# COMMUTATIVE DIRECTOIDS WITH SECTIONALLY ANTITONE BIJECTIONS* 

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#### Abstract

We study commutative directoids with a greatest element, which can be equipped with antitone bijections in every principal filter. These can be axiomatized as algebras with two binary operations satisfying four identities. A minimal subvariety of this variety is described.


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Join-semilattices whose principal filters are Boolean lattices were used by J.C. Abbott [1] for a characterization of the logic connective implication in the classical propositional logic. These semilattices also have the property that on each principal filter of them an antitone involution is defined.

[^0]Motivated by this observation, the notion of a join semilattice with sectionally antitone involutions in [2] and join semilattice with sectionally antitone bijections in [5] was defined. In this paper we introduce a further generalization of this concept defining the notion of a commutative directoid with sectionally antitone bijections. By means of these directoids we obtain "nice" algebraic structures, forming a variety with "nice" congruence properties.

The concept of directoid was introduced by J. Ježek and R. Quackenbush [8] and independently by V.M. Kopytov and Z.I. Dimitrov [9] in order to axiomatize algebraic structures defined on upward directed ordered sets. In a certain sense directoids generalize semilattices. For the reader's convenience, we repeat definitions and basic properties of these concepts.

An ordered set $(A ; \leq)$ is upward directed if $U(x, y) \neq \emptyset$ for every $x, y \in A$, where $U(x, y)=\{a \in A ; x \leq a$ and $y \leq a\}$. Elements of $U(x, y)$ are referred to as common upper bounds of $x, y$. Of course, if $(A ; \leq)$ has a greatest element then it is upward directed.

Let $(A ; \leq)$ be an upward directed set and $\sqcup$ denots a binary operation on $A$. The pair $\mathcal{A}=(A ; \sqcup)$ is called a $\sqcup$-directoid if
(i) $x \sqcup y \in U(x, y)$ for all $x, y \in A$;
(ii) if $x \leq y$ then $x \sqcup y=y$ and $y \sqcup x=y$.

The following axiomatization of directoids was established in [8]:
Proposition 1. A groupoid $\mathcal{A}=(A ; \sqcup)$ is a $\sqcup$-directoid if and only if it satisfies the following identities
(D1) $x \sqcup x=x$;
(D2) $(x \sqcup y) \sqcup x=x \sqcup y ;$
(D3) $y \sqcup(x \sqcup y)=x \sqcup y$;
(D4) $x \sqcup((x \sqcup y) \sqcup z)=(x \sqcup y) \sqcup z$.
A binary relation $\leq$ defined on $A$ by the rule

$$
x \leq y \quad \text { if and only if } \quad x \sqcup y=y
$$

is an order and $x \sqcup y \in U(x, y)$ for each $x, y \in A$.

A binary relation $\leq$ will be called the induced order of $(A ; \sqcup)$. A directoid $\mathcal{A}=(A ; \sqcup)$ is called commutative if it satisfies the identity
(D5) $x \sqcup y=y \sqcup x$.
It was shown in [8] that commutative directoids are axiomatized by the identities (D1), (D4) and (D5).

Let us denote a greatest element of an ordered set $(P ; \leq)$ by 1 . For each $a \in P$ the interval $[a, 1]$ will be called a section.

Definition 1. Let $(P ; \leq)$ be a poset with a greatest element $1 .(P ; \leq)$ is called a poset with sectionally antitone bijections, if for each element $a \in S$ there exists a bijection $f_{a}$ of the interval $[a, 1] \subseteq P$ into itself such that

$$
x \leq y \quad \Leftrightarrow \quad f_{a}(y) \leq f_{a}(x), \text { for all } x, y \in[a, 1] .
$$

Of course, the inverse $f_{a}^{-1}$ of $f_{a}$ is also an antitone bijection on $[a, 1]$. If each $f_{a}$ is an involution, i.e., $f_{a}^{2}(x)=x$, for all $x \in[a, 1]$, then $f_{a}^{-1}=f_{a}$ and $(P ; \leq)$ is called a poset with sectionally antitone involutions.

