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THE GREATEST REGULAR-SOLID VARIETY OF SEMIGROUPS

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Abstract

A regular hypersubstitution is a mapping which takes every n_i-ary operation symbol to an n_i-ary term. A variety is called regular-solid if it contains all algebras derived by regular hypersubstitutions. We determine the greatest regular-solid variety of semigroups. This result will be used to give a new proof for the equational description of the greatest solid variety of semigroups. We show that every variety of semigroups which is finitely based by hyperidentities is also finitely based by identities.

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1. Introduction

Let τ be a fixed type, with fundamental operation symbols f_i, $i \in I$, and let $W_\tau(X)$ be the set of all terms of type τ. If $\mathcal{A} = (A; (f_i^A)_{i\in I})$ is an algebra of type τ, then we can get a new algebra of type τ with the universe A if we replace the fundamental operations by term operations of $\mathcal{A}$ of the same arity. This informal definition shows that we are interested in a map which associates to every operation symbol f_i of a given type τ a term $\sigma(f_i)$ of type τ, of the same arity as f_i. Any such map is called a *hypersubstitution* (of type τ) and the algebra $\sigma(\mathcal{A}) = (A; (\sigma(f_i)^A)_{i\in I})$ is called *derived algebra.* Here $\sigma(f_i)^A$ are the n_i-ary term operations induced by the terms $\sigma(f_i)$. If the algebra $\mathcal{A}$ belongs to a given variety V of algebras of type τ, then one can ask if the derived algebra $\sigma(\mathcal{A})$ belongs also to the variety V.

Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ. Any hypersubstitution can be uniquely extended to a map $\hat{\sigma}$ on $W_\tau(X)$ defined inductively as follows:

(i) If $t = x_i$ for some $i \geq 1$, then $\hat{\sigma}[t] = x_i$.

(ii) If $t = f(t_1, \ldots, t_n)$ for some n-ary operation symbol f and some terms $t_1, \ldots, t_n$, then $\hat{\sigma}[t] = \sigma(f)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$.

Here the right hand side is the composition of the term $\sigma(f)$ with the terms $\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]$.

We can define a binary operation $\circ_h$ on the set $Hyp(\tau)$ of all hypersubstitutions of type τ, by letting $\sigma_1 \circ_h \sigma_2$ be the hypersubstitution which maps each fundamental operation symbol f to the term $\hat{\sigma}_1[\sigma_2(f)]$. The set $Hyp(\tau)$ of all hypersubstitutions of type τ is closed under this associative composition operation, and so forms a semigroup. In fact $Hyp(\tau)$ is a monoid, since the identity hypersubstitution σ_{id} (mapping every f_i to $f_i(x_1, \ldots, x_{n_i})$) acts as an identity element. A variety V is called solid if every derived algebra $\sigma(\mathcal{A})$ belongs to V and M-solid if this holds for every hypersubstitution from a submonoid M of $Hyp(\tau)$.

Now suppose that M is any submonoid of $Hyp(\tau)$. An identity $u \approx v$ of a variety V is called an M-hyperidentity, and a hyperidentity for $M = Hyp(\tau)$, of V if for every hypersubstitution $\sigma \in M$ the equation $\hat{\sigma}[u] \approx \hat{\sigma}[v]$ holds in V. M-solid (solid) varieties V are characterized by the property that every identity of V is an M-hyperidentity (hyperidentity) of V.

Alternatively M-solid varieties V can be characterized by the property that there is a set Σ of equations such that an algebra belongs to V if and only if it satisfies all equations from Σ as M-hyperidentities. In this case we write $V = H_M Mod\, \Sigma$ or simply $V = HMod\, \Sigma$ for $M = Hyp(\tau)$ and speak of an M-hyper model class (or hyper model class for $M = Hyp(\tau)$).

In this paper we are interested in varieties of type $\tau = (2)$; that is, in varieties with one binary operation symbol f. Type (2), and especially varieties of semigroups, seem simple enough to be accessible but rich enough to be interesting, and much has been done in the investigation of hyperidentities and M-solidity for these varieties. (See for example [2]).

Let SEM be the variety of all semigroups. We are looking for such subvarieties of SEM which contain with any semigroup $\mathcal{A} = (A; f^A)$ also all derived semigroups $\sigma(\mathcal{A}) = (A; \sigma(f)^A)$, i.e., such that $\sigma(f)^A$ is associative. In the variety SEM this is in general not the case as the following example shows. We consider the hypersubstitution $\sigma_{f(x,f(y,y))}$ which maps the binary operation symbol f to the binary term $f(x, f(y, y))$. Then the corresponding term operation is not associative since $\sigma(f)^A(\sigma(f)^A(a, b), c) = ab^2c^2$ and $\sigma(f)^A(a, \sigma(f)^A(b, c)) = a(bc^2)^2$ are in general not equal. So we are looking for the greatest subvariety of SEM which contains all those derived algebras. It makes sense to concentrate on hypersubstitutions which map f to binary terms containing both variables x and y. (We notice that terms which contain only one variable can also be regarded as binary). In this case the induced term operations $\sigma(f)^A$ are *essentially binary*. Hypersubstitutions of this kind are called *regular* and the corresponding M-hyperidentities are called *regular hyperidentities*. The set Reg of all regular hypersubstitutions of type $\tau = (2)$ forms a submonoid of the monoid Hyp of all hypersubstitutions of type $\tau = (2)$. M-solid varieties of semigroups for $M = Reg$ are called *regular solid* and we want to give an equational description of the greatest regular solid variety of semigroups. It turns out that this is the variety $V_{HR} = Mod\{x(yz) \approx (xy)z, xyxzxyx \approx xyzyx, x^2y^2z \approx x^2yx^2yz, xy^2z^2 \approx xyz^2yz^2\}$, i.e., the variety generated by these identities.

Our results can be used for a very short proof of the fact, proved first in [7], that the variety V_{HS} defined by the identities $x(yz) \approx (xy)z, xyxzxyx \approx xyzyx, x^2 \approx x^4, x^2y^2z \approx x^2yx^2yz, xy^2z^2 \approx xyz^2yz^2$ is the greatest solid variety of semigroups.

For more background on hypersubstitutions and M-solid varieties we refer to [2] and to [4], respectively.

2. The two-generated free algebra over V_{HR}

The Reg-hyper model class of the associative law is the greatest regular solid variety of semigroups. By definition this class $H_{Reg}ModAss$ is the class of all semigroups which satisfy the associative identity (Ass) as regular hyperidentity. The class $H_{Reg}ModAss$ is a variety (see e.g. [2]) and therefore there is some interest to find a generating system for the set of all identities satisfied in $H_{Reg}ModAss$ and to know whether or not $H_{Reg}ModAss$ is finitely axiomatizable by identities. If we apply the following regular hypersubstitutions to the associative identity, we obtain the identities listed in the following table.

hypersubstitution	identity
$\sigma_{f(x,y)}$	$x(yz) \approx (xy)z$
$\sigma_{f(f(x,y),x)}$	$xyzyx \approx xyxzxyx$
$\sigma_{f(f(x,x),y)}$	$x^2y^2z \approx x^2yx^2yz$
$\sigma_{f(x,f(y,y))}$	$xyz^2yz^2 \approx xy^2z^2$

All these identities have to be satisfied in $H_{Reg}ModAss$. Therefore we have $H_{Reg}ModAss \subseteq V_{HR} := Mod\{x(yz) \approx (xy)z,\ xyxzxyx \approx xyzyx,\ x^2y^2z \approx x^2yx^2yz,\ xy^2z^2 \approx xyz^2yz^2\}$. Our aim is to prove the converse inclusion. The basic idea is to calculate all normal forms of binary terms with respect to the variety V_{HR} and to apply the corresponding hypersubstitutions to the associative law. If all resulting identities are satisfied in the variety V_{HR}, this variety satisfies the associative law as a regular hyperidentity and $V_{HR} \subseteq H_{Reg}ModAss$. First of all we determine some more identities satisfied in V_{HR}.

Lemma 2.1. *The following equations are identities in the variety V_{HR}:*

(i) $xy^kzy^lx \approx xy^kx^azx^ay^lx, 1 \leq k, l, a \in \mathbb{N}$,

(ii) $xyzyx \approx xy^azy^ax$ *for* $a \geq 2$,

(iii) $x^5 \approx x^7$,

(iv) $xy^3zyx \approx xyzyx$,

(v) $xyzy^3x \approx xyzyx$,

(vi) $x^2yx^4 \approx x^2yx^2$,

(vii) $x^4yx^2 \approx x^2yx^2$,

(viii) $xy^2zyx \approx xyzy^2x$.

Proof.

