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DISTRIBUTIVE DIFFERENTIAL MODALS

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Abstract

A differential modal is an algebra with two binary operations such that one of the reducts is a differential groupoid and the other is a semilattice, and with the groupoid operation distributing over the semilattice operation. The aim of this paper is to show that the varieties of entropic and distributive differential modals coincide, and to describe the lattice of varieties of entropic differential modals.

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1. Introduction

A *mode* (A, Ω) is an idempotent and entropic algebra. It means that each singleton $\{a\}$ is a subalgebra and each operation $\omega \in \Omega$ is a homomorphism. These two properties may be described as identities:

$$x \dots x\omega = x$$

$$x_{11} \dots x_{1n}\omega \dots x_{m1} \dots x_{mn}\omega\eta = x_{11} \dots x_{m1}\eta \dots x_{n1} \dots x_{nm}\eta\omega$$

true for each n-ary operation ω and each m-ary operation η in Ω.

Let AP denote the set of all non-empty, finitely generated subalgebras of the algebra (A, Ω). For subalgebras $A_1, \dots, A_n \in AP$ and for n-ary operation $\omega \in \Omega$ we define the complex ω-product of the subalgebras $A_1, \dots, A_n$ as follows

$$A_1 \dots A_n\omega := \{a_1 \dots a_n\omega \ \mid \ a_i \in A_i, \ i = 1, \dots n\}.$$

If (A, Ω) is a mode, then the complex product of subalgebras of A is also a subalgebra of A.

An identity $t = s$ is said to be linear if the multiplicities of each argument of t and s are at most 1.

Lemma 1.1 [6]**.** *If (A, Ω) is a mode, then (AP, Ω) is again a mode satisfying each linear identity satisfied by (A, Ω).*

A *modal* is an algebra $(A, +, \Omega)$ for which the reduct (A, Ω) is a mode, $(A, +)$ is a semilattice and the operations $\omega \in \Omega$ distribute over $+$. It means that for n-ary operation ω and $j = 1, \ldots, n$ the following laws are true:

$$x_1 \ldots x_{j-1}(x_j + x_j')x_{j+1} \ldots x_n\omega = x_1 \ldots x_j \ldots x_n\omega + x_1 \ldots x_j' \ldots x_n\omega.$$

Lemma 1.2 [6]**.** *Let (A, Ω) be a mode. Then for $A_1, A_2 \in AP$ define $A_1 + A_2 := \langle A_1 \cup A_2 \rangle$. Then for any mode (A, Ω) the algebra $(AP, +, \Omega)$ forms a modal.*

Lemma 1.3 [6]**.** *Let $(A, +, \Omega)$ be a modal. Then for each n-ary operation $\omega \in \Omega$ and $a_1, \ldots, a_n \in A$*

$$a_1 \ldots a_n\omega \leq a_1 + \ldots + a_n.$$

Let $\underline{\underline{V}}$ denote a variety of modes defined by linear identities and let $\underline{\underline{MV}}$ denote the variety of modals whose mode reducts lie in the variety $\underline{\underline{V}}$.

Lemma 1.4 [6]**.** *The modal $(XVP, +, \Omega)$ of finitely generated nonempty subalgebras of the free $\underline{\underline{V}}$-algebra XV on X is the free $\underline{\underline{MV}}$-algebra XMV on X.*

A mode is a *semilattice mode* if some binary term interprets as a semilattice operation.

Theorem 1.5 [1]**.** *If $\underline{\underline{K}}$ is a variety of semilattice modes, and $\{x, y\}K$ is the free $\underline{\underline{K}}$-algebra on $\{x, y\}$, then $\underline{\underline{K}} = V(\{x, y\}K)$.*

With a variety $\underline{\underline{K}}$ of semilattice modes we can associate the semiring $R(\underline{\underline{K}})$, defined as follows:

(1) the universe R of $R(\underline{\underline{K}})$ is the subuniverse of $\{x, y\}K$ of all terms t such that $t \geq y$,

(2) for $s, t \in R$ we define the addition to be the semilattice addition, while the multiplication is defined by

$$s \circ t := xy\, t\, y\, s.$$

Then $R(\underline{\underline{K}})$ is the semiring $(R, \circ, +, x+y, y)$ with y as zero and $x+y$ as identity.

