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k-NORMALIZATION AND $(k+1)$-LEVEL INFLATION OF VARIETIES

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Abstract

Let τ be a type of algebras. A common measurement of the complexity of terms of type τ is the depth of a term. For $k \geq 1$, an identity $s \approx t$ of type τ is said to be k-normal (with respect to this depth complexity measurement) if either $s = t$ or both s and t have depth $\geq k$. A variety is called k-normal if all its identities are k-normal. Taking $k = 1$ with respect to the usual depth valuation of terms gives the well-known property of normality of identities or varieties. For any variety V, there is a least k-normal variety $N_k(V)$ containing V, the variety determined by the set of all k-normal identities of V. The concept of k-normalization was introduced by K. Denecke and S.L. Wismath in [5], and an algebraic characterization of the elements of $N_k(V)$ in terms of the algebras in V was given in [4]. In [1] a simplified version of this characterization of $N_k(V)$ was given, in the special case of the 2-normalization of the variety V of all lattices, using a construction called the 3-level inflation of a lattice. In this paper we show that the analogous $(k+1)$-level inflation can be used to characterize the algebras of $N_k(V)$ for any variety V having a unary term which satisfies two technical conditions. This includes any variety V which satisfies $x \approx t(x)$ for some unary term t of depth at least k, and in particular any variety, such as the variety of lattices, which satisfies an idempotent identity.

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1. Introduction

Let $\tau = (n_i)_{i \in I}$ be any type of algebras, with an operation symbol f_i of arity n_i for each $i \in I$. Let $X = \{x_1, x_2, x_3, \ldots\}$ be a set of variable symbols, and let $W_\tau(X)$ be the set of all terms of type τ formed using variables from X. We use the well-known Galois connection $Id - Mod$ between classes of algebras and sets of identities. For any class K of algebras of type τ and any set Σ of identities of type τ, we have

$$Mod\,\Sigma = \{ \text{ algebras } \mathcal{A} \text{ of type } \tau \ \mid \ \mathcal{A} \text{ satisfies all identities in } \Sigma\},$$

and

$$Id\,K = \{ \text{ identities } s \approx t \text{ of type } \tau \ \mid \ \text{ all algebras in } K \text{ satisfy } s \approx t\}.$$

For each $t \in W_\tau(X)$, we denote by $v(t)$ the depth of t, that is, the length of the longest path from root to leaves in the tree diagram for t. This defines a valuation function v on the set of all terms of type τ (see [5]). Let $k \geq 0$ be any natural number. An identity $s \approx t$ of type τ is called *k-normal* (with respect to the depth valuation) if either s and t are identical, or $v(t)$, $v(s) \geq k$.

We denote by $N_k(\tau)$ the set of all k-normal identities of type τ. This set is easily seen to be closed under the usual five rules of deduction for identities, meaning that $N_k(\tau)$ is an equational theory. Since $Id\,V$ is also an equational theory for any variety V, so is $Id^{N_k}\,V = N_k(\tau) \cap Id\,V$, the set of all k-normal identities satisfied by V. The variety determined by this set, $N_k(V) = Mod\,Id^{N_k}\,V$, is called the *$k$-normalization of* V. In the special case that $N_k(V) = V$, we say that V is a *k-normal variety*; this occurs when every identity of V is a k-normal identity. Otherwise, V is a proper subvariety of $N_k(V)$, and $N_k(V)$ is the least k-normal variety to contain V. When $k = 1$ these concepts coincide with the usual concept of normal identities and varieties, and the normalization of a variety; see for instance [6].

The variety $N_k(V)$ is defined equationally, by means of the k-normal identities of V. An algebraic characterization of the algebras in $N_k(V)$ was given by Denecke and Wismath in [4], using the concept of a k-choice algebra. They showed that any algebra in $N_k(V)$ is a homomorphic image of a k-choice algebra constructed from an algebra in V. In [1], Chajda, Cheng and Wismath also studied the algebras of the variety $N_2(L)$, the 2-normalization of the variety L of all lattices. Using the order-theoretic nature of lattices, they introduced a modification of 2-choice algebras called the 3-level inflation of an algebra, and showed that the variety $N_2(L)$ consists exactly of all 3-level inflations of lattices. In this paper we extend this result to any $k \geq 1$ and any variety V which has a term t satisfying two technical conditions, showing that for any such V the variety $N_k(V)$ equals the class of $(k+1)$-level inflations of algebras in V. This result includes any variety V which satisfies $x \approx t(x)$ for any term t of depth at least k, and in particular any variety, such as the variety of lattices, which satisfies an idempotent identity.

2. The $(k+1)$-level inflation construction

Let V be any variety of type τ, with $N_k(V)$ its k-normalization for some $k \geq 1$. In this section we introduce a construction called the $(k+1)$-level inflation construction, which we use to produce an algebra in $N_k(V)$ from any algebra in V. Our $(k+1)$-level inflation construction is a generalization of the usual inflation construction, well-known especially in semigroup theory (see for instance [3]). Given a base algebra $\mathcal{A}$, an inflation of $\mathcal{A}$ is formed by adding disjoint sets of new elements to the base set A, one set C_a (containing a) for each element a of A. The union of these new sets then forms the base set of a new algebra containing $\mathcal{A}$, in which operations are performed by the rule that any element in the set C_a always acts like a.