(i) Without restriction of the generality we may assume that $k \leq l$. Then we have $xy^kzy^lx \approx (xy)^kx^{a+l-k}zx^a(yx)^l \approx xy^kx^azx^ay^lx$ using the identity $xyxzxyx \approx xyzyx$.

(ii) Using again $xyxzxyx \approx xyzyx$ we obtain $xyzyx \approx xyxzxyx \approx xyxy^{a-1}zy^{a-1}xyx \approx xy^azy^ax$.

(iii) This follows from $(x^2y)^2z \approx x^2y^2z$ if we identify all three variables.

(iv) Here we have

$$\begin{aligned} xy^3zyx &\approx xy^8zy^6x \text{ by (ii)} \\ &\approx xy^6zy^6x \text{ by } x^5 \approx x^7 \\ &\approx xyzyx \text{ by (ii).} \end{aligned}$$

(v) Follows similar as (iv).

(vi) $x^2yx^4 \approx xxyx^3x \approx xxyxx$ by (v).

(vii) This can be derived in a similar way.

(viii) By (ii) and (iv) we have $xy^2zyx \approx xy^3zy^2x \approx xyzy^2x$. ■

Lemma 2.2. *For $a, b, c, d \geq 1, a, b, c, d \in \mathbb{N}$ the following equations are identities in the variety V_{HR}:*

(i) $xy^ax^by^cx \approx xy^{a+c+2}x$ *if b is even,*

(ii) $xy^ax^by^cx \approx xyxyx$ *if b is odd and $a + c$ is even,*

(iii) $xy^ax^by^cx \approx xy^2xyx$ *if b and $a + c$ are odd,*

(iv) $x^{a+b}yx^a \approx x^ayx^{a+b}$, *if $a \geq 2$,*

(v) $x^a y^b x^c y^d \approx x^{a+c} y^{b+d}$ *if* $a, b, c, d \geq 2$,

(vi) $xy^a x \approx xy^{a-2} x \in IdV_{HR}$ *if* $a \geq 5$,

(vii) $xy^a x^b y \approx xy^a x^{b-2} y$ *if* $a, b \geq 3$,

(viii) $xy^a x^b y \approx xy^{a-2} x^b y$ *if* $a, b \geq 3$,

(ix) $x^2 y^a z \approx x^2 y^{a-2} z$ *if* $a \geq 4$,

(x) $xy^a z^2 \approx xy^{a-2} z^2$ *if* $a \geq 4$,

(xi) $xy^3 x^a y \approx xyx^{a+2} y$,

(xii) $xy^a x^3 y \approx xy^{a+2} xy$,

(xiii) $x^a y^b xyx \approx x^{a+1} y^{b+1} x$ *if* $a \geq 2$,

(xiv) $xyxy^b x^a \approx xy^{b+1} x^{a+1}$ *if* $a \geq 2$,

(xv) $yx^a y^b x^c y^d x \approx yx^{a+c} y^{b+d} x$,

(xvi) $x^a y^b x \approx x^{a-2} y^b x$ *if* $a \geq 4, b \geq 2$,

(xvii) $xy^b x^a \approx xy^b x^{a-2}$ *if* $a \geq 4, b \geq 2$.

Proof.

(i) Assume that $a \leq c$. Then by Lemma 2.1 (i) we have

$$\begin{aligned} xy^a x^b y^c x &\approx xy^a x^2 x^b x^2 y^c x \\ &\approx xy^a x^2 y^2 x^b y^2 x^2 y^c x \text{ by } xyxzxyx \approx xyzyx \\ &\approx xy^a x^2 y^2 x^2 y^2 x^2 y^c x \text{ by Lemma 2.1 (vi) since } b \text{ is even} \\ &\approx xy^a x^2 y^2 xy^2 xy^c x \text{ by Lemma 2.1 (ii)} \\ &\approx xy^a x^2 y^2 x^2 y^c x \text{ by } (x^2 y)^2 z \approx x^2 y^2 z \\ &\approx xy^{a+c+2} x \text{ by Lemma 2.1 (i).} \end{aligned}$$

(ii) We have

$$xy^ax^by^cx \approx xy^{a+a+1}x^by^{c+a+1}x \text{ by Lemma 2.1 (ii)}$$
$$\approx xyx^byx \text{ by Lemma 2.1 (v), (iv)}$$
$$\approx xyxyx \text{ by } xyxzxyx \approx xyzyx.$$

(iii) In this case we have

$$xy^ax^by^cx \approx xy^{a+a}x^by^{c+a}x \text{ by Lemma 2.1 (ii)}$$
$$\approx xy^2x^byx \text{ by Lemma 2.1 (v)}$$
$$\approx xy^2xyx \text{ by Lemma 2.1 (i).}$$

(iv) We have

$$x^{a+b}yx^a \approx x^{a+b}yx^{b+b+a} \text{ by Lemma 2.1 (vi)}$$
$$\approx x^ayx^{a+b} \text{ by Lemma 2.1 (ii).}$$

(v) There holds

$$x^ay^bx^cy^d \approx x^{a+c-2}y^2x^2y^{d+b-2} \text{ by (iv)}$$
$$\approx x^{a+c}y^2x^2y^{d+b-2} \text{ by Lemma 2.1 (iv)}$$
$$\approx x^{a+c}yx^2yx^2y^{d+b-2}$$
$$\approx x^{a+c}yx^2y^{d+b-1}$$
$$\approx x^{a+c}y^{d+b} \text{ using } (x^2y)^2z \approx x^2y^2z$$
$$\text{and } x(yz^2)^2 \approx xy^2z^2, \text{ respectively.}$$

(vi) Using $xyxzxyx \approx xyzyx$ we get

$$xy^ax \approx xyxyy^{a-4}yxyx$$
$$\approx xyxy^{a-4}xyx$$
$$\approx xy^{a-2}x.$$

(vii) By Lemma 2.1 (i) we have

$$\begin{aligned} xy^a x^b y &\approx xy^a xyxyx^{b-2} y \\ &\approx xy^{a+2} x^{b-2} y \\ &\approx xy^a x^{b-2} y \text{ by (vi).} \end{aligned}$$

(viii) can be proved similar to (vii).

(ix) Using $(x^2y)^2z \approx x^2y^2z$ we have

$$\begin{aligned} x^2y^az &\approx x^2yx^2yx^2yx^2y^{a-3}z \\ &\approx x^2yx^2x^2x^2y^{a-3}z \text{ using } xyxzxyx \approx xyzyx \\ &\approx x^2yx^2y^{a-3}z \text{ by Lemma 2.1 (vi)} \\ &\approx x^2y^{a-2}z \text{ using } (x^2y)^2z \approx x^2y^2z. \end{aligned}$$

(x) can be proved similar to (ix).

(xi) Using Lemma 2.1 (i) we obtain

$$\begin{aligned} xy^3x^ay &\approx xyxyxyx^ay \\ &\approx xyx^{a+2}y. \end{aligned}$$

(xii) can be proved similar to (xi).

(xiii) By Lemma 2.1 (ii) we have

$$\begin{aligned} x^ay^bxyx &\approx x^ay^{b+1}xy^2x \\ &\approx x^ay^2xy^{b+1}x \text{ by (iv)} \\ &\approx x^{a+1}y^{b+1}x \text{ using } x(yz^2)^2 \approx xy^2z^2. \end{aligned}$$

(xiv) can be proved in a similar way.

(xv) There holds

$$yx^ay^bx^cy^dx \approx yx^{a+2}y^{b+2}x^{c+2}y^{d+2}x \text{ by Lemma 2.1 (ii)}$$
$$\approx yx^{a+c+2}y^2x^2y^{d+b+2}x \text{ by (iv)}$$
$$\approx yx^{a+c}y^2x^2y^{d+b}x \text{ by Lemma 2.1 (iv)}$$
$$\approx yx^{a+c+2}y^{d+b+2}x \text{ by (v)}$$
$$\approx yx^{a+c}y^{d+b}x \text{ by (ix).}$$

(xvi) We have

$$x^ay^bx \approx x^ay^{b+2}x \text{ by (x)}$$
$$\approx x^ayxy^bxyx \text{ using } xyxzxyx \approx xyzyx$$
$$\approx x^ayxy^bx^3yx \text{ by Lemma 2.1 (iv)}$$
$$\approx x^{a-2}yxy^bx^5yx \text{ by (iv)}$$
$$\approx x^{a-2}yxy^bxyx \text{ by Lemma 2.1 (iv)}$$
$$\approx x^{a-2}y^{b+2}x \text{ using } xyxzxyx \approx xyzyx$$
$$\approx x^{a-2}y^bx \text{ by (ix).}$$

(xvii) can be proved similar to (xvi). ∎

We use these identities to determine the elements of the 2-generated free algebra with respect to V_{HR}. First of all we want to reduce the length of the terms.

