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COMMUTATIVE DIRECTOIDS WITH SECTIONALLY ANTITONE BIJECTIONS*

Ivan Chajda, Miroslav Kolařík

Department of Algebra and Geometry
Palacký University Olomouc
Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: chajda@inf.upol.cz
e-mail: kolarik@inf.upol.cz

and

Sándor Radeleczki

Institute of Mathematics University of Miskolc
3515 Miskolc-Egyetemváros, Hungary
e-mail: matradi@gold.uni-miskolc.hu

Abstract

We study commutative directoids with a greatest element, which can be equipped with antitone bijections in every principal filter. These can be axiomatized as algebras with two binary operations satisfying four identities. A minimal subvariety of this variety is described.

Keywords: directoid, section antitone bijection, implication algebra, double implication algebra.

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Join-semilattices whose principal filters are Boolean lattices were used by J.C. Abbott [1] for a characterization of the logic connective implication in the classical propositional logic. These semilattices also have the property that on each principal filter of them an antitone involution is defined.

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Motivated by this observation, the notion of a *join semilattice with sectionally antitone involutions* in [2] and *join semilattice with sectionally antitone bijections* in [5] was defined. In this paper we introduce a further generalization of this concept defining the notion of a *commutative directoid with sectionally antitone bijections*. By means of these directoids we obtain "nice" algebraic structures, forming a variety with "nice" congruence properties.

The concept of directoid was introduced by J. Ježek and R. Quackenbush [8] and independently by V.M. Kopytov and Z.I. Dimitrov [9] in order to axiomatize algebraic structures defined on upward directed ordered sets. In a certain sense directoids generalize semilattices. For the reader's convenience, we repeat definitions and basic properties of these concepts.

An ordered set $(A;\leq)$ is *upward directed* if $U(x,y)\neq\emptyset$ for every $x,y\in A$, where $U(x,y)=\{a\in A;\ x\leq a \text{ and } y\leq a\}$. Elements of $U(x,y)$ are referred to as common upper bounds of x,y. Of course, if $(A;\leq)$ has a greatest element then it is upward directed.

Let $(A;\leq)$ be an upward directed set and $\sqcup$ denots a binary operation on A. The pair $\mathcal{A}=(A;\sqcup)$ is called a $\sqcup$-*directoid* if

(i) $x\sqcup y\in U(x,y)$ for all $x,y\in A$;

(ii) if $x\leq y$ then $x\sqcup y=y$ and $y\sqcup x=y$.

The following axiomatization of directoids was established in [8]:

Proposition 1. *A groupoid $\mathcal{A}=(A;\sqcup)$ is a $\sqcup$-directoid if and only if it satisfies the following identities*

(D1) $x\sqcup x=x$;

(D2) $(x\sqcup y)\sqcup x=x\sqcup y$;

(D3) $y\sqcup(x\sqcup y)=x\sqcup y$;

(D4) $x\sqcup((x\sqcup y)\sqcup z)=(x\sqcup y)\sqcup z$.

A binary relation $\leq$ defined on A by the rule

$$x\leq y \quad \textit{if and only if} \quad x\sqcup y=y$$

is an order and $x\sqcup y\in U(x,y)$ for each $x,y\in A$.

A binary relation $\leq$ will be called the *induced order* of $(A;\sqcup)$. A directoid $\mathcal{A} = (A;\sqcup)$ is called *commutative* if it satisfies the identity

(D5) $x \sqcup y = y \sqcup x$.

It was shown in [8] that commutative directoids are axiomatized by the identities (D1), (D4) and (D5).

Let us denote a greatest element of an ordered set $(P;\leq)$ by 1. For each $a \in P$ the interval $[a,1]$ will be called a *section*.

Definition 1. Let $(P;\leq)$ be a poset with a greatest element 1. $(P;\leq)$ is called a *poset with sectionally antitone bijections*, if for each element $a \in S$ there exists a bijection f_a of the interval $[a,1] \subseteq P$ into itself such that

$$x \leq y \quad \Leftrightarrow \quad f_a(y) \leq f_a(x), \text{ for all } x,y \in [a,1].$$

Of course, the inverse f_a^{-1} of f_a is also an antitone bijection on $[a,1]$. If each f_a is an *involution*, i.e., $f_a^2(x) = x$, for all $x \in [a,1]$, then $f_a^{-1} = f_a$ and $(P;\leq)$ is called a *poset with sectionally antitone involutions*.

