

ON FUZZY IDEALS OF PSEUDO MV -ALGEBRAS

GRZEGORZ DYMEK

Institute of Mathematics and Physics
University of Podlasie
3 Maja 54, 08–110 Siedlce, Poland
e-mail: gdymek@o2.pl

Abstract

Fuzzy ideals of pseudo MV -algebras are investigated. The homomorphic properties of fuzzy prime ideals are given. A one-to-one correspondence between the set of maximal ideals and the set of fuzzy maximal ideals μ satisfying $\mu(0) = 1$ and $\mu(1) = 0$ is obtained.

Keywords: pseudo MV -algebra, fuzzy (prime, maximal) ideal.

2000 Mathematics Subject Classification: 06D35.

1. INTRODUCTION

The study of pseudo MV -algebras was initiated by G. Georgescu and A. Iorgulescu in [5] and [6], and independently by J. Rachunek in [9] (there they are called generalized MV -algebras or, in short, GMV -algebras) as a non-commutative generalization of MV -algebras which were introduced by C.C. Chang in [1]. The concept of a fuzzy set was introduced by L.A. Zadeh in [10]. Since then these ideas have been applied to other algebraic structures such as semigroups, groups, rings, ideals, modules, vector spaces and topologies. In [8] Y.B. Jun and A. Walendziak applied the concept of a fuzzy set to pseudo MV -algebras. They introduced the notions of a fuzzy ideal and a fuzzy implicative ideal in a pseudo MV -algebra, gave characterizations of them and provided conditions for a fuzzy set to be a fuzzy ideal and a fuzzy implicative ideal. Recently, the author in [3] and [4] defined, investigated and characterized fuzzy prime and fuzzy maximal ideals of pseudo MV -algebras.

In the paper we conduct further investigations of these ideals in Section 3. We provide the homomorphic properties of fuzzy prime ideals. A one-to-one correspondence between the set of maximal ideals of a pseudo MV -algebra A and the set of fuzzy maximal ideals μ of A such that $\mu(0) = 1$ and $\mu(1) = 0$ is established. For the convenience of the reader, in Section 2 we give the necessary material needed in sequel, thus making our exposition self-contained.

2. PRELIMINARIES

Let $A = (A, \oplus, ^-, \sim, 0, 1)$ be an algebra of type $(2, 1, 1, 0, 0)$. For any $x, y \in A$, set $x \cdot y = (y^- \oplus x^-)^\sim$. We consider that the operation \cdot has priority to the operation \oplus , i.e., we will write $x \oplus y \cdot z$ instead of $x \oplus (y \cdot z)$. The algebra A is called a *pseudo MV -algebra* if for any $x, y, z \in A$ the following conditions are satisfied:

- (A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
- (A2) $x \oplus 0 = 0 \oplus x = x$,
- (A3) $x \oplus 1 = 1 \oplus x = 1$,
- (A4) $1^\sim = 0, 1^- = 0$,
- (A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,
- (A6) $x \oplus x^\sim \cdot y = y \oplus y^\sim \cdot x = x \cdot y^- \oplus y = y \cdot x^- \oplus x$,
- (A7) $x \cdot (x^- \oplus y) = (x \oplus y^\sim) \cdot y$,
- (A8) $(x^-)^\sim = x$.

If the addition \oplus is commutative, then both unary operations $-$ and \sim coincide and A is an MV -algebra.

Throughout this paper A will denote a pseudo MV -algebra. For any $x \in A$ and $n = 0, 1, 2, \dots$ we put

$$\begin{aligned} 0x &= 0 \text{ and } (n+1)x = nx \oplus x, \\ x^0 &= 1 \text{ and } x^{n+1} = x^n \cdot x. \end{aligned}$$

Proposition 2.1 (Georgescu and Iorgulescu [6]). *The following properties hold for any $x \in A$:*

- (a) $(x^\sim)^- = x$,
- (b) $x \oplus x^\sim = 1, x^- \oplus x = 1$,
- (c) $x \cdot x^- = 0, x^\sim \cdot x = 0$.