Remark 1. Any poset ( $P ; \leq$ ) with a greatest element possesses a commutative $\sqcup$-directoid $(P ; \sqcup)$.

Indeed, take any choice function $\varphi: 2^{P} \rightarrow P$, such that for any nonempty subset $A \subseteq P, \varphi(A) \in A$. Then define a binary operation $\sqcup$ on $P$ as follows

$$
x \sqcup y=\left\{\begin{array}{l}
\varphi(U(x, y)), \quad \text { if } x \text { and } y \text { are incomparable } \\
y \\
\text { if } x \leq y \\
x
\end{array} \quad \text { if } y \leq x . ~ \$\right.
$$

It is evident that this operation satisfies the axioms (D1)-(D5) and that

$$
x \leq y \quad \Leftrightarrow \quad x \sqcup y=y .
$$

Given a commutative $\sqcup$-directoid $\mathcal{S}$ with sectionally antitone bijections, we can introduce two new binary operations on $S$ as follows:

$$
\begin{equation*}
x \circ y=f_{y}(x \sqcup y), \quad x * y=f_{y}^{-1}(x \sqcup y) . \tag{P}
\end{equation*}
$$

Since $x \sqcup y \in[y, 1]$, ○ and $*$ are everywhere defined operations on the set $S$. Conversely, one can check immediately that for any $a \in S$ and $x \in[a, 1]$

$$
\begin{equation*}
f_{a}(x)=x \circ a, \quad f_{a}^{-1}(x)=x * a . \tag{A}
\end{equation*}
$$

Clearly, if all the mappings $f_{a}$ are involutions, then $x \circ y=x * y$, for all $x, y \in S$.

Lemma 1. Let $\mathcal{S}$ be a commutative $\sqcup$-directoid with sectionally antitone bijections and $\circ, *$ be operations defined by (P). Then

$$
a \leq b \Leftrightarrow a \circ b=1 \Leftrightarrow a * b=1
$$

Proof. Suppose $a, b \in S$ and $a \leq b$. Since $f_{b}$ is an antitone bijection on $[b, 1]$, we have $f_{b}(b)=1$. Then
(Q) $\quad a \circ b=f_{b}(a \sqcup b)=f_{b}(b)=1, \quad a * b=f_{b}^{-1}(a \sqcup b)=f_{b}^{-1}(b)=1$.

To prove the converse implication we note that for all $x \in[b, 1]$ it is $b \leq x$ and thus $f_{b}(b) \geq f_{b}(x)$. Let $a \circ b=1$. Then $f_{b}(a \sqcup b)=1=f_{b}(b)$, but $f_{b}$ is a bijection, hence $a \sqcup b=b$, i.e., $a \leq b$. Analogously we obtain $a * b=1 \Rightarrow$ $a \leq b$.

Theorem 1. Let $\mathcal{S}$ be a commutative $\sqcup$-directoid with sectionally antitone bijections and $\circ, *$ be operations defined by (P). Then
(1) $x \circ x=x * x=1, x \circ 1=x * 1=1,1 \circ x=1 * x=x$;
(2) $(x \circ y) * y=(x * y) \circ y=(y \circ x) * x=(y * x) \circ x$;
(3) $x \circ((((x \circ y) * y) \circ z) * z)=1$;
(4) $(((((x \circ z) * z) \circ y) * y) \circ z) \circ(x \circ z)=(((((x \circ z) * z) \circ y) * y) * z) \circ(x * z)=1$.

Moreover, each of the terms of (2) is equal to $x \sqcup y$.