Now we describe the $(k+1)$-level inflation of an algebra $\mathcal{A} = (A; (f_i^A)_{i\in I})$ in V. As in the usual inflation process, we inflate the set A by adding to each $a \in A$ a set C_a containing a, such that for $a \neq b \in A$ the sets C_a and C_b are disjoint. Let $A^* = \bigcup\{C_a \mid a \in A\}$. For each element $c \in A^*$, there is a unique element $\bar{c} \in A$ such that $c \in C_{\bar{c}}$. For each $a \in A$, we will refer to C_a as the class of a. These classes form a partition of A^* which induces an equivalence relation θ on A^*. A mapping ψ from the power set of A^* to A^* satisfying $\psi(C_a) \in C_a$ for all $a \in A$ will be called a *θ-choice function.* But in addition to this usual inflation of $\mathcal{A}$, for each $a \in A$ we partition the set C_a into $k+1$ subclasses or levels C_a^j, for $j = 0, 1, \ldots, k$. We impose the restriction that $|C_a^k| \geq 1$, but the other levels may be empty. Thus, $C_a = \bigcup_{j=0}^k C_a^j$. We say that the elements of C_a^j are attached to element a at level j.

Our new algebra $\mathcal{A}^*$ will have the inflated set A^* as its universe, with operations $f_i^{\mathcal{A}^*}$ for each $i \in I$ defined as follows:

Definition 2.1. Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be an algebra in V, with A^* and θ as above. Let ϕ be a θ-choice function such that for any $a \in A$, $\phi(C_a) \in C_a^k$. For each $i \in I$, we define $f_i^{\mathcal{A}^*}$ on A^* by setting, for any $a_1, \ldots, a_{n_i} \in A^*$,

$$f_i^{\mathcal{A}^*}(a_1, \ldots, a_{n_i}) = \begin{cases} \phi\left(C_{f_i^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_{n_i}})}\right) & \text{if } p \geq k-1 \\ \text{any element of } \displaystyle\bigcup_{j=1+p}^{k} C^j_{f_i^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_{n_i}})} & \text{otherwise,} \end{cases}$$

where $p =$ maximum level of $a_1, \ldots, a_{n_i}$.

The algebra $\mathcal{A}^* = (A^*; (f_i^{\mathcal{A}^*})_{i\in I}) = Inf_{k+1}(\mathcal{A}, \theta)$ will be called a $(k+1)$-*level inflation* of $\mathcal{A}$.

The key observation about our new algebra $\mathcal{A}^*$ is the following fact. In an (ordinary) inflation, each new element a is attached to and acts like an old element $\overline{a}$ from A. In our case, each new element a is also attached to an old element $\overline{a}$, but this attachment also carries with it a level indicator j, with $0 \leq j \leq k$. Definition 2.1 means that applying an operation $f_i^{\mathcal{A}^*}$ to input elements of A^* produces an element which is at a level at least one higher, to a maximum of k, than the levels of the inputs. As a consequence, any element of A^* that is an output of a term of depth r will at least be at level r. In particular, any element that is the output of a term of depth k or more has to be determined by ϕ and so must be the special element at level k selected by our θ-choice function ϕ.

We let V^* be the class of all algebras $\mathcal{A}^* = Inf_{k+1}(\mathcal{A}, \theta)$ formed from some algebra $\mathcal{A} \in V$. Our goal now is to show that $V^* \subseteq N_k(V)$, that is, that any algebra constructed as a $(k+1)$-level inflation from an algebra in V is in $N_k(V)$. Our proof will use the following lemma.

Lemma 2.2. *Let $\mathcal{A}^* = Inf_{k+1}(\mathcal{A}, \theta)$ be a $(k+1)$-level inflation of an algebra $\mathcal{A}$ in V. For any term t of arity m and any $a_1, \ldots, a_m \in A^*$, $\overline{t^{\mathcal{A}^*}(a_1, \ldots, a_m)}$ is in the θ-class of $t^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_m}) \in \mathcal{A}$, so that $\overline{t^{\mathcal{A}^*}(a_1, \ldots, a_m)} = t^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_m})$.*

Proof. We will give a proof by induction on the complexity of t. First, if $t = x_j$, for some $j \geq 1$, then $t^{\mathcal{A}^*}(a_1, \ldots, a_m) = a_j$, and hence

$$\overline{t^{\mathcal{A}^*}(a_1, \ldots, a_m)} = \overline{a_j} = t^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_m}).$$

Therefore, both a_j and $\overline{a_j}$ are in the same θ-class, $C_{\overline{a_j}}$.