Remark 1. Any poset $(P;\leq)$ with a greatest element possesses a commutative $\sqcup$-directoid $(P;\sqcup)$.

Indeed, take any choice function $\varphi : 2^P \to P$, such that for any nonempty subset $A \subseteq P$, $\varphi(A) \in A$. Then define a binary operation $\sqcup$ on P as follows

$$x \sqcup y = \begin{cases} \varphi(U(x,y)), & \text{if } x \text{ and } y \text{ are incomparable} \\ y & \text{if } x \leq y \\ x & \text{if } y \leq x. \end{cases}$$

It is evident that this operation satisfies the axioms (D1)–(D5) and that

$$x \leq y \quad \Leftrightarrow \quad x \sqcup y = y.$$

Given a commutative $\sqcup$-directoid $\mathcal{S}$ with sectionally antitone bijections, we can introduce two new binary operations on S as follows:

(P) $$x \circ y = f_y(x \sqcup y), \qquad x * y = f_y^{-1}(x \sqcup y).$$

Since $x \sqcup y \in [y,1]$, $\circ$ and $*$ are everywhere defined operations on the set S. Conversely, one can check immediately that for any $a \in S$ and $x \in [a,1]$

(A) $$f_a(x) = x \circ a, \qquad f_a^{-1}(x) = x * a.$$

Clearly, if all the mappings f_a are involutions, then $x \circ y = x * y$, for all $x, y \in S$.

Lemma 1. *Let $\mathcal{S}$ be a commutative $\sqcup$-directoid with sectionally antitone bijections and $\circ$, $*$ be operations defined by* (P). *Then*

$$a \leq b \;\Leftrightarrow\; a \circ b = 1 \;\Leftrightarrow\; a * b = 1.$$

Proof. Suppose $a, b \in S$ and $a \leq b$. Since f_b is an antitone bijection on $[b, 1]$, we have $f_b(b) = 1$. Then

(Q) $\quad a \circ b = f_b(a \sqcup b) = f_b(b) = 1, \qquad a * b = f_b^{-1}(a \sqcup b) = f_b^{-1}(b) = 1.$

To prove the converse implication we note that for all $x \in [b, 1]$ it is $b \leq x$ and thus $f_b(b) \geq f_b(x)$. Let $a \circ b = 1$. Then $f_b(a \sqcup b) = 1 = f_b(b)$, but f_b is a bijection, hence $a \sqcup b = b$, i.e., $a \leq b$. Analogously we obtain $a * b = 1 \Rightarrow a \leq b$. ■

Theorem 1. *Let $\mathcal{S}$ be a commutative $\sqcup$-directoid with sectionally antitone bijections and $\circ$, $*$ be operations defined by* (P). *Then*

(1) $x \circ x = x * x = 1$, $x \circ 1 = x * 1 = 1$, $1 \circ x = 1 * x = x$;

(2) $(x \circ y) * y = (x * y) \circ y = (y \circ x) * x = (y * x) \circ x$;

(3) $x \circ ((((x \circ y) * y) \circ z) * z) = 1$;

(4) $(((((x \circ z) * z) \circ y) * y) \circ z) \circ (x \circ z) = (((((x \circ z) * z) \circ y) * y) * z) \circ (x * z) = 1.$

Moreover, each of the terms of (2) *is equal to $x \sqcup y$.*

Proof.

(1) By Lemma 1, $x \circ x = x * x = 1$ and $x \circ 1 = x * 1 = 1$. We also conclude $1 \circ x = f_x(1) = x$ and $1 * x = f_x^{-1}(1) = x$. Thus (1) is satisfied.

(2) $(x \circ y) * y = f_y^{-1}(f_y(x \sqcup y) \sqcup y) = f_y^{-1}(f_y(x \sqcup y)) = x \sqcup y$ since $f_y(x \sqcup y) \geq y$. Analogously, we can check $(x * y) \circ y = x \sqcup y$, $(y \circ x) * x = x \sqcup y$, and $(y * x) \circ x = x \sqcup y$. Due to the fact that $\mathcal{S}$ is commutative, we can establish the last two equalities.

(3) $x \circ ((((x \circ y) * y) \circ z) * z) = x \circ ((x \sqcup y) \sqcup z) = f_{(x \sqcup y) \sqcup z}(x \sqcup ((x \sqcup y) \sqcup z)) = f_{(x \sqcup y) \sqcup z}((x \sqcup y) \sqcup z) = 1$ by (D4).