Theorem 1.6 [1]. *If $\underline{\underline{K}}$ is a variety of semilattice modes, then the lattice of equational theories extending the theory of $\underline{\underline{K}}$ is isomorphic to the congruence lattice $Con\, R(\underline{\underline{K}})$ of $R(\underline{\underline{K}})$.*

A *differential groupoid* is a groupoid $(G, \cdot)$ satisfying the following identities:

$$x \cdot x = x, \tag{1.1}$$

$$xy \cdot zt = xz \cdot yt, \tag{1.2}$$

$$x^2y := x \cdot xy = x. \tag{1.3}$$

Note also two other identities satisfied in all differential groupoids

$$x \cdot yz = xy, \tag{1.4}$$

and

$$xy \cdot z = xz \cdot y. \tag{1.5}$$

To denote the variety of differential groupoids we will use the symbol $\underline{\underline{Dm}}$. Note that differential groupoids are modes.

Lemma 1.7 [3]. *In the free differential groupoid $\{x_0, \ldots, x_{n-1}\}Dm$ on $\{x_0, \ldots, x_{n-1}\}$ each element may be expressed in the standard form*

$$xy_1^{\alpha_1} \ldots y_n^{\alpha_n} := (\ldots((\ldots(\ldots((x\,\underbrace{y_1)y_1)\ldots)y_1)}_{\alpha_1}\ldots)\,\underbrace{y_n)\ldots)y_n}_{\alpha_n},$$

where $x, y_i \in \{x_0, \ldots, x_{n-1}\}$ and $y_i \neq x \quad \forall i \in \{1, \ldots, n\}$.

Note that for $\alpha = 0$, we have $xy^\alpha = x$.

Let G be the free differential groupoid $\{x_0, \ldots, x_{n-1}\}Dm$ and $y \in G$. Let

$$R(y) : G \to G;\ x \mapsto x \cdot y$$

and

$$R : G \to End(G);\ x \mapsto R(x).$$

The set $xR(G)$ is the *orbit of* x in G. By results of [3] and [7], it is easy to see that G consists of n disjoint orbits

$$x_0R(G), \ldots, x_{n-1}R(G),$$

and an orbit $x_iR(G)$ consists of elements with x_i as the left-most variable. Each orbit is a subgroupoid and a left-zero band. Free differential groupoids may be represented by labelled directed graphs, see [2]. For example the free algebra $\{x, y\}Dm$ on two generators is illustrated in the Figure 1.

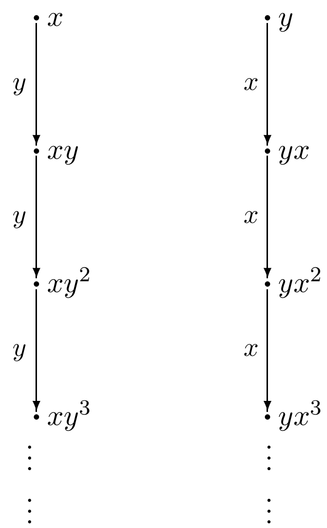

Figure 1

An arrow denotes multiplication by the generator of the other orbit.

2. Free differential modals

A *differential modal* is a modal whose mode reduct is a differential groupoid. Let $\underline{\underline{DM}}$ denote the variety of differential modals.