Inductively, let $t = f_i(t_1, \ldots, t_{n_i})$. Thus,

$$t^{\mathcal{A}^*}(a_1, \ldots, a_m) = f_i^{\mathcal{A}^*}(t_1^{\mathcal{A}^*}(a_1, \ldots, a_m), \ldots, t_{n_i}^{\mathcal{A}^*}(a_1, \ldots, a_m)).$$

By definition of $f_i^{\mathcal{A}^*}$, we have

$$f_i^{\mathcal{A}^*}(t_1^{\mathcal{A}^*}(a_1, \ldots, a_m), \ldots, t_{n_i}^{\mathcal{A}^*}(a_1, \ldots, a_m)) \in C_{f_i^{\mathcal{A}}(\overline{t_1^{\mathcal{A}^*}(a_1,\ldots,a_m)}, \ldots, \overline{t_{n_i}^{\mathcal{A}^*}(a_1,\ldots,a_m)})}.$$

By induction, $\overline{t_j^{\mathcal{A}^*}(a_1, \ldots, a_m)} = t_1^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_m})$, for all $1 \leq j \leq n_i$. Therefore, $t^{\mathcal{A}^*}(a_1, \ldots, a_m)$ is in $C_{f_i^{\mathcal{A}}(t_1^{\mathcal{A}}(\overline{a_1},\ldots,\overline{a_m}), \ldots, t_{n_i}^{\mathcal{A}}(\overline{a_1},\ldots,\overline{a_m}))}$. Now,

$$f_i^{\mathcal{A}}(t_1^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_m}), \ldots, t_{n_i}^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_m})) = t^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_m}).$$

Therefore, $t^{\mathcal{A}^*}(a_1, \ldots, a_m) \in C_{t^{\mathcal{A}}(\overline{a_1},\ldots,\overline{a_m})}$ and thus $t^{\mathcal{A}^*}(a_1, \ldots, a_m)$ is in the θ-class of $t^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_m})$, which is in A. ■

Theorem 2.3. *Any algebra $\mathcal{A}^*$ constructed as a $(k+1)$-level inflation of an algebra $\mathcal{A}$ in V is in $N_k(V)$. Consequently, $V^* \subseteq N_k(V)$.*

Proof. Let $\mathcal{A}^* = Inf_{k+1}(\mathcal{A}, \theta)$ be a $(k+1)$-level of some algebra $\mathcal{A}$ in V. We will show that $\mathcal{A}^*$ is in $N_k(V)$ by showing that it satisfies any k-normal identity $s \approx t$ of V. By Lemma 2.2, we know that $s^{\mathcal{A}^*}(a_1, \ldots, a_m)\ \theta\ s^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_m})$ and $t^{\mathcal{A}^*}(a_1, \ldots, a_m)\ \theta\ t^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_m})$. Since V satisfies $s \approx t$ and all the elements $\overline{a_1}, \ldots, \overline{a_m}$ are in A, we have $s^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_m}) = t^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_m})$. Therefore, $s^{\mathcal{A}^*}(a_1, \ldots, a_m)\ \theta\ t^{\mathcal{A}^*}(a_1, \ldots, a_m)$. That is, $s^{\mathcal{A}^*}(a_1, \ldots, a_m)$ and $t^{\mathcal{A}^*}(a_1, \ldots, a_m)$ are in the same θ-class; specifically, $s^{\mathcal{A}^*}(a_1, \ldots, a_m)$ and $t^{\mathcal{A}^*}(a_1, \ldots, a_m)$ are both in $C_{s^{\mathcal{A}}(\overline{a_1},\ldots,\overline{a_m})}$.

Moreover, we know that $v(s), v(t) \geq k$, so by the comment following Definition 2.1, $s^{\mathcal{A}^*}(a_1, \ldots, a_m) = \phi(C_{s^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_m})}) = t^{\mathcal{A}^*}(a_1, \ldots, a_m)$. Thus $s^{\mathcal{A}^*}(a_1, \ldots, a_m) = t^{\mathcal{A}^*}(a_1, \ldots, a_m)$. This shows that $\mathcal{A}^*$ satisfies $s \approx t$, as required. ■

For any $\mathcal{A} \in V$, if no new elements are added in the $(k+1)$-level inflation of $\mathcal{A}$ to $\mathcal{A}^*$, then $\mathcal{A}^*$ is just $\mathcal{A}$ again. This means that we have $V \subseteq V^* \subseteq N_k(V)$. If sufficiently many new elements are added in a $(k+1)$-level inflation of $\mathcal{A}$, then it is possible to break the non-k-normal identities of V but keep all the k-normal identities of V, and so have $\mathcal{A}^* \in N_k(V) - N_{k-1}(V)$, for $k \geq 2$ or $\mathcal{A}^* \in N_k(V) - V$, for $k = 1$.

Example 2.4. Let $\mathcal{A} = (\{a, b, c\}; f^{\mathcal{A}})$ be the three element left zero band with the Cayley table given in Figure 1. Figure 1 shows a 4-level inflation of $\mathcal{A}$, using $k = 3$, where new elements have been added as follows.

Let $C_a = \{u, q, r, a\}$, with $C_a^0 = \{u\}$, $C_a^1 = \{q\}$, $C_a^2 = \{r\}$, and $C_a^3 = \{a\}$.

Let $C_b = \{b, d, g, h\}$, with $C_b^0 = \{b\}$, $C_b^1 = \emptyset$, $C_b^2 = \{d\}$, and $C_b^3 = \{g, h\}$.

Let $C_c = \{w, c, z\}$, with $C_c^0 = \{w\}$, $C_c^1 = \{c\}$, $C_c^2 = \emptyset$, and $C_c^3 = \{z\}$.

Let $\mathcal{A}^* = (C_a \bigcup C_b \bigcup C_c; f^{\mathcal{A}^*})$ be the 4-level inflation of $\mathcal{A}$ with operation $f^{\mathcal{A}^*}$ as shown in the second table of Figure 1. For products involving elements at level 3 we use the choice function ϕ defined to have $\phi(C_a) = a$, $\phi(C_b) = g$, and $\phi(C_c) = z$.