## Proof.

(1) By Lemma 1, $x \circ x=x * x=1$ and $x \circ 1=x * 1=1$. We also conclude $1 \circ x=f_{x}(1)=x$ and $1 * x=f_{x}^{-1}(1)=x$. Thus (1) is satisfied.
(2) $(x \circ y) * y=f_{y}^{-1}\left(f_{y}(x \sqcup y) \sqcup y\right)=f_{y}^{-1}\left(f_{y}(x \sqcup y)\right)=x \sqcup y$ since $f_{y}(x \sqcup y) \geq y$. Analogously, we can check $(x * y) \circ y=x \sqcup y,(y \circ x) * x=x \sqcup y$, and $(y * x) \circ x=x \sqcup y$. Due to the fact that $\mathcal{S}$ is commutative, we can establish the last two equalities.
(3) $x \circ((((x \circ y) * y) \circ z) * z)=x \circ((x \sqcup y) \sqcup z)=f_{(x \sqcup y) \sqcup z}(x \sqcup((x \sqcup y) \sqcup z))=$ $f_{(x \sqcup y) \sqcup z}((x \sqcup y) \sqcup z)=1$ by (D4).
(4) $((((x \circ z) * z) \circ y) * y) \circ z=((x \sqcup z) \sqcup y) \circ z=f_{z}(((x \sqcup z) \sqcup y) \sqcup z)$. Since $f_{z}$ is antitone and $z \leq(x \sqcup z) \sqcup y, x \sqcup z \leq(x \sqcup z) \sqcup y$ we have:

$$
f_{z}(((x \sqcup z) \sqcup y) \sqcup z)=f_{z}((x \sqcup z) \sqcup y) \leq f_{z}(x \sqcup z)=x \circ z,
$$

thus, by (Q):

$$
(((((x \circ z) * z) \circ y) * y) \circ z) \circ(x \circ z)=1 .
$$

Analogously we obtain:

$$
(((((x \circ z) * z) \circ y) * y) * z) \circ(x * z)=1 .
$$

Theorem 2. Let $\mathcal{A}=(A ; 0, *, 1)$ be an algebra of type $(2,2,0)$ satisfying the identities (1), (2) and (3). Define a binary relation $\leq$ on $A$ as follows:

$$
\begin{equation*}
a \leq b \text { if and only if } a \circ b=1 \text {. } \tag{R}
\end{equation*}
$$

Then $\leq$ is a partial order on $A$ with a greatest element 1 and $(A ; \leq)$ is a commutative $\sqcup$-directoid $(A ; \sqcup)$ where $(x \circ y) * y=x \sqcup y$ and the following assertions are equivalent:
(i) The algebra $\mathcal{A}$ satisfies the identity (4);
(ii) for any $x, y, a \in A, x \in[a, 1]$ the following implication holds:

$$
\begin{equation*}
x \leq y \Rightarrow y \circ a \leq x \circ a \text { and } y * a \leq x * a . \tag{5}
\end{equation*}
$$

Moreover, for any $a \in A$ the mappings $f_{a}(x)=x \circ a, f_{a}^{-1}(x)=x * a$ are mutually inverse antitone bijections on $[a, 1]$.

Proof. First we prove that the relation $\leq$ defined by $(\mathrm{R})$ is a partial order.
Due to (1), $\leq$ is reflexive. Suppose $x \leq y$ and $y \leq x$. Then $x \circ y=1$ and $y \circ x=1$ hence, by (1) and (2), $x=1 * x=(y \circ x) * x=(x \circ y) * y=1 * y=y$, thus $\leq$ is antisymmetrical.

Suppose $x \leq y$ and $y \leq z$. Then $x \circ y=1$ and $y \circ z=1$. Hence $x \circ z=$ $x \circ(1 * z)=x \circ((y \circ z) * z)=x \circ(((1 * y) \circ z) * z)=x \circ((((x \circ y) * y) \circ z) * z)=1$ by (3). Thus $\leq$ is transitive, i.e., it is a partial order. Also (1) tells us that $a \circ 1=1$ for all $a$, so 1 is the greatest element.

