Discussiones Mathematicae
General Algebra and Applications 27 (2007) 263–275

PRIME IDEAL THEOREM FOR DOUBLE BOOLEAN ALGEBRAS

Léonard Kwuida

Universität Bern Mathematisches Institut
Sidlerstr. 5 CH–3012 Bern
e-mail: kwuida@math.unibe.ch

To the memory of Professor Kazimierz Głazek

Abstract

Double Boolean algebras are algebras $(D, \sqcap, \sqcup, ^{\lhd}, ^{\rhd}, \bot, \top)$ of type $(2, 2, 1, 1, 0, 0)$. They have been introduced to capture the equational theory of the algebra of protoconcepts. A filter (resp. an ideal) of a double Boolean algebra D is an upper set F (resp. down set I) closed under $\sqcap$ (resp. $\sqcup$). A filter F is called primary if $F \neq \emptyset$ and for all $x \in D$ we have $x \in F$ or $x^{\lhd} \in F$. In this note we prove that if F is a filter and I an ideal such that $F \cap I = \emptyset$ then there is a primary filter G containing F such that $G \cap I = \emptyset$ (i.e. the Prime Ideal Theorem for double Boolean algebras).

Keywords: double Boolean algebra, protoconcept algebra, concept algebra, weakly dicomplemented lattices.

2000 Mathematics Subject Classification: Primary 06E05; 02J05.

1. Introduction and motivation

1.1. Motivation

Formal Concept Analysis emerged in the early eighties from an attempt to restructure lattice theory by Rudolf Wille. To extend Formal Concept Analysis to a quite large field called Contextual Logic, a negation has to be formalized. Many propositions have been made and investigated [6].

To keep a correspondence between negation and set complementation the notion of concept as formalized in [5] has been generalized successively to the notions of semiconcept, protoconcept and preconcept. To capture their equational theory double Boolean algebras have been introduced by Rudolf Wille and coworkers. Each double Boolean algebra D contains two Boolean algebras: $D_\sqcap$ and $D_\sqcup$. To construct a suitable context $\mathbb{K}(D)$ such that D could be embedded into its algebra of protoconcepts, they used as objects filters F of D whose intersections with $D_\sqcap$ are prime filters and as attributes ideals I of D whose intersections with $D_\sqcup$ are prime ideals, and proved that D can be quasi-embedded into the algebra of protoconcepts of that context. For Boolean algebras, it is well known that prime filters can be defined as filters F satisfying

$$(\ddagger) \quad x \in F \quad \text{or} \quad x^* \in F \quad \text{for all } x,$$

where * denotes the complementation. With a similar definition we got the "prime ideal theorem" for weakly dicomplemented lattices (introduced to capture the equational theory of concept algebras, see [4]). In the search of a common definition for such filters the author was asking himself whether the prime ideal theorem can be proved for double Boolean algebras using (‡) as definition. The answer is yes, and will be presented in this note. Before that we present the algebra of protoconcepts.

1.2 Formal concepts and negation

The starting point of Formal Concept Analysis is a formal context. A *Formal context* is a triple (G, M, I) with $I \subseteq G \times M$. G is called the set of *objects* and M the set of *attributes*. The derivation operation is defined on subsets $A \subseteq G$ and $B \subseteq M$ by:

$$A' := \{m \in M \mid \ \forall g \in A \quad gIm\}$$

and

$$B' := \{g \in G \mid \ \forall m \in B \quad gIm\}.$$

The maps $A \mapsto A'$ and $B \mapsto B'$ define a Galois connection between the powerset of G and that of M. A *Formal concept* is then a pair (A, B) with $A' = B$ and $B' = A$. A is called the *extent* and B the *intent* of the concept (A, B). $\mathfrak{B}(G, M, I)$ denotes the set of all concepts of the context (G, M, I).

The *concept hierarchy* is captured by the order relation

$$(A, B) \leq (C, D) :\iff A \subseteq C \qquad (\iff D \subseteq B).$$

$\underline{\mathfrak{B}}(G, M, I)$ denotes the poset $(\mathfrak{B}(G, M, I), \leq)$. The basic theorem on concept lattices (Theorem 3 [3]) states that

> $\underline{\mathfrak{B}}(G, M, I)$ is a complete lattice (called the *concept lattice* of the context (G, M, I)) and conversely, each complete lattice is isomorphic to a concept lattice of a suitable context.