We illustrate the remaining products with some examples. The maximum of the levels of b and q is 1, and so for $f^{\mathcal{A}^*}(b, q)$ we can select any element of $C^2_{f^{\mathcal{A}}(\bar{b},\bar{q})} \bigcup C^3_{f^{\mathcal{A}}(\bar{b},\bar{q})}$ $(= C_b^2 \bigcup C_b^3)$. In this example we chose $f^{\mathcal{A}^*}(b, q) = h$. Since the maximum of the levels of u and d is 2, we have $f^{\mathcal{A}^*}(u, d) = \phi(C_{f^{\mathcal{A}}(\bar{u},\bar{d})}) = \phi(C_a) = a$.

Note also that in this example, we have $f^{\mathcal{A}^*}(f^{\mathcal{A}^*}(u, u), w) = a$, but $f^{\mathcal{A}^*}(u, f^{\mathcal{A}^*}(u, w)) = r$. This shows that $\mathcal{A}^*$ does not satisfy associativity, which is a 2-normal identity of the variety V of left zero bands and hence of $N_2(V)$. So $\mathcal{A}^*$ is in $N_3(V)$ but not in $N_2(V)$.

$f^{\mathcal{A}}$	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

$f^{\mathcal{A}^*}$	u	b	w	q	c	r	d	a	g	h	z
u	q	r	q	r	a	a	a	a	a	a	a
b	d	h	d	h	d	g	g	g	g	g	g
w	c	c	z	z	z	z	z	z	z	z	z
q	r	a	a	r	r	a	a	a	a	a	a
c	z	z	z	z	z	z	z	z	z	z	z
r	a	a	a	a	a	a	a	a	a	a	a
d	g	g	g	g	g	g	g	g	g	g	g
a	a	a	a	a	a	a	a	a	a	a	a
g	g	g	g	g	g	g	g	g	g	g	g
h	g	g	g	g	g	g	g	g	g	g	g
z	z	z	z	z	z	z	z	z	z	z	z

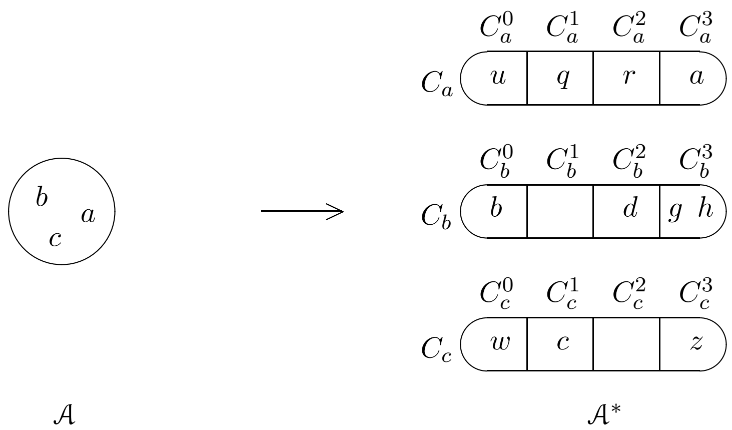

Figure 1

3. From $N_k(V)$ to $(k+1)$-level inflations

In this section, we consider the question of whether any algebra in $N_k(V)$ can be viewed as a $(k+1)$-level inflation of some algebra in V. Starting with any algebra $\mathcal{A}$ in $N_k(V)$, our first step is to produce a subalgebra $\mathcal{B}$ of $\mathcal{A}$ which is in V. Then we describe a method for attaching each element of $\mathcal{A}$ to an element of $\mathcal{B}$, and for defining a $(k+1)$-level inflation of the base set B back to the set A in such a way as to obtain $\mathcal{A}$ again. It turns out that this process does not always produce the original algebra $\mathcal{A}$; but we show that it works for certain large classes of varieties.

We begin by showing how to produce from any $\mathcal{A}$ in $N_k(V)$ a subalgebra which is in V. To do this, we use the concept of the level of an element in an algebra. This concept was used in [7] for normality and in [2] for k-normality, and is the analogue for elements of an algebra of the depth of a term. In general, let $\mathcal{D}$ be any algebra of type τ and let $d \in D$. The element d is always the output of some term operations $t^{\mathcal{D}}$ on $\mathcal{D}$, in particular, of variable terms. If the maximum depth of any term t for which d is obtainable as an output of $t^{\mathcal{D}}$ is j, for $0 \leq j \leq k-1$, then we assign d a level of j. Otherwise, we assign d a level of k. From this definition of levels of elements in an algebra it is clear that applying any operation of the algebra to elements of particular levels results in an output element whose level is at least one more than the maximum of the levels of the input elements, to a maximum level of k. Note that this definition of levels is consistent with how we defined operations on the sets C_a^j in our $(k+1)$-level inflation, in Definition 2.1.