Definition 2.3. Let t be a term built up by the variables x and y. If there are natural numbers $n, k_1, \ldots, k_n \geq 1$ such that $t \approx x_1^{k_1} \ldots x_n^{k_n}$ where $x_j \in \{x, y\}$ for $1 \leq j \leq n$ and $x_j \neq x_{j+1}$, then n is called the *periodic length* of t and is denoted by $l_p(t)$.

Theorem 2.4. *For every $t \in W(\{x,y\})$ there is a term $r \in W(\{x,y\})$ with $t \approx r$ and with $l_p(r) \leq 5$.*

Proof. Assume that there is a binary term $t \in W(\{x,y\})$ such that for all $r \in W(\{x,y\})$ with $t \approx r$ we have $l_p(r) \geq 6$. Let r' be a binary term with $t \approx r' \in V_{HR}$ where r' has minimal periodic length. Then $l_p(r') \geq 6$. By Lemma 2.2 (xv) there is a binary term s with $s \approx r' \in V_{HR}$ and $l_p(s) = l_p(r') - 2$. Since $s \approx t$ this contradicts the minimality of r'. ∎

Theorem 2.4 together with Lemma 2.1 (iii) show that there are finitely many binary terms over the variety V_{HR}, i.e., the two-generated free algebra over V_{HR} is finite. Remark that the variety V_{HR} is locally finite. This can be shown using results from [3] and the identity $xyxzxyx \approx xyzyx$ which is satisfied in V_{HR}. The word $xyxzxyx$ is said to be a *Zimin word*.

Theorem 2.4 gives us a set of hypersubstitutions which we have to apply to the associative law if we want to check if the associative law is satisfied as a regular hyperidentity. But we can reduce the number of hypersubstitutions which are needed, more. From now on we assume that the first variable of the considered term t is x, i.e., $leftmost(t) = x$. In the corresponding way one defines $rightmost(t)$.

Theorem 2.5. *Let $t \in W(\{x,y\})$ such that $l_p(t) = 5$ and $leftmost(t) = x$. Then $t \approx xy^s xyx \in IdV_{HR}$ for some $s \in \{1,2\}$ or there is a term $r \in W(\{x,y\})$ with $t \approx r$ and $l_p(r) \leq 4$.*

Proof. By Lemma 2.1 (iii) there are natural numbers $a,b,c,d,e \leq 6$ with $t \approx x^a y^b x^c y^d x^e$. Here the right hand side has to start with x since for every identity $s \approx t$ which belongs to the generating system of the set of all identities in V_{HR} we have $leftmost(s) = leftmost(t)$ and $rightmost(s) = rightmost(t)$. If c is even, then by Lemma 2.2 (i) we have $t \approx x^a y^{b+d+2} x^c$ and if c is odd, then by Lemma 2.2 (ii) we get $t \approx x^a yxyx^e$ if $b+d$ is even and $t \approx x^a y^2 xyx^e$ if $b+d$ is odd. Now for $a \geq 2$ we apply Lemma 2.2(xiii) and obtain $t \approx x^{a+1} y^2 x^e$ in the first case and $t \approx x^{a+1} y^3 x^e$ in the second one. So, it is left to consider the cases $t \approx xy^2 xyx^e$ and $t \approx xyxyx^e$. If $e \geq 2$, then by Lemma 2.1 (viii) and Lemma 2.2 (xiv) we have $t \approx xy^3 x^{e+1}$ or $t \approx xy^2 x^{e+1}$ otherwise; i.e., if $e = 1$, we have $t \approx xy^2 xyx$ or $t \approx xyxyx$. ∎

Theorem 2.6. *Let* $t \in W(\{x, y\})$ *with* $l_p(t) = 4$ *and* $leftmost(t) = x$. *Then*

$$t \approx xy^kxy \ \in IdV_{HR} \text{ for some } k \in \{2,4\} \text{ or}$$

$$t \approx xyx^ky \ \in IdV_{HR} \text{ for some } k \in \{2,4\} \text{ or}$$

$$t \approx xy^3xy \ \in IdV_{HR} \text{ or}$$

$$t \approx x^ky^lxy \in IdV_{HR} \text{ or for some } k,l \in \{2,3\}$$

$$t \approx xyx^ky^l \in IdV_{HR} \text{ for some } k,l \in \{2,3\} \text{ or}$$

$$t \approx x^2yx^2y \in IdV_{HR} \text{ or}$$

$$t \approx xy^2xy^2 \in IdV_{HR} \text{ or}$$

$$t \approx x^kyxy \ \in IdV_{HR} \text{ for some } k \in \{2,3\} \text{ or}$$

$$t \approx xyxy^k \ \in IdV_{HR} \text{ for some } k \in \{2,3\} \text{ or}$$

$$t \approx xy^2x^2y \in IdV_{HR} \text{ or}$$

$$t \approx x^2yxy^2 \in IdV_{HR} \text{ or}$$

there is an $r \in W(\{x, y\})$ *with* $t \approx r$ *and* $l_p(r) \leq 3$.

Proof. There are $a, b, c, d \in \mathbb{N}$ with $t \approx x^ay^bx^cy^d$. By Lemma 2.1 (iii) we may assume that $a, b, c, d \leq 6$. Suppose that $b, c \geq 2$. By Lemma 2.2 (ix), (x) we may assume that $b, c \leq 3$.

If $b = 3$ or $c = 3$ we get $t \approx x^ayx^{c+2}y^d$ and $t \approx x^ay^{b+2}xy^d$, respectively, by Lemma 2.2 (xi), (xii). If $b = c = 2, a = 1$ and $d \geq 2$, then

$$\begin{aligned} t &\approx xy^4x^2y^d && \text{by Lemma 2.1 (vi)} \\ &\approx xy^3x^2y^{d+1} && \text{by Lemma 2.2 (iv)} \\ &\approx xyx^4y^{d+1} && \text{by Lemma 2.2 (xi)} \\ &\approx xyx^2y^{d+1} && \text{by Lemma 2.2 (x)} \\ &\approx xyx^2y^p && \text{for some } p \leq 3 \text{ by Lemma 2.2 (xvii).} \end{aligned}$$

If $b = c = 2, d = 1$ and $a \geq 2$, then in a similar way we show the existence of a number $p \leq 3$ such that $t \approx x^p y^2 xy$.

If $b = c = 2$ and $a = d = 1$, then we get $t \approx xy^2x^2y$.

Now we consider the case $b = 1$. Then $t \approx x^a y x^c y^d \in IdV_{HR}$.

If $a, c, d \geq 2$, then

$$\begin{aligned} t &\approx x^a y x^{c+2} y^d && \text{by Lemma 2.1 (vi)} \\ &\approx x^a y^3 x^c y^d && \text{by Lemma 2.2 (xi)} \\ &\approx x^{a+c} y^{d+3} && \text{by Lemma 2.2 (v).} \end{aligned}$$

If $c = 1$, then $t \approx x^a y x y^d$. Because of Lemma 2.2 (ix),(x) we may assume that $a, d \leq 5$. Suppose that $a \geq 3$ and $d \geq 2$. Then we have

$$\begin{aligned} x^a y x y^d &\approx x^{a-1}(xy)^2 y^{d-1} \\ &\approx x^{a-1}(xy)x^2(xy)y^{d-1} && (\text{using } x^2y^2y \approx x^2yx^2z) \\ &\approx x^{a-1}yx^2yy^{d-1} && (\text{using } xyxzxyx \approx xyzyx) \\ &\approx x^{a-1}y^{d+1} && (\text{using } x^2y^2z \approx x^2yx^2yz). \end{aligned}$$

If $a \geq 2$ and $d \geq 3$, then dually we get $t \approx x^{a+1}y^{d-1}$.

For $a = 4, 5$ there holds

$$\begin{aligned} x^a yxy &\approx x^{a-3}x^2(xy)^2 \\ &\approx x^{a-3}x(xy)^2x(xy)^2 && (\text{using } x(yz^2)^2 \approx xy^2z^2) \\ &\approx x^{a-1}y^2xyxy && (\text{using } xyxzxyx \approx xyzyx) \\ &\approx x^{a-1}y^2xy^3xy && (\text{using Lemma 2.1 (v)}) \\ &\approx x^a y^3 xy && (\text{using } x(yz^2)^2 \approx xy^2z^2) \\ &\approx x^{a-2}y^3xy && (\text{using Lemma 2.2 (xvi)}). \end{aligned}$$

For $d = 4, 5$ we dually have $xyxy^d \approx xyx^3y^{d-2}$. If $c \geq 2$ and $a = d = 1$, then $t \approx xyx^cy$.

By Lemma 2.2 (vi) we may assume that $c \leq 4$. If $c = 3$, then we have $xy^3xy \approx xyx^3y$ by Lemma 2.2 (xi).