By Lemma 1.4, the free differential modal XDM on $X = \{x_0, \ldots, x_{n-1}\}$ is the modal of all finitely generated, non-empty submodes of the free differential mode XDm. By results of [6, Section 3.5], the one-to-one corespondence between elements of $XDmP$ and terms representing elements of XDM is given by

$$A = \langle a_1, \ldots, a_k \rangle := \langle \{a_1, \ldots, a_k\} \rangle \mapsto t_A = a_1 + \ldots + a_k,$$

where $\{a_1, \ldots, a_k\}$ is the (uniquely determined) minimal set of generators of A. This corespondence and Lemma 1.7 imply the following result.

Lemma 2.1. *In the free differential modal $\{x_0, \ldots, x_{n-1}\}DM$ on the set $\{x_0, \ldots, x_{n-1}\}$ each element may be expressed in the form*

$$t = y_1 z_{11}^{\alpha_{11}} \ldots z_{1k_1}^{\alpha_{1k_1}} + \ldots + y_l z_{l1}^{\alpha_{l1}} \ldots z_{lk_l}^{\alpha_{lk_l}},$$

where $y_i,\ z_{ij} \in \{x_0, \ldots, x_{n-1}\}$ and $y_i \neq z_{ij}$ for $j = 1, \ldots k_i$.

Let us describe in details the free differential modal $\{x, y\}DM$ on two generators. We have two kinds of non-empty, finitely generated subalgebras of $G = \{x, y\}Dm$:

(1) the finite subsets of one orbit ($xR(G)$ or $yR(G)$),

(2) the subalgebras with elements of both orbits, generated by two elements xy^k and yx^r for some k, r :

$$\langle xy^k, yx^r \rangle = \{xy^k, xy^{k+1}, xy^{k+2}, \ldots\} \cup \{yx^r, yx^{r+1}, yx^{r+2}, \ldots\}.$$

Let $A, B \in \{x, y\}DmP$ and $A = A_x \cup A_y,\ B = B_x \cup B_y$ where $A_x, B_x \subseteq xR(G)$ and $A_y, B_y \subseteq yR(G)$. Then the modal multiplication can be described as follows:

$$A \cdot B = (A_x \cup A_y) \cdot (B_x \cup B_y)$$

$$= A_x \cdot B_x \cup A_x \cdot B_y \cup A_y \cdot B_x \cup A_y \cdot B_y$$

assuming additionally that

$$A_x \cdot \emptyset = A_y \cdot \emptyset = \emptyset \cdot B_x = \emptyset \cdot B_y = \emptyset.$$

Note that for $A = \{xy^{i_1}, \ldots, xy^{i_k}\}$, we have $t_A = xy^{i_1} + \cdots + xy^{i_k}$, and similarly for a finite subgroupoid of $yR(G)$. For $A = \{xy^i, xy^{i+1}, \ldots\} \cup \{yx^j, yx^{j+1}, \ldots\}$, we have $t_A = xy^i + yx^j$.

3. Distributive differential modals

A *distributive differential modal* is a differential modal such that also addition distributes over multiplication i.e., the modal that satisfies the identity:

$$x + yz = (x + y)(x + z). \tag{3.1}$$

The variety of distributive differential modals will be denoted by $\underline{\underline{\underline{dDM}}}$.

Lemma 3.1. *Each distributive differential modal satisfies the identities*

$$xy^k = xy \tag{3.2}$$

for each $k \geq 2$,

$$x + xy = xy. \tag{3.3}$$

$$x + y = xy + yx, \tag{3.4}$$

and

$$x + yx = x + y. \tag{3.5}$$

Proof. Substituting xy for x, x for y and y for z in (3.1) we obtain

$$xy = xy + xy = (xy + x)(xy + y) = x + xy + xy^2.$$

Hence

$$x + xy = x + xy + xy^2 = xy,$$

which implies

$$xy^2 = xy$$

and hence (3.2) holds. Now, from the identity

$$x + xy = xy$$

we obtain

$$x + yx = x + y + yx = x + y$$

and analogously

$$xy + y = x + xy + y = x + y.$$

Hence

$$xy + yx = x + xy + y + yx = x + y.$$ ■

Theorem 3.2. *The free distributive differential modal* $\{x, y\}dDM$ *is isomorphic to the algebra* $F := (\{x, xy, y, yx, x + y\}, \cdot, +)$, *where the groupoid operation* $\cdot$ *is defined in Figure* 2, *while the semilattice operation* $+$ *is defined in Figure* 3.

