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ON FUZZY IDEALS OF PSEUDO MV-ALGEBRAS

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Abstract

Fuzzy ideals of pseudo MV-algebras are investigated. The homomorphic properties of fuzzy prime ideals are given. A one-to-one correspondence between the set of maximal ideals and the set of fuzzy maximal ideals μ satisfying $\mu(0) = 1$ and $\mu(1) = 0$ is obtained.

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1. Introduction

The study of pseudo MV-algebras was initiated by G. Georgescu and A. Iorgulescu in [5] and [6], and independently by J. Rachůnek in [9] (there they are called generalized MV-algebras or, in short, GMV-algebras) as a non-commutative generalization of MV-algebras which were introduced by C.C. Chang in [1]. The concept of a fuzzy set was introduced by L.A. Zadeh in [10]. Since then these ideas have been applied to other algebraic structures such as semigroups, groups, rings, ideals, modules, vector spaces and topologies. In [8] Y.B. Jun and A. Walendziak applied the concept of a fuzzy set to pseudo MV-algebras. They introduced the notions of a fuzzy ideal and a fuzzy implicative ideal in a pseudo MV-algebra, gave characterizations of them and provided conditions for a fuzzy set to be a fuzzy ideal and a fuzzy implicative ideal. Recently, the author in [3] and [4] defined, investigated and characterized fuzzy prime and fuzzy maximal ideals of pseudo MV-algebras.

In the paper we conduct further investigations of these ideals in Section 3. We provide the homomorphic properties of fuzzy prime ideals. A one-to-one correspondence between the set of maximal ideals of a pseudo MV-algebra A and the set of fuzzy maximal ideals μ of A such that $\mu(0) = 1$ and $\mu(1) = 0$ is established. For the convenience of the reader, in Section 2 we give the necessary material needed in sequel, thus making our exposition self-contained.

2. Preliminaries

Let $A = (A, \oplus, ^-, ^\sim, 0, 1)$ be an algebra of type $(2, 1, 1, 0, 0)$. For any $x, y \in A$, set $x \cdot y = (y^- \oplus x^-)^\sim$. We consider that the operation $\cdot$ has priority to the operation $\oplus$, i.e., we will write $x \oplus y \cdot z$ instead of $x \oplus (y \cdot z)$. The algebra A is called a *pseudo MV-algebra* if for any $x, y, z \in A$ the following conditions are satisfied:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(A2) $x \oplus 0 = 0 \oplus x = x$,

(A3) $x \oplus 1 = 1 \oplus x = 1$,

(A4) $1^\sim = 0, 1^- = 0$,

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,

(A6) $x \oplus x^\sim \cdot y = y \oplus y^\sim \cdot x = x \cdot y^- \oplus y = y \cdot x^- \oplus x$,

(A7) $x \cdot (x^- \oplus y) = (x \oplus y^\sim) \cdot y$,

(A8) $(x^-)^\sim = x$.

If the addition $\oplus$ is commutative, then both unary operations $^-$ and $^\sim$ coincide and A is an MV-algebra.

Throughout this paper A will denote a pseudo MV-algebra. For any $x \in A$ and $n = 0, 1, 2, \ldots$ we put

$$0x = 0 \text{ and } (n+1)x = nx \oplus x,$$

$$x^0 = 1 \text{ and } x^{n+1} = x^n \cdot x.$$

Proposition 2.1 (Georgescu and Iorgulescu [6]). *The following properties hold for any $x \in A$:*

(a) $(x^{\sim})^{-} = x$,

(b) $x \oplus x^{\sim} = 1, x^{-} \oplus x = 1$,

(c) $x \cdot x^{-} = 0, x^{\sim} \cdot x = 0$.