The meet and the join operations of a concept lattice will encode the conjunction and the disjunction of concepts respectively while the top and bottom element will encode the tautology and the contradiction. What about negation?

One approach (see [6]) is to use a *weak negation* $^\triangle$ and a *weak opposition* $^\triangledown$ defined by:

$$(A, B)^\triangle := ((G \setminus A)'', (G \setminus A)')$$

and

$$(A, B)^\triangledown := ((M \setminus B)', (M \setminus B)'').$$

A concept lattice equipped with these two operations is called a concept algebra. The following equations* hold in all concept algebras:

(1) $x^{\triangle\triangle} \leq x$,	(1') $x^{\triangledown\triangledown} \geq x$,
(2) $x \leq y \implies x^\triangle \geq y^\triangle$,	(2') $x \leq y \implies x^\triangledown \geq y^\triangledown$,
(3) $(x \wedge y) \vee (x \wedge y^\triangle) = x$,	(3') $(x \vee y) \wedge (x \vee y^\triangledown) = x$.

Weakly dicomplemented lattices are algebras $(L, \wedge, \vee, ^\triangle, ^\triangledown, 0, 1)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded lattice and the identities (1)–(3') hold. Finite distributive weakly dicomplemented lattices are copies of concept algebras (Theorem 4.1.7 and Corollary 4.1.8 [4]). Then the class of finite distributive concept algebras forms a pseudovariety. Until now no complete set of equations is known to generate the equational theory of concept algebras.

*Note that (2) is equivalent to $(x \wedge y)^\triangle \wedge x^\triangle = x^\triangle$ and (2') is equivalent $(x \vee y)^\triangledown \vee x^\triangledown = x^\triangledown$.

Even for finite concept algebras, it is not known whether they form a pseudovariety. Note that in this approach, the correspondence between negation and set complementation cannot be preserved. In order to keep such a correspondence, the notion of "concept" has been successively generalized to that of semiconcept, protoconcept and preconcept.

2. Algebras of protoconcepts

Let (G, M, I) be a formal context. A *preconcept* is a pair (A, B) with $A \subseteq G$ and $B \subseteq M$ such that $A \subseteq B'$ (equivalent to $B \subseteq A'$). A *protoconcept* is a pair (A, B) with $A \subseteq G$, $B \subseteq M$ and $A'' = B'$ (equivalent to $B' = A''$). The set of all protoconcepts of the context $\mathbb{K} := (G, M, I)$ is denoted by $\mathfrak{P}(\mathbb{K})$. Logical operations are defined on protoconcepts as follows:

$$\begin{aligned}
\textit{meet}: &\quad (A_1, B_1) \sqcap (A_2, B_2) := (A_1 \cap A_2, (A_1 \cap A_2)') \\
\textit{join}: &\quad (A_1, B_1) \sqcup (A_2, B_2) := ((B_1 \cap B_2)', B_1 \cap B_2) \\
\textit{negation}: &\quad (A, B)^{\lhd} := (G \setminus A, (G \setminus A)') \\
\textit{opposition}: &\quad (A, B)^{\rhd} := ((M \setminus B)', M \setminus B) \\
\textit{nothing}: &\quad \bot := (\emptyset, M) \\
\textit{all}: &\quad \top := (G, \emptyset)
\end{aligned}$$