$(F, \cdot)$

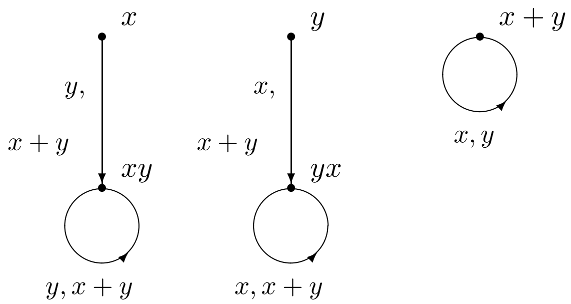

Figure 2

$(F, +)$

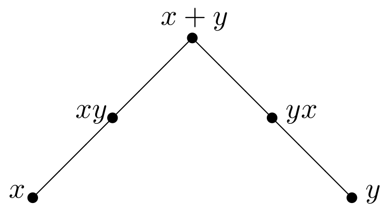

Figure 3

Proof. By Section 2, each element of $\{x, y\}DM$ is equal to

$$xy^{i_1} + \ldots + xy^{i_k}$$

or to

$$xy^i + yx^j.$$

By Lemma 3.1, the modal $\{x, y\}dDM$ satisfies the identities (3.2) for all positive integers k. Hence in $\{x, y\}dDM$, every term of the form $xy^{i_1} + \ldots + xy^{i_k}$ is equal to x or xy, and any term $xy^i + yx^j$ is equal to one of the following:

$$x + y,\ x + yx,\ xy + y,\ xy + yx.$$

However, by Lemma 3.1, $x + yx = y + xy = x + y$ and $xy + yx = x + y$. Therefore the set F can be considered as the set of elements of $\{x, y\}dDM$. The operations $+$ and $\cdot$ are defined as in the figures. Now it suffices to observe that the algebra F is distributive. We must show, that for any elements a, b, c of F

$$a + bc = (a + b)(a + c). \tag{3.6}$$

If a, b or c is equal to $x + y$, then (3.6) holds. If a and b are from different orbits, then both sides of (3.6) are $x + y$. Now assume, that both a and b are from the same orbit generated by x. (A similar proof will go for the orbit generated by y.) We have four possibilities:

(1) $a = b = x$,

(2) $a = x$ and $b = xy$,

(3) $a = xy$ and $b = x$,

(4) $a = b = xy$.

If c is from the same orbit we will have the equality $a + b = a + b$, so it suffices to consider only the case such that c is from the orbit generated by y. But then in each case both sides of (3.6) are equal to xy, so our algebra is distributive. ■

Our next aim is to describe the free distributive differential modal $\{x_0, \ldots, x_{n-1}\}dDM$ on n generators.

By Lemma 2.1, in the variety $\underline{\underline{\underline{DM}}}$ of differential modals every term with variables in $\{x_0, \ldots, x_{n-1}\}$ can be written as

$$x_{i_1} y_{11}^{\alpha_{11}} \ldots y_{1k_1}^{\alpha_{1k_1}} + \ldots + x_{i_l} y_{l1}^{\alpha_{l1}} \ldots y_{lk_l}^{\alpha_{lk_l}}, \tag{3.7}$$

where $x_{i_j},\ y_{ij} \in \{x_0, \ldots, x_{n-1}\}$ and $y_{ij} \neq y_{im}$ for $j \neq m$ and $x_{i_j} \neq y_{jm}$ for $m = 1, \ldots, k_j$. If a term t has the form as in (3.7), then we define the set t_f by

$$t_f := \bigcup_{j=1}^{l} \{x_{i_j}\},$$

the union of first variables of all summands. We will show that in the variety $\underline{\underline{\underline{dDM}}}$ each such term t equals to some term in a very special form.