(4) $((((x \circ z) * z) \circ y) * y) \circ z = ((x \sqcup z) \sqcup y) \circ z = f_z(((x \sqcup z) \sqcup y) \sqcup z)$. Since f_z is antitone and $z \leq (x \sqcup z) \sqcup y$, $x \sqcup z \leq (x \sqcup z) \sqcup y$ we have:

$$f_z(((x \sqcup z) \sqcup y) \sqcup z) = f_z((x \sqcup z) \sqcup y) \leq f_z(x \sqcup z) = x \circ z,$$

thus, by (Q):

$$(((((x \circ z) * z) \circ y) * y) \circ z) \circ (x \circ z) = 1.$$

Analogously we obtain:

$$(((((x \circ z) * z) \circ y) * y) * z) \circ (x * z) = 1.$$

■

Theorem 2. *Let $\mathcal{A} = (A; \circ, *, 1)$ be an algebra of type $(2, 2, 0)$ satisfying the identities* (1), (2) *and* (3). *Define a binary relation $\leq$ on A as follows:*

(R) $$a \leq b \text{ if and only if } a \circ b = 1.$$

Then $\leq$ is a partial order on A with a greatest element 1 and $(A; \leq)$ is a commutative $\sqcup$-directoid $(A; \sqcup)$ where $(x \circ y) * y = x \sqcup y$ and the following assertions are equivalent:

(i) *The algebra $\mathcal{A}$ satisfies the identity* (4);

(ii) *for any $x, y, a \in A$, $x \in [a, 1]$ the following implication holds*:

(5) $$x \leq y \Rightarrow y \circ a \leq x \circ a \text{ and } y * a \leq x * a.$$

Moreover, for any $a \in A$ the mappings $f_a(x) = x \circ a$, $f_a^{-1}(x) = x * a$ are mutually inverse antitone bijections on $[a, 1]$.

Proof. First we prove that the relation $\leq$ defined by (R) is a partial order.

Due to (1), $\leq$ is reflexive. Suppose $x \leq y$ and $y \leq x$. Then $x \circ y = 1$ and $y \circ x = 1$ hence, by (1) and (2), $x = 1 * x = (y \circ x) * x = (x \circ y) * y = 1 * y = y$, thus $\leq$ is antisymmetrical.

Suppose $x \leq y$ and $y \leq z$. Then $x \circ y = 1$ and $y \circ z = 1$. Hence $x \circ z = x \circ (1 * z) = x \circ ((y \circ z) * z) = x \circ (((1 * y) \circ z) * z) = x \circ ((((x \circ y) * y) \circ z) * z) = 1$ by (3). Thus $\leq$ is transitive, i.e., it is a partial order. Also (1) tells us that $a \circ 1 = 1$ for all a, so 1 is the greatest element.

Define $a \sqcup b = (a \circ b) * b$. Put $x = y = a$, $z = b$ in (3). Then, with (1), we can easily derive

$$a \circ ((a \circ b) * b) = 1.$$

Putting $x = y = b$, $z = a$ in (3) we obtain analogously (using (1) and (2)) that

$$b \circ ((a \circ b) * b) = 1.$$

Thus $a \leq a \sqcup b$ and $b \leq a \sqcup b$.

Suppose now $a \leq b$. Then $a \sqcup b = (a \circ b) * b = 1 * b = b$. Clearly, $a \sqcup b = (a \circ b) * b = (b \circ a) * a = b \sqcup a$, thus $(A; \sqcup)$ is a commutative $\sqcup$-directoid.

(i) $\Rightarrow$ (ii): Suppose now that also (4) is satisfied and $x \in [a, 1]$, $x \leq y$. Then $a \leq x$ and thus $(x \circ a) * a = x \sqcup a = x$. Therefore from (4) for $z = a$ it follows: $(((x \circ y) * y) \circ a) \circ (x \circ a) = 1$, which yields: $(y \circ a) \circ (x \circ a) = ((1 * y) \circ a) \circ (x \circ a) = (((x \circ y) * y) \circ a) \circ (x \circ a) = 1$. Thus $y \circ a \leq x \circ a$. Analogously we obtain: $y * a \leq x * a$, i.e., the condition (5) holds.