We define

$$x \leqslant y \text{ iff } x^- \oplus y = 1.$$

Proposition 2.2 (Georgescu and Iorgulescu [6]). *The following properties hold for any $a, x, y \in A$:*

- (a) if $x \leqslant y$, then $a \oplus x \leqslant a \oplus y$,
- (b) if $x \leqslant y$, then $x \oplus a \leqslant y \oplus a$.

As it is shown in [6], (A, \leqslant) is a lattice in which the join $x \vee y$ and the meet $x \wedge y$ of any two elements x and y are given by:

$$\begin{aligned} x \vee y &= x \oplus x^\sim \cdot y = x \cdot y^- \oplus y, \\ x \wedge y &= x \cdot (x^- \oplus y) = (x \oplus y^\sim) \cdot y. \end{aligned}$$

Definition 2.3. A subset I of A is called an *ideal* of A if it satisfies:

- (I1) $0 \in I$,
- (I2) if $x, y \in I$, then $x \oplus y \in I$,
- (I3) if $x \in I, y \in A$ and $y \leqslant x$, then $y \in I$.

Denote by $\mathcal{J}(A)$ the set of all ideals of A .

Remark 2.4. Let $I \in \mathcal{J}(A)$. If $x, y \in I$, then $x \cdot y, x \wedge y, x \vee y \in I$.

Definition 2.5. Let I be a proper ideal of A (i.e., $I \neq A$). Then

- (a) I is called *prime* if, for all $I_1, I_2 \in \mathcal{J}(A)$, $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$.
- (b) I is called *maximal* iff whenever J is an ideal such that $I \subseteq J \subseteq A$, then either $J = I$ or $J = A$.

Denote by $\mathcal{M}(A)$ the set of all maximal ideals of A .

Definition 2.6. The *order* of an element $x \in A$ is the least n such that $nx = 1$ if such n exists, and $\text{ord}(x) = \infty$ otherwise.

Remark 2.7. It is easy to see that for any $x \in A$, $\text{ord}(x^-) = \text{ord}(x^\sim)$.

Theorem 2.8. Let $x \in A$. Then $\text{ord}(x) = \infty$ if and only if $x \in I$ for some proper ideal I of A .

Proof. Let $x \in A$. If x belongs to a proper ideal of A , then clearly $\text{ord}(x) = \infty$. Now, assume that $\text{ord}(x) = \infty$. Let I be the set of all elements y such that $y \leq nx$ for some $n \in \mathbb{N}$. Then $x \in I$ and I is a proper ideal of A . ■

Definition 2.9. Let A and B be pseudo MV -algebras. A function $f : A \rightarrow B$ is a *homomorphism* if and only if it satisfies, for each $x, y \in A$, the following conditions:

- (H1) $f(0) = 0$,
- (H2) $f(x \oplus y) = f(x) \oplus f(y)$,
- (H3) $f(x^-) = (f(x))^-$,
- (H4) $f(x^\sim) = (f(x))^\sim$.

Remark 2.10. We also have for all $x, y \in A$:

- (a) $f(1) = 1$,
- (b) $f(x \cdot y) = f(x) \cdot f(y)$,
- (c) $f(x \vee y) = f(x) \vee f(y)$,
- (d) $f(x \wedge y) = f(x) \wedge f(y)$.

We now review some fuzzy logic concepts. Let Γ be a subset of the interval $[0, 1]$ of real numbers. We define $\bigwedge \Gamma = \inf \Gamma$ and $\bigvee \Gamma = \sup \Gamma$. Obviously, if $\Gamma = \{\alpha, \beta\}$, then $\alpha \wedge \beta = \min \{\alpha, \beta\}$ and $\alpha \vee \beta = \max \{\alpha, \beta\}$. Recall that a fuzzy set in A is a function $\mu : A \rightarrow [0, 1]$. For any fuzzy sets μ and ν in A , we define

$$\mu \leq \nu \text{ iff } \mu(x) \leq \nu(x) \text{ for all } x \in A.$$

Definition 2.11. Let A and B be any two sets, μ be any fuzzy set in A and $f : A \rightarrow B$ be any function. The fuzzy set ν in B defined by

$$\nu(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in B$, is called the *image* of μ under f and is denoted by $f(\mu)$.