Lemma 3.3. *If t and s are terms of the form* (3.7) *with variables in* $\{x_0, \ldots, x_{n-1}\}$, *then the variety* $\underline{\underline{\underline{dDM}}}$ *satisfies the identity $t = s$ if and only if $t_f = s_f$ and* $\texttt{arg}\, t = \texttt{arg}\, s$.

Proof. ($\Leftarrow$) Note that by Lemma 3.1, if t is of the form (3.7) then the variety $\underline{\underline{\underline{dDM}}}$ satisfies the identity

$$t = x_{i_1} y_{11} \ldots y_{1k_1} + \ldots x_{i_l} y_{l1} \ldots y_{lk_l}.$$

A similar identity holds for s. We will first show that the equalities $t_f = s_f = \{x\}$ and $\texttt{arg}\, t = \texttt{arg}\, s$, imply that $t = s$ holds in $\underline{\underline{dDM}}$. The proof is by induction on $k = |\texttt{arg}\, t| \leq n$. For $k = 2$ our claim follows by Theorem 3.2.

Assume that it holds for k and we will prove it for $k+1$. Let $|\texttt{arg}\, t| = k+1$. Let z be any variable in $\texttt{arg}\, t \setminus t_f$. Consider the following cases:

Case 1. Let

$$t = t_1 + t_2 z,\ \ s = s_1 + s_2 z$$

where t_1, t_2, s_1, s_2 are terms which do not contain the variable z. Then by distributivity

$$t = t_1 + t_2 z = (t_1 + t_2)(t_1 + z)$$

$$= (t_1 + t_2)t_1 + (t_1 + t_2)z = (t_1 + t_2) + (t_1 + t_2)z.$$

Similarly

$$s = (s_1 + s_2) + (s_1 + s_2)z.$$

Let us note that

$$(t_1 + t_2)_f = (s_1 + s_2)_f = \{x\},\ \ \texttt{arg}\,(t_1 + t_2) = \texttt{arg}\,(s_1 + s_2)$$

and

$$|\texttt{arg}\,(t_1 + t_2)| = k.$$

By induction hypothesis

$$t_1 + t_2 = s_1 + s_2$$

and thus

$$s = t.$$

Case 2. Let

$$t = t_1 + t_2 z,\ \ s = s_2 z,$$

where t_1, t_2, s_2 are terms which do not contain the variable z. By Lemma 3.1

$$s = s_2 z = s_2 + s_2 z$$

and we go back to the previous case.

Case 3. Let

$$t = t_2 z, \ \ s = s_2 z$$

and t_2, s_2 do not contain z. Then $\texttt{arg}\, t_2 = \texttt{arg}\, s_2$ and $|\texttt{arg}\, t_2| = k$. By the assumption $s_2 = t_2$ and hence $t = s$.

Now let

$$t_f = s_f = \{x_{i_1}, \dots, x_{i_l}\}.$$

The proof is again by induction on the number k of variables of t.

First assume that $k = l \leq n$ so that

$$t_f = s_f = \texttt{arg}\, t = \texttt{arg}\, s.$$

Then by Lemma 3.1, there exist $t_1, \dots, t_l, s_1, \dots, s_l$ such that

$$t = t_1 + \ldots + t_l, \ \ s = s_1 + \ldots + s_l,$$

where for all $j = 1, \dots, l$

$$(t_j)_f = (s_j)_f = \{x_{i_j}\}$$

and

$$\texttt{arg}\, t_j = \texttt{arg}\, s_j = \{x_{i_1}, \dots, x_{i_l}\}.$$

By the previous part of the proof $t_j = s_j$ for all $j = 1, \dots, l$ and this implies that $t = s$.

Now assume that our claim is true for $k > l$ and $|\texttt{arg}(t)| = k + 1$. Let z be any variable in $\texttt{arg}(t) \setminus t_f$. We consider similar cases as in the previous part of the proof.