Now we use this definition to determine the level of each element of our algebra $\mathcal{A}$ from $N_k(V)$. We are particularly interested in the set

$$L_k^{\mathcal{A}} \quad := \quad \{a \in \mathcal{A} \mid a \text{ has level } k \text{ in } \mathcal{A}\}.$$

Since there are terms of type τ of arbitrarily high depth, the set $L_k^{\mathcal{A}}$ is clearly a non-empty subset of A. Moreover, any application of operations of $\mathcal{A}$ to elements from $L_k^{\mathcal{A}}$ results in an element of $\mathcal{A}$ at level k, so $\mathcal{L}_k^{\mathcal{A}}$ is a subalgebra of $\mathcal{A}$. We shall refer to $\mathcal{L}_k^{\mathcal{A}}$ as the *skeleton algebra* of the original algebra $\mathcal{A}$. Now we will show that this skeleton algebra is in V.

Lemma 3.1. *Let $k \geq 1$. Let V be any variety and let $\mathcal{A}$ be any algebra in $N_k(V)$. Then the skeleton algebra $\mathcal{L}_k^{\mathcal{A}}$ of $\mathcal{A}$ is in V.*

Proof. We will show that $\mathcal{L}_k^{\mathcal{A}}$ is in V by showing that it satisfies any identity $s \approx t$ of V. Suppose that s and t have arity p, and let $b_1, \ldots, b_p \in L_k^{\mathcal{A}}$. These are level k elements of A, so we can write each $b_j = u_j^{\mathcal{A}}(a_{1_j}, \ldots, a_{m_j})$ for some elements $a_{1_j}, \ldots, a_{m_j} \in A$ and some term u_j of depth $\geq k$.

We shall denote by a^+ the total list of inputs $a_{1_1}, \ldots, a_{m_1}, \ldots, a_{1_p}, \ldots, a_{m_p}$. Let r be the number of items in this list (counting any multiplicities). By adding fictitious variables as needed, we can find terms w_j of arity r such that $w_j^{\mathcal{A}}(a^+) = b_j$ for each $1 \leq j \leq p$. That is, we set $w_1(x_1, \ldots, x_r) = u_1(x_1, \ldots, x_{m_1})$, then $w_2(x_1, \ldots, x_r) = u_2(x_{m_1+1}, \ldots, x_{m_1+m_2})$, and so on. Thus we have $w_j^{\mathcal{A}}(a+) = u_j^{\mathcal{A}}(a_{1_j}, \ldots, a_{m_j}) = b_j$ for each $1 \leq j \leq p$, and each w_j is a term of depth at least k.

Now since $s \approx t$ holds in V, its k-normal consequence $s(w_1, \ldots, w_p) \approx t(w_1, \ldots, w_p)$ holds in $N_k(V)$ and hence in $\mathcal{A}$. Therefore we have

$$\begin{aligned} &s^{\mathcal{L}_k^{\mathcal{A}}}(b_1, \ldots, b_p) = s^{\mathcal{A}}(b_1, \ldots, b_p) = s^{\mathcal{A}}(w_1^{\mathcal{A}}(a^+), \ldots, w_p^{\mathcal{A}}(a^+)) \\ &= s(w_1, \ldots, w_p)^{\mathcal{A}}(a^+) = t(w_1, \ldots, w_p)^{\mathcal{A}}(a^+) = t^{\mathcal{A}}(w_1^{\mathcal{A}}(a^+), \ldots, w_p^{\mathcal{A}}(a^+)) \\ &= t^{\mathcal{A}}(b_1, \ldots, b_p) = t^{\mathcal{L}_k^{\mathcal{A}}}(b_1, \ldots, b_p). \end{aligned}$$

This shows that $\mathcal{L}_k^{\mathcal{A}}$ satisfies $s \approx t$ and so is in V. ■

Having produced from $\mathcal{A} \in N_k(V)$ a skeleton algebra $\mathcal{L}_k^{\mathcal{A}}$ in V, we now want to form a $(k+1)$-level inflation of this skeleton which gives us back the original algebra $\mathcal{A}$. Inflating the base set $L_k^{\mathcal{A}}$ back up to the original set A requires that we attach each element a of A to some level k element $\overline{a}$ of $L_k^{\mathcal{A}}$. To do this, we use the following technique. Let t be a (fixed) unary term of type τ of depth at least k. We shall attach each element $a \in A$ to the element $\overline{a} = t(a)$, which is a level k element of A. Since level k elements should be attached to themselves, we have our first condition on the term t:

(C1) For any algebra $\mathcal{A} \in N_k(V)$ and any element $a \in A$ of level k, we need $t^A(a) = a$.

For each base element $b \in L_k^{\mathcal{A}}$, we set $C_b = \{a \in A \mid \bar{a} = b\}$. For the $(k+1)$-level inflation, we use the base set $(L_k^{\mathcal{A}})^* = \bigcup\{C_b \mid b \in L_k^{\mathcal{A}}\}$. It is clear that this set is equal to the universe A of $\mathcal{A}$. We also partition each C_b into the $k+1$ subclasses $C_b^j = \{b_1 \in C_b \mid b_1 \text{ has level } j \text{ in } \mathcal{A}\}$, for all $0 \leq j \leq k$. Note that C_b^k is the singleton set $\{b\}$, so for any $b \in L_k^{\mathcal{A}}$ we set $\phi(C_b) = b$. Finally, we need to define the operations of the $(k+1)$-level inflation on this new base set, using ϕ. For each $i \in I$ and for any $a_1, \ldots, a_{n_i} \in (L_k^{\mathcal{A}})^*$, we set

$$f_i^{(\mathcal{L}_k^{\mathcal{A}})^*}(a_1,\ldots,a_{n_i}) = \begin{cases} f_i^{\mathcal{L}_k^{\mathcal{A}}}(\overline{a_1},\ldots,\overline{a_{n_i}}) \ = \phi\left(C_{f_i^{\mathcal{L}_k^{\mathcal{A}}}(\overline{a_1},\ldots,\overline{a_{n_i}})}\right) & \text{if } p \geq k-1 \\ f_i^{\mathcal{A}}(a_1,\ldots,a_{n_i}) & \text{otherwise,} \end{cases}$$

where p is the maximum level of $a_1,\ldots,a_{n_i}$.