Definition 2.12. Let A and B be any two sets, $f : A \rightarrow B$ be any function and ν be any fuzzy set in B . The fuzzy set μ in A defined by

$$\mu(x) = \nu(f(x)) \text{ for all } x \in A$$

is called the *preimage* of ν under f and is denoted by $f^{-1}(\nu)$.

3. FUZZY IDEALS

In this section we investigate fuzzy prime ideals and fuzzy maximal ideals of a pseudo MV -algebra. First, we recall from [8] the definition and some facts concerning fuzzy ideals.

Definition 3.1. A fuzzy set μ in a pseudo MV -algebra A is called a *fuzzy ideal* of A if it satisfies for all $x, y \in A$:

$$(d1) \quad \mu(x \oplus y) \geq \mu(x) \wedge \mu(y),$$

$$(d2) \quad \text{if } y \leq x, \text{ then } \mu(y) \geq \mu(x).$$

It is easily seen that (d2) implies

$$(d3) \quad \mu(0) \geq \mu(x) \text{ for all } x \in A.$$

Denote by $\mathcal{FI}(A)$ the set of all fuzzy ideals of A .

Example 3.2. Let $A = \{(1, y) \in \mathbb{R}^2 : y \geq 0\} \cup \{(2, y) \in \mathbb{R}^2 : y \leq 0\}$, $\mathbf{0} = (1, 0)$, $\mathbf{1} = (2, 0)$. For any $(a, b), (c, d) \in A$, we define operations $\oplus, -, \sim$ as follows:

$$(a, b) \oplus (c, d) = \begin{cases} (1, b + d) & \text{if } a = c = 1, \\ (2, ad + b) & \text{if } ac = 2 \text{ and } ad + b \leq 0, \\ (2, 0) & \text{in other cases,} \end{cases}$$

$$(a, b)^- = \left(\frac{2}{a}, -\frac{2b}{a} \right),$$

$$(a, b)^\sim = \left(\frac{2}{a}, -\frac{b}{a} \right).$$

Then $A = (A, \oplus, -, \sim, \mathbf{0}, \mathbf{1})$ is a pseudo MV -algebra (see [2]). Let $A_1 = \{(1, y) \in \mathbb{R}^2 : y > 0\}$ and $A_2 = \{(2, y) \in \mathbb{R}^2 : y < 0\}$ and let $0 \leq \alpha_3 < \alpha_2 < \alpha_1 \leq 1$. We define a fuzzy set μ in A as follows:

$$\mu(x) = \begin{cases} \alpha_1 & \text{if } x = \mathbf{0}, \\ \alpha_2 & \text{if } x \in A_1, \\ \alpha_3 & \text{if } x \in A_2 \cup \{\mathbf{1}\}. \end{cases}$$

It is easily checked that μ satisfies (d1) and (d2). Thus $\mu \in \mathcal{FI}(A)$.

Proposition 3.3 (Jun and Walendziak [8]). *Every fuzzy ideal μ of A satisfies the following two inequalities:*

$$(1) \quad \mu(y) \geq \mu(x) \wedge \mu(y \cdot x^-) \text{ for all } x, y \in A,$$

$$(2) \quad \mu(y) \geq \mu(x) \wedge \mu(x^\sim \cdot y) \text{ for all } x, y \in A.$$

Proposition 3.4 (Jun and Walendziak [8]). *For a fuzzy set μ in A , the following are equivalent:*

- (a) $\mu \in \mathcal{FI}(A)$,
- (b) μ satisfies the conditions (d3) and (1),
- (c) μ satisfies the conditions (d3) and (2).

Now, we consider two special fuzzy sets in A . Let I be a subset of A . Define a fuzzy set μ_I in A by

$$\mu_I(x) = \begin{cases} \alpha & \text{if } x \in I, \\ \beta & \text{otherwise,} \end{cases}$$

where $\alpha, \beta \in [0, 1]$ with $\alpha > \beta$. The fuzzy set μ_I is a generalization of a fuzzy set χ_I which is the characteristic function of I :

$$\chi_I(x) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{otherwise.} \end{cases}$$

We have simple proposition.

Proposition 3.5. $I \in \mathcal{I}(A)$ iff $\mu_I \in \mathcal{FI}(A)$.