Case 1. Let

$$t = t_1 + t_2 z, \ \ s = s_1 + s_2 z$$

where t_1, t_2, s_1, s_2 are terms which do not contain the variable z. Then by distributivity

$$t = t_1 + t_2 z = (t_1 + t_2)(t_1 + z)$$

$$= (t_1 + t_2)t_1 + (t_1 + t_2)z$$

and similarly

$$s = (s_1 + s_2)s_1 + (s_1 + s_2)z.$$

Let us note that

$$[(t_1 + t_2)t_1]_f = [t_1 + t_2]_f = [s_1 + s_2]_f = [(s_1 + s_2)s_1]_f$$

and

$$\texttt{arg}\,[(t_1 + t_2)t_1] = \texttt{arg}\,[(s_1 + s_2)s_1],$$

thus $t = s$.

Remaining two cases are proved analogously.

($\Rightarrow$) Assume that the variety $\underline{\underline{\underline{dDM}}}$ satisfies an identity $t = s$ where t and s are terms of the form (3.7). We will show that $t_f = s_f$ and $\texttt{arg}\, t = \texttt{arg}\, s$. Suppose on the contrary that $t_f \neq s_f$ and let z be a variable such that $z \in t_f$ but $z \notin s_f$. Then substituting y for z and x for any other variable in the identity $t = s$, we obtain

$$x = x + yx^l \text{ or } x = yx^l.$$

However, by Lemma 3.1

$$x = x + yx^l = x + yx = x + y$$

whence $x = y$. Similarly,

$$x = yx^l = yx$$

whence

$$y = y \cdot yx = y \cdot x = x.$$

It follows that the identity $t = s$ implies $x = y$ and this gives a contradiction. Thus $t_f = s_f$.

Now suppose that $\texttt{arg}\, t \neq \texttt{arg}\, s$. Let z be a variable such that $z \in \texttt{arg}\, t$ but $z \notin \texttt{arg}\, s$. Substituting y for z and x for any other variable in $t = s$, we obtain

$$x = xy^{i_1} + \ldots + xy^{i_l} = x + xy = xy.$$

But the variety $\underline{\underline{\underline{dDM}}}$ does not satisfy the left-zero identity $x = xy$. It follows that $\texttt{arg}\, t = \texttt{arg}\, s$. This finishes the proof. ■

Theorem 3.4. *In the free distributive differential modal* $\{x_0, \ldots, x_{n-1}\}$ *dDM each element can be written in the standard form*

$$x_{i_1}x_{j_1}\ldots x_{j_k} + \ldots + x_{i_l}x_{j_1}\ldots x_{j_k}$$

where $j_1 < j_2 < \ldots < j_k$ *and* $i_1 < i_2 < \ldots < i_l$ *and* $x_{i_r} \notin \{x_{j_1}, \ldots, x_{j_k}\}$. *If in such a term* t, $\mathtt{arg}\, t = t_f = \{x_{i_1}, \ldots, x_{i_l}\}$, *then*

$$t = x_{i_1} + \ldots + x_{i_l}.$$

Proof. The proof of Theorem 3.4 is a direct consequence of Lemma 3.3. If t is a term of the form (3.7) and $t_f = \{x_{i_1}, \ldots, x_{i_l}\}$ and $\mathtt{arg}\, t = t_f \cup \{x_{j_1}, \ldots, x_{j_k}\}$, then by Lemma 3.3, the variety $\underline{\underline{dDM}}$ satisfies the identity

$$t = x_{i_1}x_{j_1}\ldots x_{j_k} + \ldots + x_{i_l}x_{j_1}\ldots x_{j_k}.$$

If $t_f = \mathtt{arg}\, t$, it satisfies the identity

$$t = x_{i_1} + \ldots + x_{i_l}.$$

■

Theorem 3.4 allows to provide a representation for free differential modals.