With these operations is defined the algebra $\underline{\mathfrak{P}}(\mathbb{K}) := (\mathfrak{P}(\mathbb{K}), \sqcap, \sqcup, ^{\lhd}, ^{\rhd}, \bot, \top)$ called the *algebra of protoconcepts* of $\mathbb{K}$. We set

$$\mathfrak{P}(\mathbb{K})_{\sqcap} := \{(A, A') \mid A \subseteq G\}$$

and

$$\mathfrak{P}(\mathbb{K})_{\sqcup} := \{(B', B) \mid B \subseteq M\}.$$

$\mathfrak{P}(\mathbb{K})_{\sqcap}$ and $\mathfrak{P}(\mathbb{K})_{\sqcap}$ are special subalgebras of the protoconcept algebra called respectively $\sqcap$-*semiconcept algebra* and $\sqcup$-*semiconcept algebra*. Their intersection gives the concept lattice of $\mathbb{K}$. Their union denoted by $\mathfrak{H}(\mathbb{K})$ is also a subalgebra of $\mathfrak{P}(\mathbb{K})$, called the *semiconcept algebra*. Further operations are defined on protoconcepts as follows:

$$x \oplus y := (x^{\lhd} \sqcap y^{\lhd})^{\lhd}, \quad x \odot y := (x^{\rhd} \sqcup y^{\rhd})^{\rhd}, \quad \mathtt{1} := \bot^{\lhd} \text{ and } \mathtt{0} := \top^{\rhd}.$$

The algebra $\underline{\mathfrak{P}}(\mathbb{K})_{\sqcap} := (\mathfrak{P}(\mathbb{K})_{\sqcap}, \sqcap, \oplus, ^{\lhd}, \bot, \mathtt{1})$ is a Boolean algebra isomorphic to the powerset algebra of G and the algebra $\underline{\mathfrak{P}}(\mathbb{K})_{\sqcup} := (\mathfrak{P}(\mathbb{K})_{\sqcup}, \odot, \sqcup, ^{\rhd}, \mathtt{0}, \top)$ a Boolean algebra anti-isomorphic to the powerset algebra of M.

Rudolf Wille proved (see for example [6]) that the following equations hold in the algebra of protoconcepts:

(1) $x \sqcap y = y \sqcap x$, (1') $x \sqcup y = y \sqcup x$

(2) $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$, (2') $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$

(3) $x \sqcap (x \sqcup y) = x \sqcap x$, (3') $x \sqcup (x \sqcap y) = x \sqcup x$

(4) $x \sqcap (x \oplus y) = x \sqcap x$, (4') $x \sqcup (x \odot y) = x \sqcup x$

(5) $(x \sqcap x) \sqcap y = x \sqcap y$ (5') $(x \sqcup x) \sqcup y = x \sqcup y$

(6) $x \sqcap (y \oplus z) = (x \sqcap y) \oplus (x \sqcap z)$ (6') $x \sqcup (y \odot z) = (x \sqcup y) \odot (x \sqcup z)$

(7) $(x \sqcap y)^{\lhd\lhd} = x \sqcap y$ (7') $(x \sqcup y)^{\rhd\rhd} = x \sqcup y$

(8) $(x \sqcap x)^{\lhd} = x^{\lhd}$ (8') $(x \sqcup x)^{\rhd} = x^{\rhd}$

(9) $x \sqcap x^{\lhd} = \bot$ (9') $x \sqcup x^{\rhd} = \top$

(10) $\bot^{\lhd} = \top \sqcap \top$ (10') $\top^{\rhd} = \bot \sqcup \bot$

(11) $\top^{\lhd} = \bot$ (11') $\bot^{\rhd} = \top$

(12) $(x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$.

Protoconcepts can be ordered by the relation $\leq$ defined by:

$$(A, B) \leq (C, D) :\iff A \subseteq C \text{ and } B \supseteq D.$$

Remark 2.1. Let (A, B) and (C, D) be protoconcepts of (G, M, I) such that $(A, B) \leq (C, D)$. We have:

(i) $(A,B) \sqcap (C,D) = (A,B) \sqcap (A,B)$ and $(A,B) \sqcup (C,D) = (C,D) \sqcup (C,D)$.

(ii) $(C,D)^{\triangleleft} = (G \setminus C, (G \setminus C)') \leq (G \setminus A, (G \setminus A)') = (A,B)^{\triangleleft}$ and

(iii) $(C,D)^{\triangleright} = ((M \setminus D)', M \setminus D) \leq ((M \setminus A)', M \setminus B) = (A,B)^{\triangleright}$.