Now we consider $a \in A$, and f_a, f_a^{-1} defined by (A). Assume $x \in [a, 1]$. Then

$$f_a^{-1}(f_a(x)) = (x \circ a) * a = x \sqcup a = x,$$

$$f_a(f_a^{-1}(x)) = (x * a) \circ a = x \sqcup a = x,$$

using (3). Since $x \in [a, 1]$, by [5], also $f_a(x) \in [a, 1]$. Hence f_a and f_a^{-1} are bijections on $[a, 1]$ (each being the inverse of the other).

For $x \in [a, 1]$ and $x \leq y$ we have by (5):

$$f_a(y) = y \circ a \leq x \circ a = f_a(x),$$

$$f_a^{-1}(y) = y * a \leq x * a = f_a^{-1}(x),$$

therefore f_a and f_a^{-1} are antitone bijections.

(ii) ⇒ (i): By the assumptions $(A; \sqcup)$ is a commutative $\sqcup$-directoid with sectionally antitone bijections. Take $x, y \in A$. Since

$$f_y(x \sqcup y) = (x \sqcup y) \circ y = ((x \circ y) * y) \circ y = f_y(f_y^{-1}(x \circ y)) = x \circ y,$$

$$f_y^{-1}(x \sqcup y) = (x \sqcup y) * y = ((x * y) \circ y) * y = f_y^{-1}(f_y(x * y)) = x * y,$$

$\circ$ and $*$ can be also defined by (P). Applying Theorem 1, we obtain that the algebra $(A; \circ, *, 1)$ satisfies (4). ∎

Remark 2.

(i) According to Theorem 1, to any commutative $\sqcup$-directoid $(P; \leq)$ with sectionally antitone bijections it corresponds an algebra $(P; \circ, *, 1)$ satisfying the axioms (1)–(4).

(ii) In view of Theorem 2, to any algebra $\mathcal{A} = (A; \circ, *, 1)$ satisfying the identities (1)–(4) it corresponds a commutative directoid $(A; \sqcup)$ with sectionally antitone bijections, where $\leq$ is the induced order of $(A; \sqcup)$.

The following example shows, that different algebras satisfying the identities (1)–(4) can be assigned to the same poset $(P; \leq)$ with sectionally antitone bijections.

A poset $(P; \leq)$ is called a *complete bipartite* poset if there exists two nonempty subset A, B of P such that $A \cap B = \emptyset$, $A \cup B = P$ and such that for any $x, y \in P$

$$x \leq y \quad \text{if and only if} \quad \text{either } x = y \text{ or } x \in A \text{ and } y \in B.$$

Example 1. Let us consider the finite poset $P \cup \{1\}$, where $(P; \leq)$ is a complete bipartite poset and 1 is an element with the property $x \leq 1$, for all $x \in P$. Denote the sets A and B as follows: $A = \{a_1, a_2, \ldots, a_n\}$, $B = \{b_1, b_2, \ldots, b_m\}$. Clearly $P \cup \{1\} = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m, 1\}$ and $a_i < b_j < 1$ for all $1 \leq i \leq n$, $1 \leq j \leq m$, see Figure 1. We show that $(P \cup \{1\}; \leq)$ is a poset with antitone bijections. Take an arbitrary permutation π of the set $\{b_1, b_2, \ldots, b_m\}$ and let us define an arbitrary bijection for each section $[p, 1]$, (where $p \in P \cup \{1\}$) as follows:

$$f_1(1) = 1,$$

$$f_{b_j}(1) = b_j, \quad f_{b_j}(b_j) = 1, \quad 1 \leq j \leq m,$$

$$f_{a_i}(1) = a_i, \quad f_{a_i}(a_i) = 1, \quad 1 \le i \le n$$

and $f_{a_i}(b_j) = \pi(b_j)$ for all $1 \le i \le n$ and $1 \le j \le m$.

Further, a commutative directoid $(P \cup \{1\}; \sqcup)$ can be defined on the set $P \cup \{1\}$ as follows:

Fix an arbitrary element b_k $(1 \le k \le m)$. Then define $\sqcup$ in the following way:

$$x \sqcup y = y \quad \text{iff} \quad x \le y,$$

for any b_i, b_j, $b_i \ne b_j$ let $b_i \sqcup b_j = 1$, for any a_i, a_j, $a_i \ne a_j$ let $a_i \sqcup a_j = b_k$.

According to Remark 1, $(P \cup \{1\}, \sqcup)$ is a commutative directoid with sectionally antitone bijections.

