeed, for any $n \geq 0$

$$
\begin{aligned}
C_n T_n &= \left((2X-E)^n + 2\textstyle\sum_{k=1}^{n} E^k (2X-E)^{n-k}\right) T_n \\
&= T_0 + 2\sum_{k=1}^{n} T_{2k} \\
&= U_{2n}, \quad \text{(from (d) of Lemma 6).}
\end{aligned}
$$

***Proof of Theorem* 4.** For any $n \geq 0$, we have

$$
\begin{aligned}
T_{2n+1} &= U_{2n+1} - XU_{2n}, \quad \text{(from (b) of Lemma 6)} \\
&= \sum_{k=1}^{n+1} \alpha_{n+1,k} X^k U_{2n+1-k} - XU_{2n} \quad \text{(from (12))} \\
&= \sum_{k=1}^{n+1} (\alpha_{n+1,k} - \delta_{k,1}) X^k U_{2n+1-k}.
\end{aligned}
$$

We have also, for any $n \geq 0$

$$
\begin{aligned}
U_{2n+1} &= XU_{2n} + T_{2n+1}, \quad \text{(from (b) of Lemma 6)} \\
&= \sum_{k=0}^{n} \gamma_{n,k} X^{k+1} T_{2n-k} + \sum_{k=1}^{n+1} \beta_{n,k} X^k T_{2n+1-k}, \quad \text{(from (11) and (14))} \\
&= \sum_{k=1}^{n+1} (\gamma_{n,k-1} + \beta_{n,k}) X^k T_{2n+1-k}.
\end{aligned}
$$

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