Let $X = \{x_0, \ldots, x_{n-1}\}$ be a set of variables and let F_X be the set of pairs of disjoint subsets of X

$$F_X := \{(A, B) \mid \emptyset \neq A \subseteq X,\ B \subseteq X \setminus A\}.$$

We define the operations $+$ and $\cdot$ on F_X by

$$(A, B) + (C, D) := (A \cup C, (B \cup D) \setminus (A \cup C)),$$

$$(A, B) \cdot (C, D) := (A, (B \cup C) \setminus A).$$

It is easy to check that the set F_X is closed under both operations $+$ and $\cdot$.

Lemma 3.5. *The algebra* $(F_X, +, \cdot)$ *is a distributive differential modal.*

Proof. Let $(A, B), (C, D), (E, F), (G, H)$ be any elements of the set F_X. First we show that the reduct $(F_X, +)$ is a semilattice. Note that the operation $+$ is idempotent and commutative, so it suffices to check its associativity. We have

$$(A, B) + [(C, D) + (E, F)]$$

$$= (A \cup C \cup E, \{B \cup [(D \cup F) \setminus (C \cup E)]\} \setminus (A \cup C \cup E))$$

$$= ((A \cup C \cup E, (B \cup D \cup F) \setminus (A \cup C \cup E))$$

$$= (A \cup C \cup E, \{[(B \cup D) \setminus (A \cup C)] \cup F\} \setminus (A \cup C \cup E))$$

$$= [(A, B) + (C, D)] + (E, F).$$

Now we show that the second reduct $(F_X, \cdot)$ is a differential groupoid. As the following equalities hold,

$$[(A, B) \cdot (C, D)] \cdot [(E, F) \cdot (G, H)]$$

$$= (A, \{[(B \cup C) \setminus A] \cup E\} \setminus A) = (A, (B \cup C \cup E) \setminus A)$$

$$= (A, \{[(B \cup E) \setminus A] \cup C\} \setminus A)$$

$$= [(A, B) \cdot (E, F)] \cdot [(C, D) \cdot (G, H)],$$

our algebra satisfies the identity (1.2). Moreover we have

$$(A, B) \cdot [(A, B) \cdot (C, D)] = (A, B) \cdot (A, (B \cup C) \setminus A)$$

$$= (A, (B \cup A) \setminus A) = (A, B).$$

This means that our algebra satisfies also (1.3). Since the operation $\cdot$ is idempotent, $(F_X, \cdot)$ is a differential groupoid.

Our next aim is to show that $\cdot$ distributes over $+$. Note that

$$(A,B)\cdot[(C,D)+(E,F)] = (A,(B\cup C\cup E)\setminus A)$$

$$= (A,\{[(B\cup C)\setminus A]\cup[(B\cup E)\setminus A]\}\setminus A)$$

$$= (A,B)\cdot(C,D)+(A,B)\cdot(E,F).$$

Thus $(F_X,+,\cdot)$ is a differential modal.

Now it suffices to check that this modal is distributive. Since

$$(A,B)+(C,D)\cdot(E,F)$$

$$= (A\cup C,[B\cup((D\cup E)\setminus C)]\setminus(A\cup C)) = (A\cup C,(B\cup D\cup E)\setminus(A\cup C))$$

$$= (A\cup C,\{[(B\cup D)\setminus(A\cup C)]\cup(A\cup E)\}\setminus(A\cup C))$$

$$= [(A,B)+(C,D)]\cdot[(A,B)+(E,F)],$$

our claim is proved. ■

Theorem 3.6. *The algebra $(F_X,+,\cdot)$ is isomorphic to the free distributive differential modal $XdDM$ on the set X.*

Proof. Let $f : X \longrightarrow F_X$ be the mapping such that $xf = (\{x\},\emptyset)$. Thus there exists uniquely determined homomorphism $\bar{f} : XdDM \to F_X$ extending f such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "\iota"'] & (F_X,+,\cdot) \\
& (XdDM,+,\cdot) \arrow[u, "\exists!\bar{f}"']
\end{tikzcd}$$