To ensure that these operations do define a $(k+1)$-level inflation of $\mathcal{L}_k^{\mathcal{A}}$, we note that Definition 2.1 requires that for $p < k-1$,

$$f_i^{(\mathcal{L}_k^{\mathcal{A}})^*}(a_1,\ldots,a_{n_i}) \in \bigcup_{j=1+p}^{k} C^j_{f_i^{\mathcal{L}_k^{\mathcal{A}}}(\overline{a_1},\ldots,\overline{a_{n_i}})}.$$

We will start by checking whether our element $f_i^{\mathcal{A}}(a_1,\ldots,a_{n_i})$ is in the class $C_{f_i^{\mathcal{L}_k^{\mathcal{A}}}(\overline{a_1},\ldots,\overline{a_{n_i}})}$. We have

$$\begin{aligned} & f_i^{\mathcal{A}}(a_1,\ldots,a_{n_i}) \in C_{f_i^{\mathcal{L}_k^{\mathcal{A}}}(\overline{a_1},\ldots,\overline{a_{n_i}})} \\ \text{iff} \quad & \overline{f_i^{\mathcal{A}}(a_1,\ldots,a_{n_i})} = f_i^{\mathcal{L}_k^{\mathcal{A}}}(\overline{a_1},\ldots,\overline{a_{n_i}}) \\ \text{iff} \quad & \overline{f_i^{\mathcal{A}}(a_1,\ldots,a_{n_i})} = f_i^{\mathcal{A}}(\overline{a_1},\ldots,\overline{a_{n_i}}) \\ \text{iff} \quad & t(f_i^{\mathcal{A}}(a_1,\ldots,a_{n_1})) \ = \ f_i^{\mathcal{A}}(t(a_1),\ldots,t(a_{n_i})). \end{aligned}$$

This gives us our second restriction on the term t:

(C2) For all $i \in I$, the variety $N_k(V)$ must satisfy the identity

$$t(f_i(x_1,\ldots,x_{n_i})) \ \approx \ f_i(t(x_1),\ldots,t(x_{n_i})).$$

So for any $\mathcal{A}$ in $N_k(V)$, if term t satisfies condition (C2), then our element $f_i^{\mathcal{A}}(a_1,\ldots,a_{n_i})$ is in the right class $C_{f_i^{\mathcal{L}_k^{\mathcal{A}}}(\overline{a_1},\ldots,\overline{a_{n_i}})}$. If $a_1,\ldots,a_{n_i}$ have maximum level p in $\mathcal{A}$, then $f_i^{\mathcal{A}}(a_1,\ldots,a_{n_i})$ has level at least $p+1$, and our construction ensures that our element is also at the correct level.

Hence for $p < k-1$,

$$f_i^{\mathcal{A}}(a_1, \ldots, a_{n_i}) \in \bigcup_{j=1+p}^{k} C^j_{f_i^{\mathcal{A}}(\overline{a_1}, \ldots, \overline{a_{n_i}})}.$$

This shows that our construction of $(\mathcal{L}_k^{\mathcal{A}})^*$ satisfies the conditions of the $(k+1)$-level inflation construction given in Section 2. and so $(\mathcal{L}_k^{\mathcal{A}})^*$ is a $(k+1)$-level inflation of $\mathcal{L}_k^{\mathcal{A}}$.

Now we are ready to prove that as long as there is exists a term t of V satisfying the two restrictions (C1) and (C2), any algebra $\mathcal{A}$ in $N_k(V)$ is a $(k+1)$-level inflation of its skeleton $\mathcal{L}_k^{\mathcal{A}}$.

Theorem 3.2. *Let V be a variety of type τ for which there exists a term t of depth at least k satisfying conditions* (C1) *and* (C2). *Then any algebra $\mathcal{A}$ in $N_k(V)$ is a $(k+1)$-level inflation of its skeleton algebra $\mathcal{L}_k^{\mathcal{A}}$ in V.*