We define

$$x \leqslant y \text{ iff } x^{-} \oplus y = 1.$$

Proposition 2.2 (Georgescu and Iorgulescu [6]). *The following properties hold for any $a, x, y \in A$:*

(a) *if $x \leqslant y$, then $a \oplus x \leqslant a \oplus y$,*

(b) *if $x \leqslant y$, then $x \oplus a \leqslant y \oplus a$.*

As it is shown in [6], $(A, \leqslant)$ is a lattice in which the join $x \vee y$ and the meet $x \wedge y$ of any two elements x and y are given by:

$$\begin{aligned} x \vee y &= x \oplus x^{\sim} \cdot y = x \cdot y^{-} \oplus y, \\ x \wedge y &= x \cdot (x^{-} \oplus y) = (x \oplus y^{\sim}) \cdot y. \end{aligned}$$

Definition 2.3. A subset I of A is called an *ideal* of A if it satisfies:

(I1) $0 \in I$,

(I2) if $x, y \in I$, then $x \oplus y \in I$,

(I3) if $x \in I$, $y \in A$ and $y \leqslant x$, then $y \in I$.

Denote by $\mathcal{I}(A)$ the set of all ideals of A.

Remark 2.4. Let $I \in \mathcal{I}(A)$. If $x, y \in I$, then $x \cdot y, x \wedge y, x \vee y \in I$.

Definition 2.5. Let I be a proper ideal of A (i.e., $I \neq A$). Then

(a) I is called *prime* if, for all $I_1, I_2 \in \mathcal{I}(A)$, $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$.

(b) I is called *maximal* iff whenever J is an ideal such that $I \subseteq J \subseteq A$, then either $J = I$ or $J = A$.

Denote by $\mathcal{M}(A)$ the set of all maximal ideals of A.

Definition 2.6. The *order* of an element $x \in A$ is the least n such that $nx = 1$ if such n exists, and $\mathrm{ord}(x) = \infty$ otherwise.

Remark 2.7. It is easy to see that for any $x \in A$, $\mathrm{ord}(x^-) = \mathrm{ord}(x^\sim)$.

Theorem 2.8. *Let $x \in A$. Then $\mathrm{ord}(x) = \infty$ if and only if $x \in I$ for some proper ideal I of A.*

Proof. Let $x \in A$. If x belongs to a proper ideal of A, then clearly $\mathrm{ord}(x) = \infty$. Now, assume that $\mathrm{ord}(x) = \infty$. Let I be the set of all elements y such that $y \leqslant nx$ for some $n \in \mathbb{N}$. Then $x \in I$ and I is a proper ideal of A. ∎

Definition 2.9. Let A and B be pseudo MV-algebras. A function $f : A \to B$ is a *homomorphism* if and only if it satisfies, for each $x, y \in A$, the following conditions:

(H1) $f(0) = 0$,

(H2) $f(x \oplus y) = f(x) \oplus f(y)$,

(H3) $f(x^-) = (f(x))^-$,

(H4) $f(x^\sim) = (f(x))^\sim$.

Remark 2.10. We also have for all $x, y \in A$:

(a) $f(1) = 1$,

(b) $f(x \cdot y) = f(x) \cdot f(y)$,

(c) $f(x \vee y) = f(x) \vee f(y)$,

(d) $f(x \wedge y) = f(x) \wedge f(y)$.

We now review some fuzzy logic concepts. Let Γ be a subset of the interval $[0, 1]$ of real numbers. We define $\bigwedge \Gamma = \inf \Gamma$ and $\bigvee \Gamma = \sup \Gamma$. Obviously, if $\Gamma = \{\alpha, \beta\}$, then $\alpha \wedge \beta = \min\{\alpha, \beta\}$ and $\alpha \vee \beta = \max\{\alpha, \beta\}$. Recall that a fuzzy set in A is a function $\mu : A \to [0, 1]$. For any fuzzy sets μ and ν in A, we define

$$\mu \leqslant \nu \text{ iff } \mu(x) \leqslant \nu(x) \text{ for all } x \in A.$$

Definition 2.11. Let A and B be any two sets, μ be any fuzzy set in A and $f : A \to B$ be any function. The fuzzy set ν in B defined by

$$\nu(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in B$, is called the *image* of μ under f and is denoted by $f(\mu)$.