Define $a \sqcup b=(a \circ b) * b$. Put $x=y=a, z=b$ in (3). Then, with (1), we can easily derive

$$
a \circ((a \circ b) * b)=1
$$

Putting $x=y=b, z=a$ in (3) we obtain analogously (using (1) and (2)) that

$$
b \circ((a \circ b) * b)=1
$$

Thus $a \leq a \sqcup b$ and $b \leq a \sqcup b$.
Suppose now $a \leq b$. Then $a \sqcup b=(a \circ b) * b=1 * b=b$. Clearly, $a \sqcup b=(a \circ b) * b=(b \circ a) * a=b \sqcup a$, thus $(A ; \sqcup)$ is a commutative $\sqcup$-directoid.
(i) $\Rightarrow$ (ii): Suppose now that also (4) is satisfied and $x \in[a, 1], x \leq y$. Then $a \leq x$ and thus $(x \circ a) * a=x \sqcup a=x$. Therefore from (4) for $z=a$ it follows: $(((x \circ y) * y) \circ a) \circ(x \circ a)=1$, which yields: $(y \circ a) \circ(x \circ a)=$ $((1 * y) \circ a) \circ(x \circ a)=(((x \circ y) * y) \circ a) \circ(x \circ a)=1$. Thus $y \circ a \leq x \circ a$. Analogously we obtain: $y * a \leq x * a$, i.e., the condition (5) holds.

Now we consider $a \in A$, and $f_{a}, f_{a}^{-1}$ defined by (A). Assume $x \in[a, 1]$. Then

$$
\begin{aligned}
& f_{a}^{-1}\left(f_{a}(x)\right)=(x \circ a) * a=x \sqcup a=x \\
& f_{a}\left(f_{a}^{-1}(x)\right)=(x * a) \circ a=x \sqcup a=x
\end{aligned}
$$

using (3). Since $x \in[a, 1]$, by [5], also $f_{a}(x) \in[a, 1]$. Hence $f_{a}$ and $f_{a}^{-1}$ are bijections on $[a, 1]$ (each being the inverse of the other).

For $x \in[a, 1]$ and $x \leq y$ we have by (5):

$$
\begin{gathered}
f_{a}(y)=y \circ a \leq x \circ a=f_{a}(x) \\
f_{a}^{-1}(y)=y * a \leq x * a=f_{a}^{-1}(x)
\end{gathered}
$$

therefore $f_{a}$ and $f_{a}^{-1}$ are antitone bijections.
(ii) $\Rightarrow$ (i): By the assumptions $(A ; \sqcup)$ is a commutative $\sqcup$-directoid with sectionally antitone bijections. Take $x, y \in A$. Since

$$
\begin{gathered}
f_{y}(x \sqcup y)=(x \sqcup y) \circ y=((x \circ y) * y) \circ y=f_{y}\left(f_{y}^{-1}(x \circ y)\right)=x \circ y, \\
f_{y}^{-1}(x \sqcup y)=(x \sqcup y) * y=((x * y) \circ y) * y=f_{y}^{-1}\left(f_{y}(x * y)\right)=x * y,
\end{gathered}
$$

$\circ$ and $*$ can be also defined by (P). Applying Theorem 1, we obtain that the algebra $(A ; \circ, *, 1)$ satisfies (4).

## Remark 2.

(i) According to Theorem 1 , to any commutative $\sqcup$-directoid $(P ; \leq)$ with sectionally antitone bijections it corresponds an algebra ( $P ; \circ, *, 1$ ) satisfying the axioms (1)-(4).
(ii) In view of Theorem 2, to any algebra $\mathcal{A}=(A ; 0, *, 1)$ satisfying the identities (1)-(4) it corresponds a commutative directoid $(A ; \sqcup)$ with sectionally antitone bijections, where $\leq$ is the induced order of $(A ; \sqcup)$.

The following example shows, that different algebras satisfying the identities (1)-(4) can be assigned to the same poset ( $P ; \leq$ ) with sectionally antitone bijections.