Corollary 3.6. $I \in \mathcal{I}(A)$ iff $\chi_I \in \mathcal{FI}(A)$.

For an arbitrary fuzzy set μ in A , consider the set $A_\mu = \{x \in A : \mu(x) = \mu(0)\}$. We have the following simple proposition.

Proposition 3.7. *If $\mu \in \mathcal{FI}(A)$, then $A_\mu \in \mathcal{I}(A)$.*

The following example shows that the converse of Proposition 3.7 does not hold.

Example 3.8. Let A be as in Example 3.2. Define a fuzzy set μ in A by

$$\mu(x) = \begin{cases} \frac{1}{2} & \text{if } x = \mathbf{0}, \\ \frac{2}{3} & \text{if } x \neq \mathbf{0}. \end{cases}$$

Then $A_\mu = \{\mathbf{0}\} \in \mathcal{I}(A)$ but $\mu \notin \mathcal{FI}(A)$.

Since $A_{\mu_I} = I$, we have a simple proposition.

Proposition 3.9. $\mu_I \in \mathcal{FI}(A)$ iff $A_{\mu_I} \in \mathcal{I}(A)$.

Proposition 3.10. Let $\mu, \nu \in \mathcal{FI}(A)$. If $\mu \leq \nu$ and $\mu(0) = \nu(0)$, then $A_\mu \subseteq A_\nu$.

Proof. Let $x \in A_\mu$. Then $\mu(x) = \mu(0) = \nu(0)$ and since $\mu(x) \leq \nu(x)$, we have $\nu(x) = \nu(0)$. Hence, $x \in A_\nu$. ■

Theorem 3.11. Let $x \in A$. Then $\text{ord}(x) = \infty$ if and only if $\mu(x) = \mu(0)$ for some non-constant fuzzy ideal μ of A .

Proof. Let $x \in A$. Suppose $\text{ord}(x) = \infty$. Then, by Theorem 2.8, $x \in I$ for some proper ideal I of A . Thus $\chi_I(x) = 1 = \chi_I(0)$ for the non-constant fuzzy ideal χ_I of A .

Conversely, assume that $\mu(x) = \mu(0)$ for some non-constant fuzzy ideal μ of A . Then $x \in A_\mu$ and A_μ is a proper ideal of A . Hence, by Theorem 2.8, $\text{ord}(x) = \infty$. ■

Theorem 3.12. Let $\mu \in \mathcal{FI}(A)$. Then a subset $P(\mu) = \{x \in A : \mu(x) > 0\}$ of A is an ideal when it is non-empty.

Proof. Assume that μ is a fuzzy ideal of A such that $P(\mu) \neq \emptyset$. Obviously, $0 \in P(\mu)$. Let $x, y \in A$ be such that $x, y \in P(\mu)$. Then $\mu(x) > 0$ and $\mu(y) > 0$. It follows from (d1) that $\mu(x \oplus y) \geq \mu(x) \wedge \mu(y) > 0$ so that $x \oplus y \in P(\mu)$. Now, let $x, y \in A$ be such that $x \in P(\mu)$ and $y \leq x$. Then, by (d2), we have $\mu(y) \geq \mu(x)$, and since $\mu(x) > 0$, we obtain $\mu(y) > 0$. So, $y \in P(\mu)$. Thus, $P(\mu)$ is the ideal of A . ■

Proposition 3.13 (Dymek [3]). Let $f : A \rightarrow B$ be a homomorphism, $\mu \in \mathcal{FI}(A)$ and $\nu \in \mathcal{FI}(B)$. Then:

- (a) if μ is constant on $\text{Ker} f$, then $f^{-1}(f(\mu)) = \mu$,
- (a) if f is surjective, then $f(f^{-1}(\nu)) = \nu$.

Proposition 3.14 (Dymek [3]). Let $f : A \rightarrow B$ be a surjective homomorphism and $\nu \in \mathcal{FI}(B)$. Then $f^{-1}(\nu) \in \mathcal{FI}(A)$.