Proof. Let $\mathcal{A}$ be an element of $N_k(V)$, and let t be a term satisfying (C1) and (C2). Let $\mathcal{L}_k^{\mathcal{A}}$ be the skeleton algebra of $\mathcal{A}$ and let $(\mathcal{L}_k^{\mathcal{A}})^* = Inf_{k+1}(\mathcal{L}_k^{\mathcal{A}}, \theta) = ((\mathcal{L}_k^{\mathcal{A}})^*; (f_i^{(\mathcal{L}_k^{\mathcal{A}})^*})_{i\in I})$ be the $(k+1)$-level inflation described above, where each element a is attached to $\overline{a} = t^A(a)$. It is clear from our construction that the base set $(L_k^{\mathcal{A}})^* = \bigcup\{C_b \mid b \in L_k^{\mathcal{A}}\}$ of $(\mathcal{L}_k^{\mathcal{A}})^*$ is equal to the base set A of $\mathcal{A}$, and we want to show that for all $i \in I$, the operations $f_i\{_k^{\mathcal{A}}\}^*$ and $f^{\mathcal{A}}$ coincide. Let $i \in I$ and $a_1, \ldots, a_{n_i} \in L_k^{\mathcal{A}}$. If the maximum of the levels of $a_1, \ldots, a_{n_i}$ is $< k-1$, then by definition $f_i^{(\mathcal{L}_k^{\mathcal{A}})^*}(a_1, \ldots, a_{n_i}) = f_i^{\mathcal{A}}(a_1, \ldots, a_{n_i})$. If the maximum of the levels of $a_1, \ldots, a_{n_i}$ is $\geq k-1$, then $f_i^{(\mathcal{L}_k^{\mathcal{A}})^*}(a_1, \ldots, a_{n_i}) = f_i^{\mathcal{L}_k^{\mathcal{A}}}(\overline{a_1}, \ldots, \overline{a_{n_i}})$. In this case, $f_i^{\mathcal{A}}(a_1, \ldots, a_{n_i})$ is in $C_{f_i^{\mathcal{L}_k^{\mathcal{A}}}(\overline{a_1}, \ldots, \overline{a_{n_i}})}$ and has level k.

But $C^k_{f_i^{\mathcal{L}_k^{\mathcal{A}}}(\overline{a_1}, \ldots, \overline{a_{n_i}})}$ is a singleton set, containing only $f_i^{\mathcal{L}_k^{\mathcal{A}}}(\overline{a_1}, \ldots, \overline{a_{n_i}})$. Hence, $f_i^{\mathcal{A}}(a_1, \ldots, a_{n_i}) = f_i^{\mathcal{L}_k^{\mathcal{A}}}(\overline{a_1}, \ldots, \overline{a_{n_i}})$ and we have $f_i^{\mathcal{A}}(a_1, \ldots, a_{n_i}) = f_i^{(\mathcal{L}_k^{\mathcal{A}})^*}(a_1, \ldots, a_{n_i})$. Therefore, $\mathcal{A} = (\mathcal{L}_k^{\mathcal{A}})^*$. ■

Corollary 3.3. *Let V be a variety of type τ for which there exists a term t of depth at least k satisfying conditions* (C1) *and* (C2). *Then the class $N_k(V)$ is precisely the class V^*.*

Corollary 3.4. *Let t be any unary term of type τ having depth at least k. Let $V_1 = Mod\{x \approx t(x)\}$ be the variety of type τ determined by the identity $x \approx t(x)$. Then any subvariety V of V_1 satisfies conditions* (C1) *and* (C2) *for t, and has $N_k(V) = V^*$.*

Proof. Condition (C2) is clearly met, since $t(f_i(x_1, \ldots, x_{n_i})) \approx f_i(t(x_1), \ldots, t(x_{n_i}))$ is a k-normal identity which is a consequence of $x \approx t(x)$. So we show that (C1) holds, that is, that $t(a) = a$ for any level k element a of any algebra $\mathcal{A}$ in $N_k(V)$. If $a \in A$ is at level k, we can write $a = s^{\mathcal{A}}(a_1, \ldots, a_m)$, for some term s of depth $\geq k$ and some $a_1, \ldots, a_m \in A$. Then $s \approx t(s)$ is a k-normal consequence of $x \approx t(x)$ and also holds in $\mathcal{A}$. So we have $a = s^{\mathcal{A}}(a_1, \ldots, a_m) = t^{\mathcal{A}}(s^{\mathcal{A}}(a_1, \ldots, a_m)) = t^{\mathcal{A}}(a)$. ■

Example 3.5. For this example we use type (2) and varieties of semigroups. As is customary for such varieties, we will write terms and identities with the binary operation symbol omitted, writing x_1x_2 for the product $f(x_1, x_2)$.

In any variety V of semigroups, the identity $s_1 \approx s_2$ holds in $Id\,V$ for any two terms s_1 and s_2 of depth $\geq k$ having exactly $k+1$ occurrences of x and no other variables. Assuming associativity, we will use x^{k+1} to represent any such term. We will use as our special term t the term $t(x) = f(f(\ldots(f(x,x),x),\ldots),x) = x^{k+1}$. We will show that although from Corollary 3.4 it is sufficient to have V satisfy $x \approx t(x)$, this condition is not necessary. Instead we can impose two weaker conditions: $x_1 \cdots x_k \approx t(x_1 \cdots x_k)$ and $t(x_1x_2) \approx t(x_1)t(x_2)$. That is, we let V_2 be the variety of semigroups determined by the identities $x_1 \cdots x_{k+1} \approx (x_1 \cdots x_{k+1})^{k+1}$ and $(x_1)^{k+1}(x_2)^{k+1} \approx (x_1x_2)^{k+1}$.

The identity $(x_1)^{k+1}(x_2)^{k+1} \approx (x_1x_2)^{k+1}$ is precisely the type (2) analogue of condition (C2), so it suffices to verify that condition (C1) also holds for any algebra $\mathcal{A}$ in $N_k(V_2)$. From the identity $x_1 \cdots x_{k+1} \approx (x_1 \cdots x_{k+1})^{k+1}$ of V_2 we can deduce as a k-normal consequence that for any term s of depth $\geq k$, the identity $s \approx s^{k+1}$ holds in $N_k(V_2)$. The proof of (C1) then proceeds exactly as in the proof of Corollary 3.4.