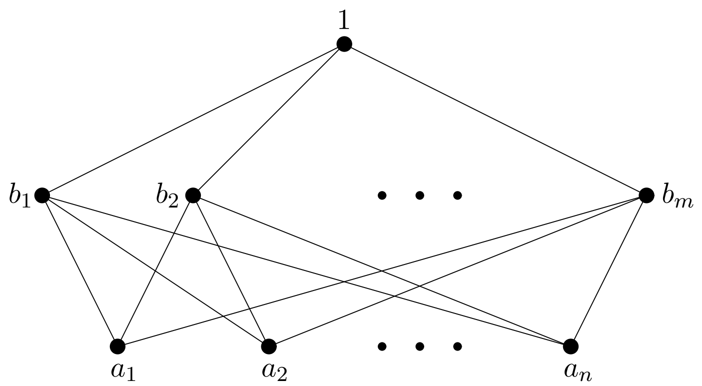

Figure 1

In view of Theorem 1, to any permutation π on the set $\{b_1, \dots, b_m\}$ and to any $b_k \in \{b_1, b_2, \dots, b_m\}$ there corresponds an algebra $\mathcal{A}$ satisfying the identities (1)–(4). In fact we have $a_i \circ a_i = b_i \circ b_i = a_i \circ b_j = b_j \circ 1 = 1 \circ 1 = 1$ and $a_i * a_i = b_i * b_i = a_i * b_j = b_j * 1 = 1 * 1 = 1$ for all $i \ne j$ such that $1 \le i \le n$ and $1 \le j \le m$. Further, we have $b_i \circ b_j = b_i * b_j = b_j$, for all $i \ne j$ such that $1 \le i, j \le m$, $a_i \circ a_j = \pi(b_k)$, $a_i * a_j = \pi^{-1}(b_k)$, for all $1 \le i, j \le n$ and $b_j \circ a_i = \pi(b_j)$, $b_j * a_i = \pi^{-1}(b_j)$, for all $1 \le i \le m$, $1 \le j \le n$.

Let us recall that an algebra $(L; \sqcup, \sqcap)$ of type $(2, 2)$ is called a λ-*lattice* if its both reducts $(L; \sqcup)$ and $(L; \sqcap)$ are commutative directoids and hence the operations are connected by the absorption laws, see e.g. [11].

Corollary 1. *Let $\mathcal{A} = (A; \circ, *, 1)$ be an algebra of type $(2,2,0)$ satisfying the identities (1), (2), (3) and (4) and let $\leq$ be the induced order. Then $(A; \leq)$ is a commutative $\sqcup$-directoid with 1, where $x \sqcup y = (x \circ y) * y$ and for each $a \in A$ the section $[a,1]$ becomes a λ-lattice where*

$$x \sqcap_a y = (((x \circ a) \circ (y \circ a)) * (y \circ a)) * a.$$

Proof. In view of Theorem 2 (ii), the section $[a,1]$ is a commutative $\sqcup$-directoid in which $x \sqcup y = (x \circ y) * y$. Since f_a, f_a^{-1} are antitone bijections it follows that letting $x \sqcap_a y = f_a^{-1}(f_a(x) \sqcup f_a(y))$ we get a commutative $\sqcap$-directoid with the same order. Hence $([a,1]; \sqcup, \sqcap_a)$ is a λ-lattice.

We have

$$x \sqcap_a y = f_a^{-1}(f_a(x) \sqcup f_a(y)) = ((x \circ a) \sqcup (y \circ a)) * a =$$

$$= (((x \circ a) \circ (y \circ a)) * (y \circ a)) * a.$$

■

When $\mathcal{A} = (A; \sqcup, \sqcap)$ is a λ-lattice then the induced operation $\sqcap_a$ in a section $[a,1]$ need not coincide with the operation $\sqcap$, see e.g. the following:

Example 2. Let $\mathcal{L} = (L; \sqcup, \sqcap)$, $L = \{0, w, z, y, x, 1\}$ be a λ-lattice depicted in Figure 2. Pick $w = x \sqcap y$ and $z \sqcup w = x$.