Proposition 3.15 (Dymek [3]). *Let $f : A \rightarrow B$ be a surjective homomorphism and $\mu \in \mathcal{FI}(A)$ be such that $A_\mu \supseteq \text{Ker} f$. Then $f(\mu) \in \mathcal{FI}(B)$.*

Now, we establish the analogous homomorphic properties of fuzzy prime ideals. First, we recall from [4] the definition and some characterizations of a fuzzy prime ideal.

Definition 3.16. A fuzzy ideal μ of A is said to be *fuzzy prime* if it is non-constant and satisfies:

$$\mu(x \wedge y) = \mu(x) \vee \mu(y)$$

for all $x, y \in A$.

Proposition 3.17 (Dymek [4]). *Let μ be a non-constant fuzzy ideal of A . Then the following are equivalent:*

- (a) μ is a fuzzy prime ideal of A ,
- (b) for all $x, y \in A$, if $\mu(x \wedge y) = \mu(0)$, then $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$,
- (c) for all $x, y \in A$, $\mu(x \cdot y^-) = \mu(0)$ or $\mu(y \cdot x^-) = \mu(0)$,
- (d) for all $x, y \in A$, $\mu(x^\sim \cdot y) = \mu(0)$ or $\mu(y^\sim \cdot x) = \mu(0)$.

The following two theorems give the homomorphic properties of fuzzy prime ideals and they are a supplement of the Section 4 of [4].

Theorem 3.18. *Let $f : A \rightarrow B$ be a surjective homomorphism and ν be a fuzzy prime ideal of B . Then $f^{-1}(\nu)$ is a fuzzy prime ideal of A .*

Proof. From Proposition 3.14 we know that $f^{-1}(\nu) \in \mathcal{FI}(A)$. Obviously, $f^{-1}(\nu)$ is non-constant. Let $x, y \in A$ be such that $(f^{-1}(\nu))(x \wedge y) = (f^{-1}(\nu))(0)$. Then $\nu(f(x) \wedge f(y)) = \nu(f(0)) = \nu(0)$. So, by Proposition 3.17, $\nu(f(x)) = \nu(f(0))$ or $\nu(f(y)) = \nu(f(0))$, i.e., $(f^{-1}(\nu))(x) = (f^{-1}(\nu))(0)$ or $(f^{-1}(\nu))(y) = (f^{-1}(\nu))(0)$. Therefore, from Proposition 3.17 it follows that $f^{-1}(\nu)$ is a fuzzy prime ideal of A . ■

Theorem 3.19. *Let $f : A \rightarrow B$ be a surjective homomorphism and μ a fuzzy prime ideal of A such that $A_\mu \supseteq \text{Ker} f$. Then $f(\mu)$ is a fuzzy prime ideal of B when it is non-constant.*

Proof. From Proposition 3.15 we know that $f(\mu) \in \mathcal{FI}(A)$. Assume that $f(\mu)$ is non-constant. Let $x_B, y_B \in B$ be such that $(f(\mu))(x_B \wedge y_B) = (f(\mu))(0)$. Since f is surjective, there exist $x_A, y_A \in A$ such that $f(x_A) = x_B$ and $f(y_A) = y_B$. Since $A_\mu \supseteq \text{Ker} f$, μ is constant on $\text{Ker} f$. Hence, by Proposition 3.13(a), we have

$$\begin{aligned} \mu(0) &= (f(\mu))(0) = (f(\mu))(x_B \wedge y_B) = (f(\mu))(f(x_A \wedge y_A)) \\ &= (f^{-1}(f(\mu)))(x_A \wedge y_A) = \mu(x_A \wedge y_A). \end{aligned}$$

Since μ is fuzzy prime, from Proposition 3.17 we conclude that $\mu(x_A) = \mu(0)$ or $\mu(y_A) = \mu(0)$. Thus

$$\begin{aligned} (f(\mu))(0) &= \mu(0) = \mu(x_A) = (f^{-1}(f(\mu)))(x_A) \\ &= (f(\mu))(f(x_A)) = (f(\mu))(x_B) \text{ or} \\ (f(\mu))(0) &= \mu(0) = \mu(y_A) = (f^{-1}(f(\mu)))(y_A) \\ &= (f(\mu))(f(y_A)) = (f(\mu))(y_B). \end{aligned}$$

Therefore, from Proposition 3.17 it follows that $f(\mu)$ is a fuzzy prime ideal of A . ■

Now, we investigate fuzzy maximal ideals of a pseudo MV -algebra. The investigations are a continuation of the Section 4 of [3].