If $c, d \geq 2$ and $a = 1$, then $t \approx xyx^cy^d$. Here by Lemma 2.2 (xvi) and Lemma 2.2 (x) we may assume that $d \leq 3$ and $c \leq 3$.

If $c, a \geq 2$ and $d = 1$, then $t \approx x^ayx^cy \approx x^2yx^{c+a-2}y$ by Lemma 2.2 (iv). Using Lemma 2.1 (vi) we get for some $p \in \{2, 3\}$ the identity $t \approx x^2yx^py$. If $p = 3$, then $x^2yx^3y \approx x^2y^3xy$ by Lemma 2.2 (xi).

In the case $c = 1$ in a similar way we get $t \approx xy^kxy$ for some $k \in \{2, 4\}$ or $t \approx xyx^ky^l$ for some $k, l \in \{2, 3\}$ or $t \approx xy^2xy^2$ or $t \approx x^2yxy^2$ or $t \approx x^kyxy \in V_{HR}$ or $t \approx xyxy^k$ for some $k \in \{1, 2, 3\}$ or there is an $r \in W(\{x, y\})$ with $l_p(r) \leq 3$ and $t \approx r$. ■

Theorem 2.7. *Let $t \in W(\{x, y\})$ with $l_p(t) = 3$ and $leftmost(t) = x$. Then*

$$t \approx x^kyx \ \in IdV_{HR} \text{ for some } k \in \{1, \ldots, 5\} \text{ or}$$

$$t \approx xyx^k \ \in IdV_{HR} \text{ for some } k \in \{1, \ldots, 5\} \text{ or}$$

$$t \approx x^ky^lx \in IdV_{HR} \text{ for some } k, l \in \{2, 3\} \text{ or}$$

$$t \approx xy^lx^k \ \in IdV_{HR} \text{ for some } k, l \in \{2, 3\} \text{ or}$$

$$t \approx xy^kx \ \in IdV_{HR} \text{ for some } k \in \{2, 3, 4\} \text{ or}$$

$$t \approx x^2y^lx^k \in IdV_{HR} \text{ for some } l \in \{1, 2, 3\}, k \in \{2, 3\}.$$

Proof. There are natural numbers a, b, c with $t \approx x^ay^bx^c$. By Lemma 2.1 (iii) we may assume that $1 \leq a, b, c \leq 6$.

If $a, c \geq 2$, then

$$\begin{aligned} t &\approx x^2y^bx^{c+a-2} && \text{by Lemma 2.2 (iv)} \\ &\approx x^2y^bx^p && \text{for some } p \in \{2,3\} \text{ by Lemma 2.1 (vi)} \\ &\approx x^2y^qx^p && \text{for some } q \in \{1,2,3\} \text{ by Lemma 2.2 (ix).} \end{aligned}$$

If $a = 1$ and $b, c \geq 2$, then $t \approx xy^bx^p$ for some $p \in \{2, 3\}$ by Lemma 2.2 (xvii) and then $t \approx xy^qx^p$ for some $q \in \{2, 3\}$ by Lemma 2.2 (x).

If $a = b = 1$, then $t \approx xyx^c$.

If $c = 6$, then $t \approx xyx^4$ by Lemma 2.2 (x).

If $a = c = 1$, then $t \approx xy^b x$.

If $b = 5, 6$, then $t \approx xy^{b-2}x$ by Lemma 2.2 (vi).

If $c = 1$ and $a, b \geq 2$, then $t \approx x^p y^q x$ for some $p, q \in \{2, 3\}$ by Lemma 2.2 (ix) and (xvi), respectively.

If $a \geq 2$ and $b = c = 1$, then $t \approx x^a yx$.

If $a = 6$, then $t \approx x^4 yx$ by Lemma 2.2 (ix). ■

Theorem 2.8. *Let $t \in W(\{x, y\})$ with $l_p(t) = 2$ and $leftmost(t) = x$. Then*

$$t \approx xy^k \in IdV_{HR} \text{ for some } k \in \{1, \ldots, 5\} \text{ or}$$

$$t \approx x^k y \in IdV_{HR} \text{ for some } k \in \{1, \ldots, 5\} \text{ or}$$

$$t \approx x^k y^l \in IdV_{HR} \text{ for some } k, l \in \{2, 3\} \text{ or}$$

$$t \approx x^2 y^4 \in IdV_{HR}.$$

Proof. There are natural numbers a, b with $t \approx x^a y^b$. We may assume that $1 \leq a, b \leq 6$. If $a, b \geq 2$, then $t \approx x^p y^q$ for some $p, q \in \{2, 3, 4\}$ by Lemma 2.2 (ix), (x). If $a = 1$, then $t \approx xy^b$. If $b = 6$ then we get $t \approx xy^4$ by Lemma 2.2 (x). If $b = 1$, then dually we get $t \approx x^k y$. Moreover, we have

$$\begin{aligned} x^2y^4 &\approx x^2y^6 && \text{(using Lemma 2.2 (ix))} \\ &\approx x^2y^2x^2y^4 && \text{(using } x^2y^2z \approx x^2yx^2yz\text{)} \\ &\approx x^4y^2x^2y^2 && \text{(using Lemma 2.1 (vi), (vii))} \\ &\approx x^6y^2 && \text{(using } x(yz^2)^2 \approx xy^2z^2\text{)} \\ &\approx x^4y^2 && \text{(using Lemma 2.2 (x)).} \end{aligned}$$

■

Theorems 2.4–2.8 allow us to determine a set of binary terms. To prove that no proper subset of this set represents all binary terms in V_{HR} we need some technical lemmas.

Proposition 2.9. *Every equation $s \approx t \in IdV_{HR}$ satisfies the following condition* $(*)$:

$(*)$ (i) *The first letter in s agrees with the first letter in t and the second letter in s agrees with the second letter in t.*

(ii) *The last letter in s agrees with the last letter in t and the second last letter in s agrees with the second last letter in t.*

Proof. Every equation from the set consisting of the four equations which generate IdV_{HR} has this property. If we can show that all equations satisfying $(*)$ form an equational theory, then IdV_{HR} satisfies condition $(*)$. But this becomes clear if we check the five derivation rules for identities. ■

If we denote by $c_x(s)$ the number of occurrences of the variable x in the term s, then IdV_{HR} satisfies the following condition $(**)$:

Proposition 2.10. *Every equation $s \approx t \in IdV_{HR}$ and every $x \in X$ satisfies the following condition* $(**)$:

$(**)$ (i) $c_x(s) \equiv c_x(t) mod\ 2$,

(ii) $c_x(s) = 1$ iff $c_x(t) = 1$.

Proof. We will give a proof by induction on the length of a proof. If $s \approx t$ belongs to the generating system of IdV_{HR}, i.e., if $s \approx t \in \{(xy)z \approx x(yz), (x^2y)^2z \approx x^2y^2z, x(yz^2)^2 \approx xy^2z^2, xyxzxyx \approx xyzyx\}$, then obviously $s \approx t$ satisfies (i) and (ii). For every term r the identity $r \approx r$ satisfies $(**)$. If $s \approx t, t \approx w \in IdV_{HR}$ satisfy $(**)$, then $t \approx s$ and $s \approx w$ satisfy $(**)$ too. By $sub_r^w(s)$ we denote the term which arises from s if we substitute for $r \in X$ the term $w \in W(X)$. Let $s \approx t \in IdV_{HR}$ satisfying $(**)$, $r \in X$ and $w \in W(X)$. If $r = x$, then $c_x(sub_r^w(s)) = c_x(w)c_x(s)$ and $c_x(sub_r^w(t)) = c_x(w)c_x(t)$. From $c_x(s) \equiv c_x(t) mod\ 2$ it follows $c_x(w)c_x(s) \equiv c_x(w)c_x(t) mod\ 2$, i.e., $c_x(sub_r^w(s)) \equiv c_x(sub_r^w(t)) mod\ 2$. Moreover, from $c_x(s) = 1$ iff $c_x(t) = 1$ there follows $c_x(w)c_x(s) = 1$ iff $c_x(w) = 1$ and $c_x(s) = 1$ iff $c_x(w)c_x(t) = 1$. Thus $c_x(sub_r^w(s)) = 1$ iff $c_x(sub_r^w(t)) = 1$.

If r is a variable different from x, then $c_x(sub_r^w(s)) = c_x(w)c_r(s) + c_x(s)$ and $c_x(sub_r^w(t)) = c_x(w)c_r(t) + c_x(t)$. From $c_x(s) \equiv c_x(t) mod\ 2$ and $c_r(s) \equiv c_r(t) mod\ 2$ there follows $c_x(w)c_r(s) + c_x(s) \equiv c_x(w)c_r(t) + c_x(t) mod\ 2$, i.e., $c_x(sub_r^w(s)) \equiv c_x(sub_r^w(t)) mod\ 2$.