A poset $(P ; \leq)$ is called a complete bipartite poset if there exists two nonempty subset $A, B$ of $P$ such that $A \cap B=\emptyset, A \cup B=P$ and such that for any $x, y \in P$

$$
x \leq y \quad \text { if and only if } \quad \text { either } x=y \text { or } x \in A \text { and } y \in B .
$$

Example 1. Let us consider the finite poset $P \cup\{1\}$, where $(P ; \leq)$ is a complete bipartite poset and 1 is an element with the property $x \leq 1$, for all $x \in P$. Denote the sets $A$ and $B$ as follows: $A=\left\{a_{1}, a_{2}, \ldots, a_{n}\right\}$, $B=\left\{b_{1}, b_{2}, \ldots, b_{m}\right\}$. Clearly $P \cup\{1\}=\left\{a_{1}, a_{2}, \ldots, a_{n}, b_{1}, b_{2}, \ldots, b_{m}, 1\right\}$ and $a_{i}<b_{j}<1$ for all $1 \leq i \leq n, 1 \leq j \leq m$, see Figure 1 . We show that ( $P \cup$ $\{1\} ; \leq)$ is a poset with antitone bijections. Take an arbitrary permutation $\pi$ of the set $\left\{b_{1}, b_{2}, \ldots, b_{m}\right\}$ and let us define an arbitrary bijection for each section $[p, 1]$, (where $p \in P \cup\{1\}$ ) as follows:

$$
\begin{gathered}
f_{1}(1)=1 \\
f_{b_{j}}(1)=b_{j}, \quad f_{b_{j}}\left(b_{j}\right)=1, \quad 1 \leq j \leq m
\end{gathered}
$$

$$
f_{a_{i}}(1)=a_{i}, \quad f_{a_{i}}\left(a_{i}\right)=1, \quad 1 \leq i \leq n
$$

and $f_{a_{i}}\left(b_{j}\right)=\pi\left(b_{j}\right)$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$.
Further, a commutative directoid $(P \cup\{1\} ; \sqcup)$ can be defined on the set $P \cup\{1\}$ as follows:

Fix an arbitrary element $b_{k}(1 \leq k \leq m)$. Then define $\sqcup$ in the following way:

$$
x \sqcup y=y \quad \text { iff } \quad x \leq y
$$

for any $b_{i}, b_{j}, b_{i} \neq b_{j}$ let $b_{i} \sqcup b_{j}=1$, for any $a_{i}, a_{j}, a_{i} \neq a_{j}$ let $a_{i} \sqcup a_{j}=b_{k}$.
According to Remark 1, $(P \cup\{1\}, \sqcup)$ is a commutative directoid with sectionally antitone bijections.


Figure 1

In view of Theorem 1 , to any permutation $\pi$ on the set $\left\{b_{1}, \ldots, b_{m}\right\}$ and to any $b_{k} \in\left\{b_{1}, b_{2}, \ldots, b_{m}\right\}$ there corresponds an algebra $\mathcal{A}$ satisfying the identities (1)-(4). In fact we have $a_{i} \circ a_{i}=b_{i} \circ b_{i}=a_{i} \circ b_{j}=b_{j} \circ 1=1 \circ 1=1$ and $a_{i} * a_{i}=b_{i} * b_{i}=a_{i} * b_{j}=b_{j} * 1=1 * 1=1$ for all $i \neq j$ such that $1 \leq i \leq n$ and $1 \leq j \leq m$. Further, we have $b_{i} \circ b_{j}=b_{i} * b_{j}=b_{j}$, for all $i \neq j$ such that $1 \leq i, j \leq m, a_{i} \circ a_{j}=\pi\left(b_{k}\right), a_{i} * a_{j}=\pi^{-1}\left(b_{k}\right)$, for all $1 \leq i, j \leq n$ and $b_{j} \circ a_{i}=\pi\left(b_{j}\right), b_{j} * a_{i}=\pi^{-1}\left(b_{j}\right)$, for all $1 \leq i \leq m, 1 \leq j \leq n$.