Definition 2.12. Let A and B be any two sets, $f : A \to B$ be any function and ν be any fuzzy set in $f(A)$. The fuzzy set μ in A defined by

$$\mu(x) = \nu(f(x)) \text{ for all } x \in A$$

is called the *preimage* of ν under f and is denoted by $f^{-1}(\nu)$.

3. Fuzzy ideals

In this section we investigate fuzzy prime ideals and fuzzy maximal ideals of a pseudo MV-algebra. First, we recall from [8] the definition and some facts concerning fuzzy ideals.

Definition 3.1. A fuzzy set μ in a pseudo MV-algebra A is called a *fuzzy ideal* of A if it satisfies for all $x, y \in A$:

(d1) $\mu(x \oplus y) \geqslant \mu(x) \wedge \mu(y)$,

(d2) if $y \leqslant x$, then $\mu(y) \geqslant \mu(x)$.

It is easily seen that (d2) implies

(d3) $\mu(0) \geqslant \mu(x)$ for all $x \in A$.

Denote by $\mathfrak{FI}(A)$ the set of all fuzzy ideals of A.

Example 3.2. Let $A = \{(1, y) \in \mathbb{R}^2 : y \geqslant 0\} \cup \{(2, y) \in \mathbb{R}^2 : y \leqslant 0\}$, $\mathbf{0} = (1, 0)$, $\mathbf{1} = (2, 0)$. For any $(a, b), (c, d) \in A$, we define operations $\oplus, ^{-}, ^{\sim}$ as follows:

$$(a, b) \oplus (c, d) = \begin{cases} (1, b + d) & \text{if } a = c = 1, \\ (2, ad + b) & \text{if } ac = 2 \text{ and } ad + b \leqslant 0, \\ (2, 0) & \text{in other cases,} \end{cases}$$

$$(a, b)^{-} = \left(\frac{2}{a}, -\frac{2b}{a}\right),$$

$$(a, b)^{\sim} = \left(\frac{2}{a}, -\frac{b}{a}\right).$$

Then $A = (A, \oplus, ^{-}, ^{\sim}, \mathbf{0}, \mathbf{1})$ is a pseudo MV-algebra (see [2]). Let $A_1 = \{(1, y) \in \mathbb{R}^2 : y > 0\}$ and $A_2 = \{(2, y) \in \mathbb{R}^2 : y < 0\}$ and let $0 \leqslant \alpha_3 < \alpha_2 < \alpha_1 \leqslant 1$. We define a fuzzy set μ in A as follows:

$$\mu(x) = \begin{cases} \alpha_1 & \text{if } x = \mathbf{0}, \\ \alpha_2 & \text{if } x \in A_1, \\ \alpha_3 & \text{if } x \in A_2 \cup \{\mathbf{1}\}. \end{cases}$$

It is easily checked that μ satisfies (d1) and (d2). Thus $\mu \in \mathfrak{FI}(A)$.

Proposition 3.3 (Jun and Walendziak [8])**.** *Every fuzzy ideal μ of A satisfies the following two inequalities:*

$$\mu(y) \geqslant \mu(x) \wedge \mu\left(y \cdot x^{-}\right) \text{ for all } x, y \in A, \tag{1}$$

$$\mu(y) \geqslant \mu(x) \wedge \mu(x^{\sim} \cdot y) \text{ for all } x, y \in A. \tag{2}$$

Proposition 3.4 (Jun and Walendziak [8]). *For a fuzzy set μ in A, the following are equivalent:*

(a) $\mu \in \mathfrak{FI}(A)$,

(b) μ *satisfies the conditions* (d3) and (1),

(c) μ *satisfies the conditions* (d3) and (2).