Definition 3.20. A fuzzy ideal μ of A is called *fuzzy maximal* iff A_μ is a maximal ideal of A .

Denote by $\mathcal{FM}(A)$ the set of all fuzzy maximal ideals of A .

Proposition 3.21 (Dymek [3]). *Let $I \in \mathcal{I}(A)$. Then $I \in \mathcal{M}(A)$ if and only if $\mu_I \in \mathcal{FM}(A)$.*

Corollary 3.22. *Let $I \in \mathcal{I}(A)$. Then $I \in \mathcal{M}(A)$ if and only if $\chi_I \in \mathcal{FM}(A)$.*

Proposition 3.23 (Dymek [3]). *If $\mu \in \mathcal{FM}(A)$, then μ has exactly two values.*

Now, denote by $\mathcal{FM}_0(A)$ the set of all fuzzy maximal ideals μ of A such that $\mu(0) = 1$ and $\mu(1) = 0$. Obviously, $\mathcal{FM}_0(A) \subseteq \mathcal{FM}(A)$. From Proposition 3.23 we immediately have the following theorem.

Theorem 3.24. *If $\mu \in \mathcal{FM}_0(A)$, then $\text{Im}\mu = \{0, 1\}$.*

Theorem 3.25. *If $\mu \in \mathcal{FM}_0(A)$, then $\mu = \chi_{A_\mu}$.*

Proof. Let $x \in A$. Since

$$\chi_{A_\mu}(x) = \begin{cases} 1 & \text{if } \mu(x) = 1, \\ 0 & \text{if } \mu(x) = 0, \end{cases}$$

we have the result. ■

Theorem 3.26. *If $\mu \in \mathcal{FM}_0(A)$, then $A_\mu = P(\mu)$.*

Proof. It is straightforward. ■

Theorem 3.27. *Let $\mu \in \mathcal{FM}_0(A)$. If there exists a fuzzy ideal ν of A such that $\nu(0) = 1, \nu(1) = 0$ and $\mu \leq \nu$, then $\nu \in \mathcal{FM}_0(A)$ and $\mu = \nu = \chi_{A_\mu} = \chi_{A_\nu}$.*

Proof. From Proposition 3.10 we know that $A_\mu \subseteq A_\nu$. Since A_μ is maximal, it follows that $A_\mu = A_\nu$ because $A_\nu \neq A$. Thus A_ν is also maximal. Hence ν is fuzzy maximal, and so $\nu \in \mathcal{FM}_0(A)$. Since $\mu, \nu \in \mathcal{FM}_0(A)$, by Theorem 3.25, $\mu = \chi_{A_\mu}$ and $\nu = \chi_{A_\nu}$. Thus $\mu = \chi_{A_\mu} = \chi_{A_\nu} = \nu$. ■

Theorem 3.28. *Let $\mu \in \mathcal{FM}(A)$ and define a fuzzy set $\hat{\mu}$ in A by*

$$\hat{\mu}(x) = \frac{\mu(x) - \mu(1)}{\mu(0) - \mu(1)}$$

for all $x \in A$. Then $\hat{\mu} \in \mathcal{FM}_0(A)$.

Proof. Since $\mu(0) \geq \mu(x)$ for all $x \in A$ and $\mu(0) \neq \mu(1)$, $\hat{\mu}$ is well-defined. Clearly, $\hat{\mu}(1) = 0$ and $\hat{\mu}(0) = 1 \geq \hat{\mu}(x)$ for all $x \in A$. Thus $\hat{\mu}$ satisfies (d3).