Let us recall that an algebra $(L ; \sqcup, \sqcap)$ of type $(2,2)$ is called a $\lambda$-lattice if its both reducts $(L ; \sqcup)$ and $(L ; \sqcap)$ are commutative directoids and hence the operations are connected by the absorption laws, see e.g. [11].

Corollary 1. Let $\mathcal{A}=(A ; \circ, *, 1)$ be an algebra of type $(2,2,0)$ satisfying the identities (1), (2), (3) and (4) and let $\leq$ be the induced order. Then $(A ; \leq)$ is a commutative $\sqcup$-directoid with 1 , where $x \sqcup y=(x \circ y) * y$ and for each $a \in A$ the section $[a, 1]$ becomes a $\lambda$-lattice where

$$
x \sqcap_{a} y=(((x \circ a) \circ(y \circ a)) *(y \circ a)) * a .
$$

Proof. In view of Theorem 2 (ii), the section $[a, 1]$ is a commutative $\sqcup-$ directoid in which $x \sqcup y=(x \circ y) * y$. Since $f_{a}, f_{a}^{-1}$ are antitone bijections it follows that letting $x \sqcap_{a} y=f_{a}^{-1}\left(f_{a}(x) \sqcup f_{a}(y)\right)$ we get a commutative $\sqcap$-directoid with the same order. Hence $\left([a, 1] ; \sqcup, \sqcap_{a}\right)$ is a $\lambda$-lattice.

We have

$$
\begin{gathered}
x \sqcap_{a} y=f_{a}^{-1}\left(f_{a}(x) \sqcup f_{a}(y)\right)=((x \circ a) \sqcup(y \circ a)) * a= \\
=(((x \circ a) \circ(y \circ a)) *(y \circ a)) * a .
\end{gathered}
$$

When $\mathcal{A}=(A ; \sqcup, \sqcap)$ is a $\lambda$-lattice then the induced operation $\Pi_{a}$ in a section $[a, 1]$ need not coincide with the operation $\sqcap$, see e.g. the following:

Example 2. Let $\mathcal{L}=(L ; \sqcup, \sqcap), L=\{0, w, z, y, x, 1\}$ be a $\lambda$-lattice depicted in Figure 2. Pick $w=x \sqcap y$ and $z \sqcup w=x$.


Figure 2

Further, let $f_{z}(1)=z, f_{z}(z)=1, f_{z}(x)=x, f_{z}(y)=y$ and $f_{z}^{-1}=f_{z}$. Then $x=x \circ z, y=y \circ z, z=(x \sqcup y) \circ z=(x \circ z) \sqcap_{z}(y \circ z)$, i.e., $x \sqcap_{z} y=z$ but $w=x \sqcap y=(x \circ z) \sqcap(y \circ z)$, thus $\sqcap_{z}$ is not the same as $\sqcap$.

An implication algebra [1] is an algebra $(A ; \circ)$ satisfying the identities
(I1) $(x \circ y) \circ x=x$;
(I2) $(x \circ y) \circ y=(y \circ x) \circ x$;
(I3) $x \circ(y \circ z)=y \circ(x \circ z)$.

It is well-known that any implication algebra $(A ; \circ)$ contains an element $1 \in A$ such that $x \circ x=1$, for all $x \in A$ (see e.g. [1]). The algebra $(A ; \circ, \circ, 1)$ of type $(2,2,0)$ where the operation $\circ$ is doubled will be called a double implication algebra.

It is easy to see that any double implication algebra satisfies the identities (1)-(4).