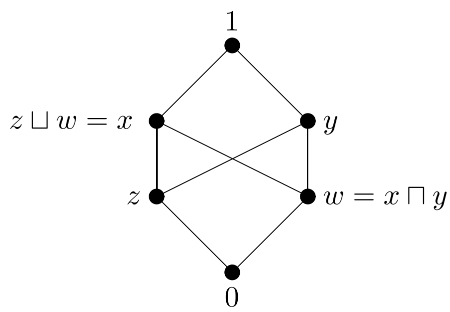

Figure 2

Further, let $f_z(1) = z$, $f_z(z) = 1$, $f_z(x) = x$, $f_z(y) = y$ and $f_z^{-1} = f_z$. Then $x = x \circ z$, $y = y \circ z$, $z = (x \sqcup y) \circ z = (x \circ z) \sqcap_z (y \circ z)$, i.e., $x \sqcap_z y = z$ but $w = x \sqcap y = (x \circ z) \sqcap (y \circ z)$, thus $\sqcap_z$ is not the same as $\sqcap$.

An *implication algebra* [1] is an algebra $(A;\circ)$ satisfying the identities

(I1) $(x \circ y) \circ x = x$;

(I2) $(x \circ y) \circ y = (y \circ x) \circ x$;

(I3) $x \circ (y \circ z) = y \circ (x \circ z)$.

It is well-known that any implication algebra $(A;\circ)$ contains an element $1 \in A$ such that $x \circ x = 1$, for all $x \in A$ (see e.g. [1]). The algebra $(A;\circ,\circ,1)$ of type (2,2,0) where the operation $\circ$ is doubled will be called a *double implication algebra*.

It is easy to see that any double implication algebra satisfies the identities (1)–(4).

Indeed, for the algebra $\mathcal{A} = (A;\circ,\circ,1)$ the identity (2) is the same as (I2) and we have $x \circ x = 1$, $1 \circ x = x$ and $x \circ 1 = (1 \circ x) \circ (1 \circ 1) = 1 \circ ((1 \circ x) \circ 1) = 1 \circ 1 = 1$, i.e., (1) is also satisfied by $\mathcal{A}$. Clearly also (3) holds. According to [1], (R) defines a partial order $\leq$ with the property (5) on any implication algebra $(A;\circ)$. Hence, by Theorem 2, we obtain that $(A;\circ,\circ,1)$ satisfies the identity (4), as well.

Example 3. Observe that for a two-element chain $(\{0,1\};\leq)$ the algebra $\mathcal{S}_2 = (\{0,1\};\circ,\circ,1)$ (where, of course, $f_0(0) = 1$, $f_0(1) = 0$) is a double implication algebra. The operation $\circ$ is given by the table

$\circ$	0	1
0	1	1
1	0	1.

It is proved in [10] that the implication algebras form a minimal quasivariety which is generated by the two-element implication algebra $(\{0,1\};\circ)$. Hence it is not hard to see that the variety generated by the algebra $\mathcal{S}_2$ is also a minimal quasivariety and it coincides with the variety of all double implication algebras.

Proposition 2. *The variety* $\mathcal{V}$ *of the algebras* $(A;\circ,*,1)$ *of type* $(2,2,0)$ *satisfying* (1)–(4) *contains a single minimal quasivariety, namely the variety of double implication algebras.*

Proof. Let $\mathcal{W}$ be a nontrivial subquasivariety of $\mathcal{V}$ and $\mathcal{A} = (A;\circ,*,1)$ a nontrivial algebra in $\mathcal{W}$. In view of Theorem 1, the corresponding poset $(A;\leq)$ is a commutative $\sqcup$-directoid with sectionally antitone bijections. As $|A| \geq 2$, there exists an element $a \in A$ with $a \neq 1$. In view of Theorem 1 we have $a \circ a = a * a = 1 \circ 1 = 1 * 1 = 1$, $a \circ 1 = a * 1 = 1$ and $1 \circ a = 1 * a = a$. Hence $(\{a,1\};\circ,*,1)$ is a subalgebra of $(A;\circ,*,1)$. Since $(\{a,1\};\leq)$ is a two-element chain, the algebra $\mathcal{S}_2 = (\{a,1\};\circ,*,1)$ is a double implication algebra with two elements (see Example 3). Denote the variety generated by it as $\mathcal{V}_2$. We already shown that $\mathcal{V}_2$ is the variety of all double implication algebras. Since $\mathcal{V}_2$ is a minimal quasivariety as well, we have $\mathcal{V}_2 = Q(\mathcal{S}_2)$, where $Q(\mathcal{S}_2)$ denotes the quasivariety generated by the algebra $\mathcal{S}_2$. As $\mathcal{S}_2 \in \mathcal{W}$ we get $\mathcal{V}_2 = Q(\mathcal{S}_2) \subseteq \mathcal{W}$, and this proves that $\mathcal{V}_2$ is the unique minimal quasivariety contained in variety $\mathcal{V}$. ∎