Now, we consider two special fuzzy sets in A. Let I be a subset of A. Define a fuzzy set μ_I in A by

$$\mu_I(x) = \begin{cases} \alpha & \text{if } x \in I, \\ \beta & \text{otherwise,} \end{cases}$$

where $\alpha, \beta \in [0,1]$ with $\alpha > \beta$. The fuzzy set μ_I is a generalization of a fuzzy set χ_I which is the characteristic function of I:

$$\chi_I(x) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{otherwise.} \end{cases}$$

We have simple proposition.

Proposition 3.5. *$I \in \mathfrak{I}(A)$ iff $\mu_I \in \mathfrak{FI}(A)$.*

Corollary 3.6. *$I \in \mathfrak{I}(A)$ iff $\chi_I \in \mathfrak{FI}(A)$.*

For an arbitrary fuzzy set μ in A, consider the set $A_\mu = \{x \in A : \mu(x) = \mu(0)\}$. We have the following simple proposition.

Proposition 3.7. *If $\mu \in \mathfrak{FI}(A)$, then $A_\mu \in \mathfrak{I}(A)$.*

The following example shows that the converse of Proposition 3.7 does not hold.

Example 3.8. Let A be as in Example 3.2. Define a fuzzy set μ in A by

$$\mu(x) = \begin{cases} \dfrac{1}{2} & \text{if } x = \mathbf{0}, \\[2ex] \dfrac{2}{3} & \text{if } x \neq \mathbf{0}. \end{cases}$$

Then $A_\mu = \{\mathbf{0}\} \in \mathfrak{I}(A)$ but $\mu \notin \mathfrak{FI}(A)$.

Since $A_{\mu_I} = I$, we have a simple proposition.

Proposition 3.9. *$\mu_I \in \mathfrak{FI}(A)$ iff $A_{\mu_I} \in \mathfrak{I}(A)$.*

Proposition 3.10. *Let $\mu, \nu \in \mathfrak{FI}(A)$. If $\mu \leqslant \nu$ and $\mu(0) = \nu(0)$, then $A_\mu \subseteq A_\nu$.*

Proof. Let $x \in A_\mu$. Then $\mu(x) = \mu(0) = \nu(0)$ and since $\mu(x) \leqslant \nu(x)$, we have $\nu(x) = \nu(0)$. Hence, $x \in A_\nu$. ■

Theorem 3.11. *Let $x \in A$. Then $\operatorname{ord}(x) = \infty$ if and only if $\mu(x) = \mu(0)$ for some non-constant fuzzy ideal μ of A.*

Proof. Let $x \in A$. Suppose $\operatorname{ord}(x) = \infty$. Then, by Theorem 2.8, $x \in I$ for some proper ideal I of A. Thus $\chi_I(x) = 1 = \chi_I(0)$ for the non-constant fuzzy ideal χ_I of A.

Conversely, assume that $\mu(x) = \mu(0)$ for some non-constant fuzzy ideal μ of A. Then $x \in A_\mu$ and A_μ is a proper ideal of A. Hence, by Theorem 2.8, $\operatorname{ord}(x) = \infty$. ■

Theorem 3.12. *Let $\mu \in \mathfrak{FI}(A)$. Then a subset $P(\mu) = \{x \in A : \mu(x) > 0\}$ of A is an ideal when it is non-empty.*

Proof. Assume that μ is a fuzzy ideal of A such that $P(\mu) \neq \varnothing$. Obviously, $0 \in P(\mu)$. Let $x, y \in A$ be such that $x, y \in P(\mu)$. Then $\mu(x) > 0$ and $\mu(y) > 0$. It follows from (d1) that $\mu(x \oplus y) \geqslant \mu(x) \wedge \mu(y) > 0$ so that $x \oplus y \in P(\mu)$. Now, let $x, y \in A$ be such that $x \in P(\mu)$ and $y \leqslant x$. Then, by (d2), we have $\mu(y) \geqslant \mu(x)$, and since $\mu(x) > 0$, we obtain $\mu(y) > 0$. So, $y \in P(\mu)$. Thus, $P(\mu)$ is the ideal of A. ■