Example 3.6. Let $V = Mod(xy \approx x)$ be the semigroup variety of left-zero bands. In this case, $V \subseteq V_1$ and $V \subseteq V_2$. Let $\mathcal{A} = (\{a, a^2, a^3, b, b^2, b^3, ab, ba\}; f^{\mathcal{A}})$ be the relatively free algebra on $\{a, b\}$ for the variety $N_2(V)$ of 2-normalized left zero bands. The Cayley table for $\mathcal{A}$ is given in Figure 2. We start by determining the levels of the elements of A. It is easy to see that a and b have level 0, while a^2, b^2, ab and ba have level 1, and a^3 and

b^3 have level 2. We set $\mathcal{L}_2^{\mathcal{A}} = (\{a^3, b^3\}; f^{\mathcal{L}_2^{\mathcal{A}}})$. Now we inflate $\mathcal{L}_2^{\mathcal{A}}$ to $(\mathcal{L}_2^{\mathcal{A}})^*$ $= (\{a, a^2, a^3, b, b^2, b^3, ab, ba\}; f^{(\mathcal{L}_2^{\mathcal{A}})^*})$ by attaching the level 0 and 1 elements of A to the level 2 elements using the term $t(x) = f(f(x,x),x)) = x^3$ and attaching each level 2 element to itself (see Figure 2). For example, $\overline{ab}$ $= (ab)^3 = ababab = a^3$ and so ab is attached to a^3. Elements a, a^2, ab and a^3 are attached to a^3 while b, b^2, ba and b^3 are attached to b^3. Let $C_{a^3} = \{a, a^2, ab, a^3\}$, with $C_{a^3}^0 = \{a\}$, $C_{a^3}^1 = \{a^2, ab\}$, and $C_{a^3}^2 = \{a^3\}$ and let $C_{b^3} = \{b, b^2, ba, b^3\}$, with $C_{b^3}^0 = \{b\}$, $C_{b^3}^1 = \{b^2, ba\}$, and $C_{b^3}^2 = \{b^3\}$. Let $\phi(C_{a^3}) = a^3$ and $\phi(C_{b^3}) = b^3$. Since a, b have maximum level 0, we have $f^{(\mathcal{L}_2^{\mathcal{A}})^*}(a,b) = f^{\mathcal{A}}(a,b) = ab$. The elements b^2, ab have maximum level 1, and so $f^{(\mathcal{L}_2^{\mathcal{A}})^*}(b^2, ab) = \phi(C_{f^{\mathcal{L}_2^{\mathcal{A}}}(\overline{b^2}, \overline{ab})}) = f^{\mathcal{L}_2^{\mathcal{A}}}(\overline{b^2}, \overline{ab}) = b^3 (= f^{\mathcal{A}}(b^2, ab))$. The Cayley table for $f^{(\mathcal{L}_2^{\mathcal{A}})^*}$ is the same as the Cayley table for $f^{\mathcal{A}}$.

$f^{\mathcal{A}}/f^{(\mathcal{L}_2^{\mathcal{A}})^*}$	a	b	a^2	b^2	ab	ba	a^3	b^3
a	a^2	ab	a^3	a^3	a^3	a^3	a^3	a^3
b	ba	b^2	b^3	b^3	b^3	b^3	b^3	b^3
a^2	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
b^2	b^3	b^3	b^3	b^3	b^3	b^3	b^3	b^3
ab	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
ba	b^3	b^3	b^3	b^3	b^3	b^3	b^3	b^3
a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
b^3	b^3	b^3	b^3	b^3	b^3	b^3	b^3	b^3

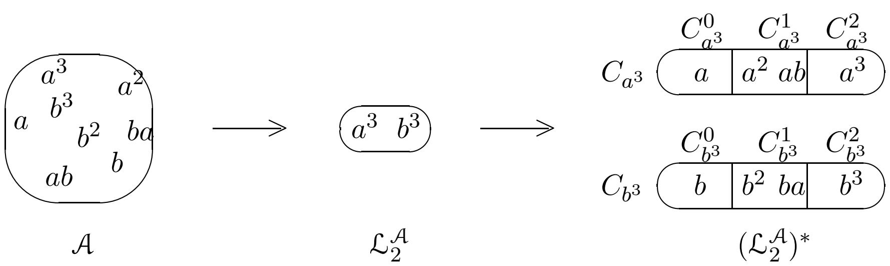

Figure 2

There are many classes of varieties which satisfy the conditions of Theorem 3.2 or Corollary 3.4. These include any variety which satisfies an idempotent identity, or even a consequence of idempotence such as $x \approx t(x)$ for a term t of depth at least k. As a special case, we see that the construction from [1] for 2-normalizations of lattices did not actually need the order-theoretic property of lattices, but only the idempotence of the meet and join operations. However, there are many varieties for which there is no term t fulfilling conditions (C1) and (C2). For instance, if V is the variety of all semigroups and $k = 3$, there is no term t of depth 3 or more for which V satisfies (C2), that $t(x, y) \approx t(x)t(y)$. It may still be true in this case that $N_k(V) = V^*$, but our construction does not give a proof of this.

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