Indeed, for the algebra $\mathcal{A}=(A ; \circ, \circ, 1)$ the identity (2) is the same as (I2) and we have $x \circ x=1,1 \circ x=x$ and $x \circ 1=(1 \circ x) \circ(1 \circ 1)=$ $1 \circ((1 \circ x) \circ 1)=1 \circ 1=1$, i.e., (1) is also satisfied by $\mathcal{A}$. Clearly also (3) holds. According to [1], (R) defines a partial order $\leq$ with the property (5) on any implication algebra ( $A ; \circ$ ). Hence, by Theorem 2 , we obtain that $(A ; \circ, \circ, 1)$ satisfies the identity (4), as well.

Example 3. Observe that for a two-element chain $(\{0,1\} ; \leq)$ the algebra $\mathcal{S}_{2}=(\{0,1\} ; \circ, \circ, 1)$ (where, of course, $f_{0}(0)=1, f_{0}(1)=0$ ) is a double implication algebra. The operation $\circ$ is given by the table

| $\circ$ | 0 | 1 |
| :---: | :---: | :---: |
| 0 | 1 | 1 |
| 1 | 0 | 1. |

It is proved in [10] that the implication algebras form a minimal quasivariety which is generated by the two-element implication algebra ( $\{0,1\} ; \circ$ ). Hence it is not hard to see that the variety generated by the algebra $\mathcal{S}_{2}$ is also a minimal quasivariety and it coincides with the variety of all double implication algebras.

Proposition 2. The variety $\mathcal{V}$ of the algebras $(A ; \circ, *, 1)$ of type $(2,2,0)$ satisfying (1)-(4) contains a single minimal quasivariety, namely the variety of double implication algebras.

Proof. Let $\mathcal{W}$ be a nontrivial subquasivariety of $\mathcal{V}$ and $\mathcal{A}=(A ; \circ, *, 1)$ a nontrivial algebra in $\mathcal{W}$. In view of Theorem 1 , the corresponding poset $(A ; \leq)$ is a commutative $\sqcup$-directoid with sectionally antitone bijections. As $|A| \geq 2$, there exists an element $a \in A$ with $a \neq 1$. In view of Theorem 1 we have $a \circ a=a * a=1 \circ 1=1 * 1=1, a \circ 1=a * 1=1$ and $1 \circ a=1 * a=a$. Hence $(\{a, 1\} ; \circ, *, 1)$ is a subalgebra of $(A ; \circ, *, 1)$. Since $(\{a, 1\} ; \leq)$ is a two-element chain, the algebra $\mathcal{S}_{2}=(\{a, 1\} ; \circ, *, 1)$ is a double implication algebra with two elements (see Example 3). Denote the variety generated by it as $\mathcal{V}_{2}$. We already shown that $\mathcal{V}_{2}$ is the variety of all double implication algebras. Since $\mathcal{V}_{2}$ is a minimal quasivariety as well, we have $\mathcal{V}_{2}=Q\left(\mathcal{S}_{2}\right)$, where $Q\left(\mathcal{S}_{2}\right)$ denotes the quasivariety generated by the algebra $\mathcal{S}_{2}$. As $\mathcal{S}_{2} \in \mathcal{W}$ we get $\mathcal{V}_{2}=Q\left(\mathcal{S}_{2}\right) \subseteq \mathcal{W}$, and this proves that $\mathcal{V}_{2}$ is the unique minimal quasivariety contained in variety $\mathcal{V}$.

Let $\mathcal{A}=(A ; \circ, *, 1)$ be an algebra of type $(2,2,0)$. A nonempty subset $K \subseteq A$ is called a congruence kernel of $\mathcal{A}$ if $K=[1]_{\Theta}=\{x \in A ;(x, 1) \in \Theta\}$ for some congruence $\Theta$ of $\mathcal{A}$. Recall that $\mathcal{A}$ is called 3-permutable if $\Theta_{1} \circ \Theta_{2} \circ \Theta_{1}=$ $\Theta_{2} \circ \Theta_{1} \circ \Theta_{2}$ holds for every $\Theta_{1}, \Theta_{2} \in \operatorname{ConA}$. According to J. Hagemann and A. Mitschke [7], a variety $\mathcal{V}$ of algebras is 3-permutable if and only if there exist ternary terms $p_{0}, p_{1}, p_{2}$ and $p_{3}$ in $\mathcal{V}$ such that the following identities hold in $\mathcal{V}$ :