Let $x, y \in A$. Then

$$\begin{aligned}
 \widehat{\mu}(x) \wedge \widehat{\mu}(y \cdot x^-) &= \frac{\mu(x) - \mu(1)}{\mu(0) - \mu(1)} \wedge \frac{\mu(y \cdot x^-) - \mu(1)}{\mu(0) - \mu(1)} \\
 &= \frac{1}{\mu(0) - \mu(1)} [(\mu(x) - \mu(1)) \wedge (\mu(y \cdot x^-) - \mu(1))] \\
 &= \frac{1}{\mu(0) - \mu(1)} [(\mu(x) \wedge \mu(y \cdot x^-)) - \mu(1)] \\
 &\leq \frac{1}{\mu(0) - \mu(1)} [\mu(y) - \mu(1)] = \frac{\mu(y) - \mu(1)}{\mu(0) - \mu(1)} = \widehat{\mu}(y).
 \end{aligned}$$

Thus $\widehat{\mu}$ satisfies (1). Therefore, $\widehat{\mu}$ is the fuzzy ideal of A satisfying $\widehat{\mu}(0) = 1$ and $\widehat{\mu}(1) = 0$. Moreover, it is easily seen, that $A_{\widehat{\mu}} = A_{\mu}$. Hence, $\widehat{\mu} \in \mathcal{FM}_0(A)$. ■

Corollary 3.29. *If $\mu \in \mathcal{FM}_0(A)$, then $\mu = \widehat{\mu}$.*

Now, we show a one-to-one correspondence between the sets $\mathcal{M}(A)$ and $\mathcal{FM}_0(A)$.

Theorem 3.30. *Let A be a pseudo MV-algebra. Then functions $\varphi : \mathcal{M}(A) \rightarrow \mathcal{FM}_0(A)$ defined by $\varphi(M) = \chi_M$ for all $M \in \mathcal{M}(A)$ and $\psi : \mathcal{FM}_0(A) \rightarrow \mathcal{M}(A)$ defined by $\psi(\mu) = A_{\mu}$ for all $\mu \in \mathcal{FM}_0(A)$ are inverses of each other.*

Proof. Let $M \in \mathcal{M}(A)$. Then $\psi\varphi(M) = \psi(\chi_M) = A_{\chi_M} = M$. Now, let $\mu \in \mathcal{FM}_0(A)$. Then we also have $\varphi\psi(\mu) = \varphi(A_{\mu}) = \chi_{A_{\mu}} = \mu$ by Theorem 3.25. Therefore φ and ψ are inverses of each other. ■

From Theorem 3.30 we obtain the following theorem.

Theorem 3.31. *There is a one-to-one correspondence between the set of maximal ideals of a pseudo MV-algebra A and the set of fuzzy maximal ideals μ of A such that $\mu(0) = 1$ and $\mu(1) = 0$.*

Remark 3.32. Theorem 3.31 implies Theorem 3.22 of [7], the analogous one for MV-algebras.

Acknowledgements

The author thanks Professor A. Walendziak for his helpful comments.

REFERENCES

- [1] C.C. Chang, *Algebraic analysis of many valued logics*, Trans. Amer. Math. Soc. **88** (1958), 467–490.
- [2] A. Dvurečenskij, *States on pseudo MV -algebras*, Studia Logica **68** (2001), 301–327.
- [3] G. Dymek, *Fuzzy maximal ideals of pseudo MV -algebras*, Comment. Math. **47** (2007), 31–46.
- [4] G. Dymek, *Fuzzy prime ideals of pseudo- MV algebras*, Soft Comput. **12** (2008), 365–372.
- [5] G. Georgescu and A. Iorgulescu, *Pseudo MV -algebras: a non-commutative extension of MV -algebras*, pp. 961–968 in: “*Proceedings of the Fourth International Symposium on Economic Informatics*”, Bucharest, Romania, May 1999.
- [6] G. Georgescu and A. Iorgulescu, *Pseudo MV -algebras*, Multi. Val. Logic **6** (2001), 95–135.
- [7] C.S. Hoo and S. Sessa, *Fuzzy maximal ideals of BCI and MV -algebras*, Inform. Sci. **80** (1994), 299–309.
- [8] Y.B. Jun and A. Walendziak, *Fuzzy ideals of pseudo MV -algebras*, Inter. Rev. Fuzzy Math. **1** (2006), 21–31.
- [9] J. Rachůnek, *A non-commutative generalization of MV -algebras*, Czechoslovak Math. J. **52** (2002), 255–273.
- [10] L.A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338–353.

Received 23 February 2007

Revised 4 April 2007