We remark that the variety SL of all semilattices is contained in V_{HR}, i.e., $IdV_{HR} \subseteq Id\ SL$. The set $Id\ SL$ consists of exactly all regular equations of type $\tau = (2)$, i.e., if $s \approx t \in IdV_{HR}$ then $c_y(s) = 0$ iff $c_y(t) = 0$ for every variable y. Then from $c_x(s) = 1$ iff $c_x(t) = 1$ we obtain $c_x(sub_r^w(t)) = c_x(w)c_r(t) + c_x(t) = 1$ iff $c_x(t) = 1$ and $c_x(w)c_r(t) = 0$ or $c_x(t)c_r(t) = 1$ and $c_x(t) = 0$. This is satisfied if and only if $c_x(s) = 1$ and $c_x(w)c_r(s) = 0$ or $c_x(w)c_r(s) = 1$ and $c_x(s) = 0$ iff $c_x(w)c_r(s) + c_x(s) = 1 = c_x(sub_r^w(s))$. Therefore the condition $(**)$ is satisfied after application of the substitution rule.

Assume now that $s \approx t, u \approx w \in IdV_{HR}$ satisfy $(**)$. Then $c_x(s) \equiv c_x(t) mod\ 2$ and $c_x(u) \equiv c_x(v) mod\ 2$, i.e., $c_x(f(s,u)) \equiv c_x(f(t,w)) mod\ 2$. Moreover we have $c_x(s) = 1$ iff $c_x(t) = 1$ and $c_x(u) = 1$ iff $c_x(w) = 1$. This gives

$$\begin{aligned} c_x(s) + c_x(u) = 1 &\Leftrightarrow (c_x(s) = 1 \wedge c_x(u) = 0) \vee (c_x(s) = 0 \wedge c_x(u) = 1) \\ &\Leftrightarrow (c_x(t) = 1 \wedge c_x(w) = 0) \vee (c_x(t) = 0 \wedge c_x(w) = 1) \\ &\Leftrightarrow c_x(t) + c_x(w) = 1. \end{aligned}$$

This means, $c_x(f(s,u)) = 1$ iff $c_x(f(t,w)) = 1$ and the condition $(**)$ is satisfied after application of the replacement rule. ■

By $l(s)$ we denote the length of the term s. Then we have

Proposition 2.11. *For $s \approx t \in IdV_{HR}$ the following condition is satisfied: If $s \approx t \notin IdSEM$, i.e., if $s \approx t$ is not derivable only from the associative law, then*

$(***)$ (i) $l(s), l(t) \geq 5$,

(ii) *If $l(s) = 5$, then s is of the form a) x^2y^2z or xy^2z^2 or $xyzyx$,*

(iii) *If $l(t) = 5$, then t is of the form a),*

(iv) *If $l(s) = 6$ then s is of the form b) wxy^2z^2 or $wxyzyx$ or x^2y^2zw or xy^2z^2w or $xyzyxw$ or $xyzwyx$,*

(v) *If $l(t) = 6$, then t is of the form b)*

Proof. We will give a proof by induction on the length of a proof. If $s \approx t \in \{(xy)z \approx x(yz), (x^2y)^2z \approx x^2y^2z, x(yz^2)^2 \approx xy^2z^2, xyxzxyx \approx xyzyx\}$, then $s \approx t$ satisfies $(***)$. For $r \in W(X)$ the identity $r \approx r$ satisfies $(***)$. If $s \approx t, t \approx w \in IdV_{HR}$ satisfy $(***)$, then $t \approx s$ and $s \approx w$ satisfy $(***)$ too. Let $s \approx t \in IdV_{HR}$ be an identity which satisfies $(***)$ and assume that $r \in X$ and $w \in W(X)$ and that $sub_r^w(s) \approx sub_r^w(t) \notin IdSEM$. Then $s \approx t \notin IdSEM$, i.e., $l(s), l(t) \geq 5$ and thus $l(sub_r^w(s)), l(sub_r^w(t)) \geq 5$. Assume that $l(sub_r^w(s)) = 5$. This is only possible if $l(s) = 5$ and $w \in X$. From $l(s) = 5$ it follows that s is of the form a). Consequently, $sub_r^w(s)$ is of the form a). For $l(sub_r^w(t)) = 5$ we conclude in the same way. Let now $l(sub_r^w(s)) = 6$. This is only possible, if

(α) $l(s) = 5$ and $l(w) = 2$ and $c_r(s) = 1$ or

(β) $l(s) = 6$ and $w \in X$.

We consider the case (α). From $l(s) = 5$ there follows that s is of the form a). Since $l(w) = 2$, there are $u, v \in X$ such that $w = uv$. Thus $sub_r^w(s)$ is of the form x^2y^2uv or uvy^2z^2 or $xyuvyx$. In the case (β) from $l(s) = 6$ there follows that s is of the form b). Consequently, $sub_r^w(s)$ is of the form b). In a similar way one shows that $sub_r^w(t)$ is of the form b) if $l(sub_r^w(t)) = 6$. Now we check the replacement rule. Let $s \approx t, u \approx w \in IdV_{HR}$ be identities satisfying $(***)$. If $f(s,u) \approx f(t,w) \notin IdSEM$, then $s \approx t \notin IdSEM$ or $u \approx w \notin IdSEM$. We consider the following cases:

Case 1. If $s \approx t, u \approx w \notin IdSEM$, then $l(t), l(s), l(u), l(w) \geq 5$ and thus $l(f(s,u)), l(f(t,w)) \geq 10$.

Case 2. If $s \approx t \notin IdSEM, u \approx w \in IdSEM$, then we have $l(s), l(t) \geq 5$ and thus $l(f(s,u)), l(f(t,w)) \geq 6$. If $l(f(s,u)) = 6$, then $l(s) = 5$ and $u \in X$, i.e., s is of the form a). This yields that $f(s,u)$ is of the form x^2y^2zw or xy^2z^2w or $xyzyxw$

Case 3. If $s \approx t \in IdSEM, u \approx w \notin IdSEM$, then similar we have that $f(s,u) \approx f(t,v)$ satisfies $(***)$. ■

Now we can prove:

Theorem 2.12. *The free algebra $F_{V_{HR}}(\{x,y\})$ consists of exactly* 128 *elements which can be represented by the following terms:*

(1) $xyxyx$,
(2) xy^2xyx,
(3) xy^2xy,
(4) xy^3xy,
(5) xy^4xy,
(6) xyx^2y,
(7) xyx^4y,
(8) x^2y^2xy,
(9) x^2y^3xy,
(10) x^3y^2xy,
(11) x^3y^3xy,
(12) xyx^2y^2,
(13) xyx^2y^3,
(14) xyx^3y^2,
(15) xyx^3y^3,
(16) x^2yx^2y,
(17) xy^2xy^2,
(18) $xyxy$,
(19) x^2yxy,
(20) x^3yxy,
(21) $xyxy^2$,
(22) $xyxy^3$,
(23) x^2yxy^2,
(24) xy^2x^2y,
(25) xyx,
(26) x^2yx,
(27) x^3yx,
(28) x^4yx,
(29) x^5yx,
(30) x^2yx^2,
(31) x^2yx^3,
(32) xyx^2,
(33) xyx^3,
(34) xyx^4,
(35) xyx^5,
(36) x^2y^2x,
(37) x^3y^2x,
(38) x^3y^3x,
(39) x^2y^3x,
(40) xy^2x^2,
(41) xy^2x^3,
(42) xy^3x^2,
(43) xy^3x^3,
(44) xy^2x,
(45) xy^3x,
(46) xy^4x,
(47) $x^2y^2x^2$,
(48) $x^2y^3x^2$,
(49) $x^2y^2x^3$,
(50) $x^2y^3x^3$,
(51) xy,
(52) x^2y,
(53) x^3y,
(54) x^4y,
(55) x^5y,
(56) xy^2,
(57) xy^3,
(58) xy^4,
(59) xy^5,
(60) x^2y^2,
(61) x^2y^3,
(62) x^2y^4,
(63) x^3y^2,
(64) x^3y^3

and all terms arising from the terms (1)–(64) *by exchanging x and y.*

Proof. We show that any two different terms of this list cannot form an identity in V_{HR}. Using Proposition 2.9 we partition at first the set of the terms of our list into classes with the property that two terms in different classes cannot form an identity since the condition from Proposition 2.9 is not satisfied. This gives exactly the following classes:

$$\{(30),(35),(47),(48),(49),(50)\}$$
$$\cup\{(8),(9),(10),(11),(16),(19),(20),(52),(53),(54),(55)\}$$
$$\cup\{(23),(60),(61),(62),(63),(64)\}$$
$$\cup\{(26),(27),(28),(29),(36),(37),(38),(39)\}$$
$$\cup\{(31),(32),(33),(34),(40),(41),(42),(43)\}$$
$$\cup\{(3),(4),(5),(6),(7),(18),(24),(51)\}$$
$$\cup\{(12),(13),(14),(15),(17),(21),(22),(56),(57),(58),(59)\}$$
$$\cup\{(1),(2),(25),(44),(45),(46)\}$$

and the dual classes.