Figure 4

We will identify elements of $XdDM$ with terms in standard form. Let t be a term in the standard form $x_{i_1}x_{j_1}\dots x_{j_k}+\dots+x_{i_l}x_{j_1}\dots x_{j_k}$. Then

$$t\bar{f}=(\{x_{i_1}\},\emptyset)(\{x_{j_1}\},\emptyset)\dots(\{x_{j_k}\},\emptyset)+\dots+(\{x_{i_l}\},\emptyset)(\{x_{j_1}\},\emptyset)\dots(\{x_{j_k}\},\emptyset)$$

$$=(\{x_{i_1}\},\{x_{j_1},\dots,x_{j_k}\})+\dots+(\{x_{i_l}\},\{x_{j_1},\dots,x_{j_k}\})$$

$$=(\{x_{i_1},\dots,x_{i_l}\},\{x_{j_1},\dots,x_{j_k}\}).$$

Clearly $\bar{f}$ is injective.

Now let $(A,B)\in F_X$. There exists a standard term t such that $t_f=A$ and $\mathtt{arg}\,t=A\cup B$. Note that $(A,B)=t\bar{f}$. Hence $\bar{f}$ is "onto" and thus it is an isomorphism. ■

Now let us consider entropic differential modals, i.e., modals which satisfy the entropic law:

$$(x+y)(z+w)=xz+yw. \tag{3.8}$$

Theorem 3.7. *The varieties $\underline{\underline{eDM}}$ of entropic differential modals and $\underline{\underline{dDM}}$ of distributive differential modals coincide.*

Proof. Since in the variety of differential modals the entropic identity (1.2) implies distributivity (3.1) of $+$ over $\cdot$, it follows that the variety $\underline{\underline{eDM}}$ is included in the variety $\underline{\underline{dDM}}$.

Let $t=(x+y)(z+w)$ and $s=xz+yw$. Note that $t_f=s_f=\{x,y\}$ and $\mathtt{arg}\,t=\mathtt{arg}\,s=\{x,y,z,w\}$. By Lemma 3.3, $t=s$ holds in $\underline{\underline{dDM}}$. Hence the variety $\underline{\underline{dDM}}$ satisfies the entropic law (3.8). This finishes the proof. ■

Let $\underline{\underline{LzDM}}$ denote the variety of differential modals satisfying the left-zero identity

$$xy=x,$$

and let $\underline{\underline{T}}$ denote its trivial subvariety.

Theorem 3.8. *The lattice of subvarieties of $\underline{\underline{dDM}}$ forms the chain*

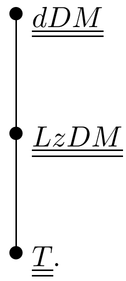

Figure 5

Proof. By Theorem 1.5, the variety $\underline{\underline{dDM}} = \underline{\underline{eDM}}$ is generated by the single algebra $\{x, y\}eDM$. By Theorem 1.6 the lattice of subvarieties of this variety is dually isomorphic to the congruence lattice of the semiring $(\{y, yx, x+y\}, \circ, +)$ with multiplication defined by the table

$\circ$	y	yx	$x+y$
y	y	y	y
yx	y	y	yx
$x+y$	y	yx	$x+y$

and addition given by the (join) semilattice of Figure 6.

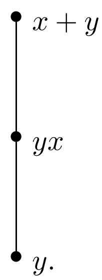

Figure 6

This semiring has precisely three congruences: the universal congruence $\nabla = \mathbb{R}^2$, the equality relation Δ, and the congruence θ with precisely two congruence classes

$$[y]_\theta = \{y, yx\} \text{ and } [x+y]_\theta = \{x+y\}.$$

Hence its congruence lattice forms the chain in Figure 7.

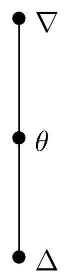

Figure 7

This finishes the proof. ■

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