Proposition 3.13 (Dymek [3])**.** *Let $f : A \to B$ be a homomorphism, $\mu \in \mathfrak{FI}(A)$ and $\nu \in \mathfrak{FI}(B)$. Then:*

(a) *if μ is constant on* Ker*f, then $f^{-1}(f(\mu)) = \mu$,*

(a) *if f is surjective, then $f\left(f^{-1}(\nu)\right) = \nu$.*

Proposition 3.14 (Dymek [3])**.** *Let $f : A \to B$ be a surjective homomorphism and $\nu \in \mathfrak{FI}(B)$. Then $f^{-1}(\nu) \in \mathfrak{FI}(A)$.*

Proposition 3.15 (Dymek [3]). *Let $f : A \to B$ be a surjective homomorphism and $\mu \in \mathfrak{FI}(A)$ be such that $A_\mu \supseteq \operatorname{Ker} f$. Then $f(\mu) \in \mathfrak{FI}(B)$.*

Now, we establish the analogous homomorphic properties of fuzzy prime ideals. First, we recall from [4] the definition and some characterizations of a fuzzy prime ideal.

Definition 3.16. A fuzzy ideal μ *of* A is said to be *fuzzy prime* if it is non-constant and satisfies:

$$\mu(x \wedge y) = \mu(x) \vee \mu(y)$$

for all $x, y \in A$.

Proposition 3.17 (Dymek [4]). *Let μ be a non-constant fuzzy ideal of A. Then the following are equivalent:*

(a) *μ is a fuzzy prime ideal of A,*

(b) *for all $x, y \in A$, if $\mu(x \wedge y) = \mu(0)$, then $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$,*

(c) *for all $x, y \in A$, $\mu(x \cdot y^-) = \mu(0)$ or $\mu(y \cdot x^-) = \mu(0)$,*

(d) *for all $x, y \in A$, $\mu(x^\sim \cdot y) = \mu(0)$ or $\mu(y^\sim \cdot x) = \mu(0)$.*

The following two theorems give the homomorphic properties of fuzzy prime ideals and they are a supplement of the Section 4 of [4].

Theorem 3.18. *Let $f : A \to B$ be a surjective homomorphism and ν be a fuzzy prime ideal of B. Then $f^{-1}(\nu)$ is a fuzzy prime ideal of A.*

Proof. From Proposition 3.14 we know that $f^{-1}(\nu) \in \mathfrak{FI}(A)$. Obviously, $f^{-1}(\nu)$ is non-constant. Let $x, y \in A$ be such that $(f^{-1}(\nu))(x \wedge y) = (f^{-1}(\nu))(0)$. Then $\nu(f(x) \wedge f(y)) = \nu(f(0)) = \nu(0)$. So, by Proposition 3.17, $\nu(f(x)) = \nu(f(0))$ or $\nu(f(y)) = \nu(f(0))$, i.e., $(f^{-1}(\nu))(x) = (f^{-1}(\nu))(0)$ or $(f^{-1}(\nu))(y) = (f^{-1}(\nu))(0)$. Therefore, from Proposition 3.17 it follows that $f^{-1}(\nu)$ is a fuzzy prime ideal of A. ∎

Theorem 3.19. *Let $f : A \to B$ be a surjective homomorphism and μ a fuzzy prime ideal of A such that $A_\mu \supseteq \operatorname{Ker} f$. Then $f(\mu)$ is a fuzzy prime ideal of B when it is non-constant.*