Our aim is to divide these classes in singleton classes. We may restrict ourselves to the classes which contain the terms (1)–(64). For the other classes we can use dual arguments.

Using Proposition 2.10 we get the following finer partitions:

The class $\{(30),(35),(47),(48),(49),(50)\}$ is divided into $\{(47)\}\cup\{(48)\}\cup\{(49)\}\cup\{(50)\}\cup\{(30)\}\cup\{(35)\}$.

The class $\{(8),(9),(10),(11),(16),(19),(20),(52),(53),(54),(55)\}$ is divided into $\{(8)\}\cup\{(53),(55)\}\cup\{(9),(19)\}\cup\{(10)\}\cup\{(52),(54)\}\cup\{(11),(16),(20)\}$.

The class $\{(23),(60),(61),(62),(63),(64)\}$ splits into $\{(23),(64)\}\cup\{(60),(62)\}\cup\{(61)\}\cup\{(63)\}$.

The class $\{(26),(27),(28),(29),(36),(37),(38),(39)\}$ can be divided into $\{(26),(28)\}\cup\{(27),(29)\}\cup\{(38)\}\cup\{(39)\}\cup\{(36)\}\cup\{(37)\}$.

The class $\{(31),(32),(33),(34),(40),(41),(42),(43)\}$ splits into $\{(31),(33)\}\cup\{(42)\}\cup\{(32),(34)\}\cup\{(43)\}\cup\{(40)\}\cup\{(41)\}$.

The class $\{(3),(4),(5),(6),(7),(18),(24),(51)\}$ splits into $\{(3),(5)\}\cup\{(4),(18)\}\cup\{(6),(7)\}\cup\{(24)\}\cup\{(51)\}$.

The class $\{(12),(13),(14),(15),(17),(21),(22),(56),(57),(58),(59)\}$ can be divided into $\{(12)\}\cup\{(57),(59)\}\cup\{(13)\}\cup\{(56),(58)\}\cup\{(14),(21)\}\cup\{(15),(17),(22)\}$.

The class $\{(1),(2),(25),(44),(45),(46)\}$ splits into $\{(1)\}\cup\{(2)\}\cup\{(25)\}\cup\{(45)\}\cup\{(44),(46)\}$.

Now the following non-singleton classes are left

$\{(53),(55)\}$,$\{(9),(19)\}$,$\{(52),(54)\}$,$\{(11),(16),(20)\}$,$\{(23),(64)\}$,$\{(60),(62)\}$, $\{(26),(28)\}$, $\{(27),(29)\}$, $\{(31),(33)\}$, $\{(32),(34)\}$, $\{(3),(5)\}$, $\{(4),(18)\}$, $\{(6),(7)\}$,$\{(57),(59)\}$, $\{(56),(58)\}$, $\{(14),(21)\}$, $\{(15),(17),(22)\}$,$\{(44),(46)\}$.

To separate $\{(53),(55)\}$, $\{(52),(54)\}$,$\{(60),(62)\}$,$\{(26),(28)\}\{(31),(33)\}$, $\{(4),(18)\}$, $\{(57),(59)\}$, $\{(56),(58)\}$, $\{(44),(46)\}$ we use $(***)$ (i).

For $\{(9),(19)\}$, $\{(11),(16),(20)\}$, $\{(23),(64)\}$, $\{(27),(29)\}$, $\{(32),(34)\}$, $\{(3),(5)\}$, $\{(6),(7)\}$, $\{(14),(21)\}$, $\{(15),(17),(22)\}$ we use $(***)$ (ii) or (iv). This finishes the proof. ■

3. The greatest regular-solid variety of semigroups

To prove that $V_{HR} \subseteq H_{Reg}ModAss$ we have to apply all regular hypersubstitutions to the associative identity and to check whether the resulting equations are satisfied in V_{HR}. The following relation on the set Reg of all regular hypersubstitutions simplifies this procedure.

Definition 3.1. For any two hypersubstitutions σ_1, σ_2 of type τ and for a variety V of type τ we define

$$\sigma_1 \sim_V \sigma_2 \Longleftrightarrow \sigma_1(f) \approx \sigma_2(f) \in Id\ V.$$

Then Płonka proved in [6] the following proposition:

Proposition 3.2. *If $s \approx t \in IdV$ for a variety V of type τ, if σ_1, σ_2 are hypersubstitutions of type τ with $\sigma_1 \sim_V \sigma_2$ and if $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$, then also $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$.*

Therefore we can partition the set Hyp of all hypersubstitutions of type $\tau = (2)$ or its submonoid Reg of all regular hypersubstitutions into equivalence classes with respect to $\sim_{V_{HR}}$ and have to check the associative law only for one representative from each class. If $\sigma_{(i)}$ denotes the hypersubstitution which maps the operation symbol f to one of the terms (i) where i is one of the numbers 128 denoting the elements of $F_{V_{HR}}(\{x,y\})$, then it is enough to consider the hypersubstitutions $\sigma_{(i)}$ representing the elements of $Reg/\sim_{V_{HR}} = \{[\sigma_{(i)}] \mid i = 1,\ldots,128\}$. First of all we prove some more useful identities in V_{HR}.

Lemma 3.3. *For $1 \leq k \in \mathbb{N}$ there holds*

(i) $(x^k y)^k z \approx x^k y^k z \in IdV_{HR}$,

(ii) $z(xy^k)^k \approx zx^k y^k \in IdV_{HR}$.

Proof. If $k = 1$, then all is clear. If $k \geq 3$ is odd, then

$$\begin{aligned}(x^k y)^k z &\approx x^k y(x^2 y)^{k-1} z \text{ by Lemma 2.1 (ii)}\\ &\approx x^k y^k z \text{ if we apply } (x^2y)^2 z \approx x^2 y^2 z \ (k-1)- \text{ times.}\end{aligned}$$

If k is even, then there is a natural number p with $2p = k$ and $(x^k y)^k z \approx ((x^p)^2 y)^k z \approx (x^p)^2 y^k z$ if we apply $(x^2 y)^2 z \approx x^2 y^2 z$ $(k-1)$ times.

(ii) can be proved similarly. ∎

Lemma 3.4. *For $1 \leq k \in \mathbb{N}$ there holds*

(i) $r(xy)^k zxy \approx rx^k y^k zxy \in IdV_{HR}$,

(ii) $xyz(xy)^k r \approx xyzx^k y^k r \in IdV_{HR}$.

Proof. We may assume that $k \geq 2$. Then we have:

$$\begin{aligned}r(xy)^k zxy &\approx rxy(xy^k)^{(k-1)} zy^{(k-1)(k-1)} xy \text{ using } xyxzxyx \approx xyzyx\\ &\approx rxy^k(xy^k)^{(k-1)} zy^{(k-1)(k-1)+(k-1)} xy \text{ by Lemma 2.1 (ii)}\\ &\approx r(xy^k)^{k-1} xy^{k+2} zy^2 xy \text{ by Lemma 2.1 (vi), (vii)}\\ &\approx r(xy^k)^{k-1} xy^k zxy \text{ (using } xyxzxyx \approx xyzyx)\\ &\approx rx^k y^k zxy \text{ (by Lemma 3.3).}\end{aligned}$$

The second identity can be proved similarly. ∎

Lemma 3.5. *For* $1 \leq k \in I\!N$ *and* $2 \leq a \in I\!N$ *there holds*

(i) $(x^a y)^k z \approx x^{ak} y^k z \in IdV_{HR}$.

(ii) $z(xy^a)^k \approx zx^k y^{ak} \in IdV_{HR}$.

Proof. We may assume that $k \geq 2$. If k is odd, then

$(x^a y)^k z \approx x^a y (x^2 y)^{k-1} z$ by Lemma 2.1 (ii)

$\approx x^a y^k z$ by $(k-1)$- fold application of $(x^2 y)^2 z \approx x^2 y^2 z$

$\approx x^{ak} y^k z$ by Lemma 2.2 (xvi) and by the fact $a \equiv ka \ mod\ 2$.