Let $\mathcal{A} = (A;\circ,*,1)$ be an algebra of type (2,2,0). A nonempty subset $K \subseteq A$ is called a *congruence kernel* of $\mathcal{A}$ if $K = [1]_\Theta = \{x \in A; (x,1) \in \Theta\}$ for some congruence Θ of $\mathcal{A}$. Recall that $\mathcal{A}$ is called 3-*permutable* if $\Theta_1 \circ \Theta_2 \circ \Theta_1 = \Theta_2 \circ \Theta_1 \circ \Theta_2$ holds for every $\Theta_1, \Theta_2 \in Con\mathcal{A}$. According to J. Hagemann and A. Mitschke [7], a variety $\mathcal{V}$ of algebras is 3-permutable if and only if there exist ternary terms p_0, p_1, p_2 and p_3 in $\mathcal{V}$ such that the following identities hold in $\mathcal{V}$:

$$\text{(B)} \qquad \begin{cases} p_0(x,y,z) = x,\ p_3(x,y,z) = z, \\ p_i(x,x,y) = p_{i+1}(x,y,y) \text{ for } i \in \{0,1,2\}. \end{cases}$$

An algebra $\mathcal{A}$ with a constant 1 is called *weakly regular* if every $\Theta \in Con\mathcal{A}$ is determined by its kernel, i.e., if $[1]_\Theta = [1]_\Phi$ implies $\Theta = \Phi$ for every $\Phi, \Theta \in Con\mathcal{A}$. A variety $\mathcal{V}$ is weakly regular if every algebra $\mathcal{A} \in \mathcal{V}$ has this property. The following characterization of weakly regular varieties was established by B. Csákány in [6].

Proposition 3 [6]**.** A variety $\mathcal{V}$ with 1 is weakly regular if and only if there exists $n \in \mathbb{N}$ and binary terms $q_1(x,y)$, $q_2(x,y)$, $\ldots, q_n(x,y)$ such that

(C) $$q_1(x,y) = q_2(x,y) = \cdots = q_n(x,y) = 1 \Leftrightarrow x = y$$

is satisfied for every algebra $\mathcal{A} \in \mathcal{V}$.

For other congruence conditions, the reader is asked to consult e.g. [3].

Theorem 3. *The variety $\mathcal{V}$ of the algebras $(A; \circ, *, 1)$ of type $(2,2,0)$ satisfying the identities (1)–(4) is weakly regular, 3-permutable, arithmetical at 1 and congruence distributive.*

Proof. Consider the terms $q_1(x,y) = x \circ y$ and $q_2(x,y) = y \circ x$. Then $q_1(x,x) = q_2(x,x) = x \circ x = 1$. If $q_1(x,y) = 1$ and $q_2(x,y) = 1$, then by (R) we have $x \leq y$ and $y \leq x$ thus $x = y$. In view of Proposition 3, we conclude that $\mathcal{V}$ is weakly regular (at 1).

Now, let $p_0(x,y,z) = x$, $p_3(x,y,z) = z$ and $p_1(x,y,z) = (z * y) \circ x$, $p_2(x,y,z) = (x * y) \circ z$. It is easy to see that these terms satisfy the identities (B). Consequently, $\mathcal{V}$ is congruence 3-permutable.

It is proved in [3] that a variety is arithmetical at 1 if and only if there exists a binary term $b(x,y)$ of it such that $b(x,x) = b(1,x) = 1$ and $b(x,1) = x$. Obviously, we can take $b(x,y) = y \circ x$.

Since $\mathcal{V}$ is arithmetical at 1 it is congruence distributive at 1, i.e., $[1]_{\Theta \cap (\Phi \vee \Psi)} = [1]_{(\Theta \cap \Phi) \vee (\Theta \cap \Psi)}$, for all $\Theta, \Phi, \Psi \in Con\mathcal{A}$ for $\mathcal{A} \in \mathcal{V}$.

As $\mathcal{V}$ is weakly regular, this equality implies $\Theta \cap (\Phi \vee \Psi) = (\Theta \cap \Phi) \vee (\Theta \cap \Psi)$ (for all $\Theta, \Phi, \Psi \in Con\mathcal{A}$ and $\mathcal{A} \in \mathcal{V}$), thus the variety $\mathcal{V}$ is congruence distributive. ∎

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