Proof. From Proposition 3.15 we know that $f(\mu) \in \mathfrak{FI}(A)$. Assume that $f(\mu)$ is non-constant. Let $x_B, y_B \in B$ be such that $(f(\mu))(x_B \wedge y_B) = (f(\mu))(0)$. Since f is surjective, there exist $x_A, y_A \in A$ such that $f(x_A) = x_B$ and $f(y_A) = y_B$. Since $A_\mu \supseteq \mathrm{Ker}f$, μ is constant on $\mathrm{Ker}f$. Hence, by Proposition 3.13(a), we have

$$\mu(0) = (f(\mu))(0) = (f(\mu))(x_B \wedge y_B) = (f(\mu))(f(x_A \wedge y_A))$$

$$= (f^{-1}(f(\mu)))(x_A \wedge y_A) = \mu(x_A \wedge y_A).$$

Since μ is fuzzy prime, from Proposition 3.17 we conclude that $\mu(x_A) = \mu(0)$ or $\mu(y_A) = \mu(0)$. Thus

$$\begin{aligned}(f(\mu))(0) &= \mu(0) = \mu(x_A) = (f^{-1}(f(\mu)))(x_A) \\ &= (f(\mu))(f(x_A)) = (f(\mu))(x_B) \text{ or} \\ (f(\mu))(0) &= \mu(0) = \mu(y_A) = (f^{-1}(f(\mu)))(y_A) \\ &= (f(\mu))(f(y_A)) = (f(\mu))(y_B).\end{aligned}$$

Therefore, from Proposition 3.17 it follows that $f(\mu)$ is a fuzzy prime ideal of A. ■

Now, we investigate fuzzy maximal ideals of a pseudo MV-algebra. The investigations are a continuation of the Section 4 of [3].

Definition 3.20. A fuzzy ideal μ of A is called *fuzzy maximal* iff A_μ is a maximal ideal of A.

Denote by $\mathfrak{FM}(A)$ the set of all fuzzy maximal ideals of A.

Proposition 3.21 (Dymek [3])**.** *Let $I \in \mathfrak{I}(A)$. Then $I \in \mathfrak{M}(A)$ if and only if $\mu_I \in \mathfrak{FM}(A)$.*

Corollary 3.22. *Let $I \in \mathfrak{I}(A)$. Then $I \in \mathfrak{M}(A)$ if and only if $\chi_I \in \mathfrak{FM}(A)$.*

Proposition 3.23 (Dymek [3])**.** *If $\mu \in \mathfrak{FM}(A)$, then μ has exactly two values.*

Now, denote by $\mathcal{FM}_0(A)$ the set of all fuzzy maximal ideals μ of A such that $\mu(0) = 1$ and $\mu(1) = 0$. Obviously, $\mathcal{FM}_0(A) \subseteq \mathcal{FM}(A)$. From Proposition 3.23 we immediately have the following theorem.

Theorem 3.24. *If* $\mu \in \mathcal{FM}_0(A)$, *then* $\mathrm{Im}\mu = \{0,1\}$.

Theorem 3.25. *If* $\mu \in \mathcal{FM}_0(A)$, *then* $\mu = \chi_{A_\mu}$.

Proof. Let $x \in A$. Since

$$\chi_{A_\mu}(x) = \begin{cases} 1 & \text{if } \mu(x) = 1, \\ 0 & \text{if } \mu(x) = 0, \end{cases}$$

we have the result. ∎

Theorem 3.26. *If* $\mu \in \mathcal{FM}_0(A)$, *then* $A_\mu = P(\mu)$.

Proof. It is straightforward. ∎

Theorem 3.27. *Let* $\mu \in \mathcal{FM}_0(A)$. *If there exists a fuzzy ideal* ν *of* A *such that* $\nu(0) = 1, \nu(1) = 0$ *and* $\mu \leqslant \nu$, *then* $\nu \in \mathcal{FM}_0(A)$ *and* $\mu = \nu = \chi_{A_\mu} = \chi_{A_\nu}$.