If k is even, then

$(x^a y)^k z \approx (x^k y)^k z$ by Lemma 2.1 (ii)

$\approx x^k y^k z$ by Lemma 3.3

$\approx x^{ak} y^k z$ by Lemma 2.2 (xvi) and by the fact that $k \equiv ka \ mod\ 2$.

The proof of (ii) is similar. ■

For our checking it is enough to select one hypersubstitution from each $\sim_{V_{HR}}$-class. The selected hypersubstitutions are called *normal form hypersubstitutions*. Now we apply all normal form hypersubstitutions to the associative identity.

Lemma 3.6. *For every hypersubstitution* $\sigma_{x^k y^l}$ *with* $l = 1, k = 1, \ldots, 5$ *or with* $k = 1, l = 1, \ldots, 5$ *or with* $l = 2, k = 2, 3, 4$ *or with* $l = 3, k = 2, 3$ *we get*

$$\hat{\sigma}_{x^l y^k}[x(yz)] \approx \hat{\sigma}_{x^k y^l}[(xy)z] \in IdV_{HR}.$$

Proof. For $l = 1$ or $k = 1$ everything is clear by Lemma 3.3. If $l, k \geq 2$, we have $\hat{\sigma}_{x^k y^l}[x(yz)] \approx x^k (y^k z^l)^l \approx x^k y^{kl} z^l \approx (x^k y^l)^k z^l \approx \hat{\sigma}_{x^k y^l}[(xy)z] \in IdV_{HR}$ by Lemma 3.3. ■

Now we consider all hypersubstitutions such that the image is one of the terms (25)–(50).

Lemma 3.7. *For* $1 \leq k, l, m \leq 6$ *there holds*

$$\hat{\sigma}_{x^k y^l x^m}[x(yz)] \approx \hat{\sigma}_{x^k y^l x^m}[(xy)z] \in IdV_{HR}.$$

Proof. We have

$$\begin{aligned}\hat{\sigma}_{x^k y^l x^m}[(xy)z] &= (x^k y^l x^m)^k z^l (x^k y^l x^m)^m \\ &\approx x^k (y^l x^m)^k z^l (x^k y^l)^m x^m \text{ by Lemma 3.3} \\ &\approx x^k y^{lk} x^{mk} z^l x^{km} y^{lm} x^m \text{ by Lemma 3.4} \\ &\approx x^k y^{lk} z^l y^{lm} x^m \text{ by Lemma 2.1 (i).}\end{aligned}$$

If $m \geq 2$, then

$$\begin{aligned}x^k y^{lk} z^l y^{lm} x^m &\approx x^k (y^k z^l)^l y^{lm} x^m \text{ by Lemma 3.3} \\ &\approx x^k (y^k z^l y^m)^l x^m \text{ by Lemma 3.5} \\ &= \hat{\sigma}_{x^k y^l x^m}[x(yz)].\end{aligned}$$

If $k \geq 2$, then we get dually $x^k y^{lk} z^l y^{lm} x^m \approx \hat{\sigma}_{x^k y^l x^m}[x(yz)] \in V_{HR}$.
If $k = m = 1$, then we have

$$\begin{aligned}xy^l z^l y^l x &\approx xy^l z^{ll} y^l x \text{ by Lemma 2.2 (ix) and the fact that } l \equiv ll \ mod\ 2 \\ &\approx xyz^{ll} yx \text{ by Lemma 2.1 (ii)} \\ &\approx x(yz^l y)^l x \text{ if we apply Lemma 2.1 (i) } (l-1)\text{-times} \\ &= \hat{\sigma}_{x^k y^l x^m}[x(yz)].\end{aligned}$$

■

Now we consider all hypersubstitutions which map the operation symbol f to one of the terms of the forms (3)–(24).

Lemma 3.8. *For* $1 \leq k, l, m, n \leq 6$ *there holds*

$$\hat{\sigma}_{x^k y^l x^m y^n}[x(yz)] \approx \hat{\sigma}_{x^k y^l x^m y^n}[(xy)z] \in IdV_{HR}.$$

Proof. We have

$$\begin{aligned}
\hat{\sigma}_{x^k y^l x^m y^n}[x(yz)] &= x^k(y^k z^l y^m z^n)^l x^m (y^k z^l y^m z^n)^n \\
&\approx x^k y^{kl}(z^l y^m z^n)^l x^m (y^k z^l y^m z^n)^n \text{ by Lemma 3.4} \\
&\approx x^k y^{kl} z^l (y^m z^n)^l x^m (y^k z^l y^m)^n z^n \text{ by Lemma 3.3} \\
&\approx x^k y^{kl} z^l y^{ml} z^{nl} x^m y^{kn} (z^l y^m)^n z^n \text{ by Lemma 3.3 and 3.4} \\
&\approx x^k y^{kl} z^l y^{ml} z^{nl} x^m y^{kn} z^{ln} y^{mn} z^n \text{ by and Lemma 3.4} \\
&\approx x^k y^{kl} z^l y^{ml} z^{nl} y x^m y^{kn+1} z^{ln} y^{mn} z^n \text{ using } xyxzxyx \approx xyzyx \\
&\approx x^k y^{kl} x^{km} z^l y^{ml} z^{nl} x^{km} y x^m y^{kn+1} z^{ln} y^{mn} z^n \text{ by Lemma 2.1 (i)} \\
&\approx x^k y^{kl} x^{km} y^{kn} z^l y^{ml} z^{nl} x^{km} y^{kn+1} x^m y^{kn+1} z^{ln} y^{mn} z^n \text{ by Lemma 2.1 (i)} \\
&\approx x^k y^{kl} x^{km} y^{kn} z^l y^{ml} x^{km} y^{kn+1} x^m y^{kn+mn+1} z^n \text{ by Lemma 2.1 (i)} \\
&\approx x^k y^{kl} x^{km} y^{kn+mn} z^l y^{ml} x^{km} y^{kn+mn+1} x^m y^{kn+1+mn} z^n \text{ by Lemma 2.1 (i)} \\
&\approx x^k y^{kl} x^{km} y^{kn+mn} z^l y^{ml} x^{km} y^{2mn} x^m y^{2mn} z^n \text{ by Lemma 2.1 (ii)} \\
&\approx x^k y^{kl} x^{km} y^{kn+mn} z^l y^{lm} x^{km+m} y^{2mn} z^n \text{ using } x(yz^2)^2 \approx xy^2z^2 \\
&\approx x^k y^{kl} x^{km} y^{kn+mn} z^l y^{lm} x^{km+m} y^{mn} x^2 y^{mn} z^n \text{ using } (x^2y)^2 z \approx x^2y^2z \\
&\approx x^k y^{kl} x^{km} y^{kn} z^l y^{lm} x^{km+m+2} y^{mn} z^n \text{ by Lemma 2.1 (i)} \\
&\approx x^k y^{kl} x^{km} y^{kn} z^l y^{lm} x^{km+m} y^{mn} z^n \text{ by Lemma 2.1 (v)} \\
&\approx x^k y^{kl} x^{km} y^{kn+1} z^l y^{lm} x^{km} y x^m y^{mn} z^n \text{ by Lemma 2.1 (i)} \\
&\approx x^k y^{kl} x^{km} y^{kn+1} x^{km} z^l x^{km} y^{lm} x^{km} y x^m y^{mn} z^n \text{ by Lemma 2.1 (i)} \\
&\approx x^k y^{kl} x^{km} y^{kn+1} z^l x^{km} y^{lm} y x^m y^{mn} z^n \text{ by Lemma 2.1 (i)} \\
&\approx x^k y^{kl} x^{km} y^{kn} z^l x^{km} y^{lm} x^m y^{mn} z^n \text{ by Lemma 2.1 (i).}
\end{aligned}$$

In a similar way we can show $\hat{\sigma}_{x^k y^l x^m y^n}[(xy)z] \approx x^k y^{kl} x^{km} y^{kn} z^l x^{km} y^{lm} x^m y^{mn} z^n \in IdV_{HR}$, consequently, $\hat{\sigma}_{x^k y^l x^m y^n}[x(yz)] \approx \hat{\sigma}_{x^k y^l x^m y^n}[(xy)z] \in IdV_{HR}$. ■

Lemma 3.9.

(i) $\hat{\sigma}_{xyxyx}[x(yz)] \approx \hat{\sigma}_{xyxyx}[(xy)z] \in V_{HR}$.

(ii) $\hat{\sigma}_{xy^2xyx}[x(yz)] \approx \hat{\sigma}_{xy^2xyx}[(xy)z] \in V_{HR}$.

Proof. (i) Using the identity $xyxzxyx \approx xyzyx$ we get $\hat{\sigma}_{xyxyx}[(xy)z)] \approx xyxyxzxyxyxzxyxyx \approx xyzxzyx \approx xyzyzyxyzyzyx \approx \hat{\sigma}_{xyxyx}[x(yz)] \in V_{HR}$.