$$
\left\{\begin{array}{l}
p_{0}(x, y, z)=x, p_{3}(x, y, z)=z  \tag{B}\\
p_{i}(x, x, y)=p_{i+1}(x, y, y) \text { for } i \in\{0,1,2\}
\end{array}\right.
$$

An algebra $\mathcal{A}$ with a constant 1 is called weakly regular if every $\Theta \in C o n \mathcal{A}$ is determined by its kernel, i.e., if $[1]_{\Theta}=[1]_{\Phi}$ implies $\Theta=\Phi$ for every $\Phi, \Theta \in \operatorname{ConA}$. A variety $\mathcal{V}$ is weakly regular if every algebra $\mathcal{A} \in \mathcal{V}$ has this property. The following characterization of weakly regular varieties was established by B. Csákány in [6].

Proposition 3 [6]. A variety $\mathcal{V}$ with 1 is weakly regular if and only if there exists $n \in \mathbb{N}$ and binary terms $q_{1}(x, y), q_{2}(x, y), \ldots, q_{n}(x, y)$ such that

$$
\begin{equation*}
q_{1}(x, y)=q_{2}(x, y)=\cdots=q_{n}(x, y)=1 \Leftrightarrow x=y \tag{C}
\end{equation*}
$$

is satisfied for every algebra $\mathcal{A} \in \mathcal{V}$.

For other congruence conditions, the reader is asked to consult e.g. [3].
Theorem 3. The variety $\mathcal{V}$ of the algebras $(A ; \circ, *, 1)$ of type $(2,2,0)$ satisfying the identities (1)-(4) is weakly regular, 3-permutable, arithmetical at 1 and congruence distributive.

Proof. Consider the terms $q_{1}(x, y)=x \circ y$ and $q_{2}(x, y)=y \circ x$. Then $q_{1}(x, x)=q_{2}(x, x)=x \circ x=1$. If $q_{1}(x, y)=1$ and $q_{2}(x, y)=1$, then by ( R ) we have $x \leq y$ and $y \leq x$ thus $x=y$. In view of Proposition 3, we conclude that $\mathcal{V}$ is weakly regular (at 1 ).

Now, let $p_{0}(x, y, z)=x, p_{3}(x, y, z)=z$ and $p_{1}(x, y, z)=(z * y) \circ x$, $p_{2}(x, y, z)=(x * y) \circ z$. It is easy to see that these terms satisfy the identities (B). Consequently, $\mathcal{V}$ is congruence 3 -permutable.

It is proved in [3] that a variety is arithmetical at 1 if and only if there exists a binary term $b(x, y)$ of it such that $b(x, x)=b(1, x)=1$ and $b(x, 1)=$ $x$. Obviously, we can take $b(x, y)=y \circ x$.

Since $\mathcal{V}$ is arithmetical at 1 it is congruence distributive at 1, i.e., $[1]_{\Theta \cap(\Phi \vee \Psi)}=[1]_{(\Theta \cap \Phi) \vee(\Theta \cap \Psi)}$, for all $\Theta, \Phi, \Psi \in \operatorname{ConA}$ for $\mathcal{A} \in \mathcal{V}$.

As $\mathcal{V}$ is weakly regular, this equality implies $\Theta \cap(\Phi \vee \Psi)=(\Theta \cap \Phi) \vee(\Theta \cap$ $\Psi$ ) (for all $\Theta, \Phi, \Psi \in \operatorname{ConA}$ and $\mathcal{A} \in \mathcal{V}$ ), thus the variety $\mathcal{V}$ is congruence distributive.

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