Proof. From Proposition 3.10 we know that $A_\mu \subseteq A_\nu$. Since A_μ is maximal, it follows that $A_\mu = A_\nu$ because $A_\nu \neq A$. Thus A_ν is also maximal. Hence ν is fuzzy maximal, and so $\nu \in \mathcal{FM}_0(A)$. Since $\mu, \nu \in \mathcal{FM}_0(A)$, by Theorem 3.25, $\mu = \chi_{A_\mu}$ and $\nu = \chi_{A_\nu}$. Thus $\mu = \chi_{A_\mu} = \chi_{A_\nu} = \nu$. ∎

Theorem 3.28. *Let* $\mu \in \mathcal{FM}(A)$ *and define a fuzzy set* $\widehat{\mu}$ *in* A *by*

$$\widehat{\mu}(x) = \frac{\mu(x) - \mu(1)}{\mu(0) - \mu(1)}$$

for all $x \in A$. *Then* $\widehat{\mu} \in \mathcal{FM}_0(A)$.

Proof. Since $\mu(0) \geqslant \mu(x)$ for all $x \in A$ and $\mu(0) \neq \mu(1)$, $\widehat{\mu}$ is well-defined. Clearly, $\widehat{\mu}(1) = 0$ and $\widehat{\mu}(0) = 1 \geqslant \widehat{\mu}(x)$ for all $x \in A$. Thus $\widehat{\mu}$ satisfies (d3).

Let $x, y \in A$. Then

$$\begin{aligned}
\widehat{\mu}(x) \wedge \widehat{\mu}\left(y \cdot x^{-}\right) &= \frac{\mu(x)-\mu(1)}{\mu(0)-\mu(1)} \wedge \frac{\mu\left(y \cdot x^{-}\right)-\mu(1)}{\mu(0)-\mu(1)} \\
&= \frac{1}{\mu(0)-\mu(1)}\left[(\mu(x)-\mu(1)) \wedge \left(\mu\left(y \cdot x^{-}\right)-\mu(1)\right)\right] \\
&= \frac{1}{\mu(0)-\mu(1)}\left[\left(\mu(x) \wedge \mu\left(y \cdot x^{-}\right)\right)-\mu(1)\right] \\
&\leqslant \frac{1}{\mu(0)-\mu(1)}[\mu(y)-\mu(1)] = \frac{\mu(y)-\mu(1)}{\mu(0)-\mu(1)} = \widehat{\mu}(y).
\end{aligned}$$

Thus $\widehat{\mu}$ satisfies (1). Therefore, $\widehat{\mu}$ is the fuzzy ideal of A satisfying $\widehat{\mu}(0) = 1$ and $\widehat{\mu}(1) = 0$. Moreover, it is easily seen, that $A_{\widehat{\mu}} = A_{\mu}$. Hence, $\widehat{\mu} \in \mathcal{FM}_0(A)$. ■

Corollary 3.29. *If $\mu \in \mathcal{FM}_0(A)$, then $\mu = \widehat{\mu}$.*

Now, we show a one-to-one correspondence between the sets $\mathcal{M}(A)$ and $\mathcal{FM}_0(A)$.

Theorem 3.30. *Let A be a pseudo MV-algebra. Then functions $\varphi : \mathcal{M}(A) \to \mathcal{FM}_0(A)$ defined by $\varphi(M) = \chi_M$ for all $M \in \mathcal{M}(A)$ and $\psi : \mathcal{FM}_0(A) \to \mathcal{M}(A)$ defined by $\psi(\mu) = A_\mu$ for all $\mu \in \mathcal{FM}_0(A)$ are inverses of each other.*

Proof. Let $M \in \mathcal{M}(A)$. Then $\psi\varphi(M) = \psi(\chi_M) = A_{\chi_M} = M$. Now, let $\mu \in \mathcal{FM}_0(A)$. Then we also have $\varphi\psi(\mu) = \varphi(A_\mu) = \chi_{A_\mu} = \mu$ by Theorem 3.25. Therefore φ and ψ are inverses of each other. ■

From Theorem 3.30 we obtain the following theorem.

Theorem 3.31. *There is a one-to-one correspondence between the set of maximal ideals of a pseudo MV-algebra A and the set of fuzzy maximal ideals μ of A such that $\mu(0) = 1$ and $\mu(1) = 0$.*

Remark 3.32. Theorem 3.31 implies Theorem 3.22 of [7], the analogous one for MV-algebras.

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