(ii) We have

$$
\begin{aligned}
\hat{\sigma}_{xy^2xyx}[(xy)z] &\approx xy^2xyxz^2xy^2xyxzxy^2xyx \\
&\approx xyxyxyxz^2xy^2xyxzxyxyxyx \text{ (using } xyxzxyx \approx xyzyx) \\
&\approx xyxyxyxz^2y^2xyzxyxyxyx \text{ by Lemma 2.1 (i)} \\
&\approx xyxyxyz^2y^2xyzyxyxyx \text{ (using } xyxzxyx \approx xyzyx) \\
&\approx xyxyxyz^2yxzyxyxyx \text{ by Lemma 2.1 (i)} \\
&\approx xyz^2yxzyx \text{ (using } xyxzxyx \approx xyzyx) \\
&\approx xyz^2yzyxzyzyx \text{ by Lemma 2.1 (i)} \\
&\approx xyz^2yzyzxz^2yzyx \text{ by Lemma 2.1 (i)} \\
&\approx xyz^2yzyzyxyz^2yzyx \text{ by Lemma 2.1 (i)} \\
&\approx xyz^2yzyz^2y^2zyxyz^2yzyx \text{ by Lemma 2.1 (i)} \\
&\approx xyz^2yzy^2z^2yzyxyz^2yzyx \text{ by Lemma 2.1 (viii)} \\
&\approx \hat{\sigma}_{xy^2xyx}[x(yz)] \in V_{HR}.
\end{aligned}
$$

■

Using all these results we obtain:

Theorem 3.10. *V_{HR} is the greatest solid variety of semigroups.*

Proof. By 3.6–3.9 for every hypersubstitution $\sigma_{(j)}$ which maps the binary operation symbol f to one of the terms (j) for $j = 1, \ldots, 64$ the equations $\hat{\sigma}_{(j)}[x(yz)] \approx \hat{\sigma}_{(j)}[(xy)z]$ are satisfied in V_{HR}. If $s \approx t \in \{(xy)z \approx x(yz), (x^2y)^2z \approx x^2y^2z, x(y^2z)^2 \approx xy^2z^2, xyxzxyx \approx xyzyx\}$, then $\hat{\sigma}_{yx}[s] \approx \hat{\sigma}_{yx}[t]$ belongs also to this set. Therefore, this is also true for every identity $s \approx t \in IdV_{HR}$. For the other hypersubstitutions we use dual arguments and this finishes the proof. ∎

4. The greatest solid variety of semigroups

As a corollary of Theorem 3.10 we determine an equational basis for the greatest solid variety $HMod\{x(yz) \approx (xy)z\}$ of semigroups, i.e., for the variety which satisfies the associative law as a hyperidentity. Clearly, the variety $HMod\{x(yz) \approx (xy)z\}$ satisfies the identities $x(yz) \approx (xy)z$, $(x^2y)^2z \approx x^2y^2z, x(yz^2)^2 \approx xy^2z^2, xyzxyx \approx xyzyx$. Applying the hypersubstitution σ_{x^2} to the associative law one obtains the identity $x^2 \approx x^4$ and we may consider the variety $V_{HS} = Mod\{x(yz) \approx (xy)z, x^2 \approx x^4, (x^2y)^2z \approx x^2y^2z, x(yz^2)^2 \approx xy^2z^2, xyxzxyx \approx xyzyx\}$. The hypermodel class of the associative law $HMod\{x(yz) \approx (xy)z\}$ is included in V_{HS}. To show the converse inclusion we have to prove that the associative law is a hyperidentity in the variety V_{HS}. As a first step we determine all elements of the two-generated free algebra with respect to V_{HS}.

Theorem 4.1. *The free algebra $\mathcal{F}_{V_{HS}}(\{x, y\})$ consists exactly of the terms* (1), (2), (3), (6), (8), (10), (12), (13), (18), (19), (20), (21), (24), (25), (26), (27), (30), (31), (32), (35), (36), (37), (38), (39), (40), (41), (42), (43), (44), (45), (47), (48), (49), (50), (51), (52), (53), (56), (57), (60), (61), (63), (64), $(65)x, (66)x^2, (67)x^3$ *and all terms arising from the given ones by permuting x and y.*

Proof. Since V_{HS} is a subvariety of V_{HR}, the universe of $\mathcal{F}_{V_{HS}}(\{x, y\})$ is a homomorphic image of $\mathcal{F}_{V_{HR}}(\{x, y\})$. Using the additional identity $x^2 \approx x^4$ we obtain the given list of terms. Since the Propositions 2.9, 2.10 are also valid for the variety V_{HS}, no two of the given terms can form an identity in V_{HS}. ∎

Corollary 4.2. *The variety V_{HS} is the greatest solid variety of semigroups.*

Proof. We know already that the application of each of the hypersubstitutions different from $\sigma_x, \sigma_{x^2}, \sigma_{x^3}, \sigma_y, \sigma_{y^2}, \sigma_{y^3}$ to the associative law gives an identity which is satisfied in V_{HS}. Application of σ_x gives $x \approx x$, application of σ_{x^2} gives $x^2 \approx x^4$ which belongs to the generating system of IdV_{HS}, and application of σ_{x^3} gives $x^3 \approx x^9$, which can be derived from $x^2 \approx x^4$. This finishes the proof. ∎

The equational basis of V_{HS} was given first by Polák in [7]. One has to apply all hypersubstitutions σ_t, where t is a binary term over the variety V_{HS} to the associative law and has to prove that all resulting identities can be derived from the identities $x(yz) \approx (xy)z$, $x^2 \approx x^4$, $xyxzxyx \approx xyzyx$, $x^2y^2z \approx (xy^2)^2z$, $xy^2z^2 \approx x(yz^2)^2$. Therefore the main problem is to determine the elements of $\mathcal{F}_{V_{HS}}(\{x, y\})$. This can also be done by using a computer programme as St. Niwczyk did. The problem is that sometimes one has to make terms at first longer to be able to apply $xyxzxyx \approx xyzyx$. This seems to be a difficult programming problem. The list of terms produced by a computer consisted of more than 700 terms. The third author reduced this list to the list given in Theorem 4.1.

5. FINITE AXIOMATIZABILITY

In [9] the author gave an example for a variety of type $\tau = (2, 1)$ which is not finitely based by identities but is finitely based by hyperidentities. Let $D := \{x(yz) \approx (xy)z, xyzw \approx xzyw, yx^2y \approx xy^2x, yG(x)x^2y \approx xyG(x)yx\}$ a set of equations of type $\tau = (2, 1)$ where G is a unary operation symbol. If we replace $G(x)$ by $x^k, k \in \mathbb{N}$, then we get an infinite set E of identities which has no finite basis ([5]). But E has the set D as a finite basis of hyperidentities.

The derivation concept for hyperidentities contains one more rule of consequences, the so-called hypersubstitution rule which means that one can substitute for operation symbols terms of the same arity. For varieties of semigroups this additional rule has no influence on the problem of finite axiomatizability by equations. Indeed, we have the following consequence of Corollary 4.2

Theorem 5.1. *If a variety of semigroups is finitely axiomatizable by hyperidentities then it is also finitely axiomatizable by identities.*

Proof. Let V be a variety of semigroups which is finitely axiomatizable by hyperidentities, i.e., there is a finite set Σ of equations such that $V = HMod\Sigma$. Since V is the hypermodel class of a set Σ of equations, V is a solid variety, i.e., every identity in V is a hyperidentity (see [2]). If we define an operator $\chi : \mathcal{P}(W_\tau(X)^2) \to \mathcal{P}(W_\tau(X)^2)$, where $\mathcal{P}$ denotes the formation of the power set, then one can prove that $HMod\Sigma = Mod\chi[\Sigma]$ ([2]). Let $\sim_V$ be the equivalence relation on Hyp defined in 3.1. Let $Hyp/ \sim_V$ be the quotient set defined by this equivalence relation. Now from each equivalence class we select one hypersubstitution and form the set $\chi_\sim[\Sigma]$ of all equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$, where $s \approx t \in \Sigma$ and where σ are the selected hypersubstitutions. In [1] was proved that $Mod\chi[\Sigma] = Mod\chi_\sim[\Sigma]$ and therefore $HMod\Sigma = Mod\chi_\sim[\Sigma]$. Since Σ contains the associative identity, a set of all representatives of $Hyp(\tau)/ \sim_V$ is a subset of the finite set listed in Theorem 4.1 and then $\chi_\sim[\Sigma]$ is finite since Σ is finite and $V = HMod\Sigma = Mod\chi_\sim[\Sigma]$ is axiomatizable by the finite set $\chi_\sim[\Sigma]$ of identities. ∎

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