We write $\sqsubseteq$ to mean that the equalities in Remark 2.1 (i) hold. i.e. For protoconcepts x and y, we have

$$x \sqsubseteq y \iff x \sqcap y = x \sqcap x \text{ and } x \sqcup y = y \sqcup y.$$

The relation $\sqsubseteq$ is a quasi-order that is by Remark 2.1 (i) an extension of the above defined order relation $\leq$. The equivalence relation $\sim$ induced by the quasi-order $\sqsubseteq$ (i.e. $x \sim y :\iff x \sqsubseteq y$ and $y \sqsubseteq x$) satisfies

$$x \sim y \iff x \sqcap x = y \sqcap y \text{ and } x \sqcup x = y \sqcup y.$$

Moreover, concepts are protoconcepts x such that $x \sqcap x = x$ and $x \sqcup x = x$. This equivalence partitions the protoconcepts in such a way that each equivalence class contains at most one concept.

Lemma 2.1. *In the algebra of protoconcepts the following formulae hold:*

$$(13) \quad x \sqcap x \leq (x \sqcap y) \sqcup (x \sqcap y^{\triangleleft}) \quad \textit{and} \quad (13') \quad x \sqcup x \geq (x \sqcup y) \sqcap (x \sqcup y^{\triangleright}).$$

Proof. We set $x := (A,B)$ and $y := (C,D)$. Then we have $y^{\triangleleft} = (G \setminus C, (G \setminus C)')$ and $x \sqcap x = (A, A')$. Therefore

$$\begin{aligned}(x \sqcap y) \sqcup (x \sqcap y^{\triangleleft}) &= (A \cap C, (A \cap C)') \sqcup \big(A \cap (G \setminus C), (A \cap (G \setminus C))'\big)\\ &= \Big(\big((A \cap C)' \cap (A \cap (G \setminus C))'\big)', ((A \cap C)' \cap (A \cap (G \setminus C)')\Big)\\ &= \big((A \cap C) \cup (A \cap (G \setminus C))'', ((A \cap C)' \cap (A \cap (G \setminus C)')\big)\\ &= (A'', A') \geq (A, A') = x \sqcap x.\end{aligned}$$

The rest of the statement is proved dually. ■

It would be interesting to investigate which relationship does exist between the operations $\sqcap$, $\sqcup$ and the order relation $\leq$. To capture the equational theory of protoconcept algebras Rudolf Wille introduced double Boolean algebras.

3. Prime ideal theorem for double Boolean algebras

In this note we call an algebra $(D, \sqcap, \sqcup, ^{\lhd}, ^{\rhd}, \bot, \top)$ of type $(2,2,1,1,0,0)$ that satisfies (1) to (13) and (1') to (13') a *double Boolean algebra*[†]. A double Boolean algebra is called *pure* if it satisfies

(14) $x \sqcap x = x$ or $x \sqcup x = x$.

Note that (14) holds in the algebra of semiconcepts. The following notations are adopted:

$$x_\sqcap := x \sqcap x, \quad D_\sqcap := \{x_\sqcap \mid x \in D\}$$

and

$$x_\sqcup := x \sqcup x, \quad D_\sqcup := \{x_\sqcup \mid x \in D\}.$$

The algebras $\underline{D}_\sqcap := (D_\sqcap, \sqcap, \oplus, ^{\lhd}, \bot, \mathtt{1})$ and $\underline{D}_\sqcup := (D_\sqcup, \odot, \sqcup, ^{\rhd}, \mathtt{0}, \top)$ are Boolean algebras, where $x \oplus y := (x^{\lhd} \sqcap y^{\lhd})^{\lhd}$, $x \odot y := (x^{\rhd} \sqcup y^{\rhd})^{\rhd}$, $\mathtt{1} := \bot^{\lhd}$ and $\mathtt{0} := \top^{\rhd}$ as introduced before on protoconcepts.

Now, how can we capture the order relation on the protoconcept algebra for double Boolean algebras? The relation $\sqsubseteq$ defined on D by

$$x \sqsubseteq y :\iff x \sqcap y = x \sqcap x \quad \text{and} \quad x \sqcup y = y \sqcup y$$

is a quasi-order. For x and y in D, we have

$$x \sqsubseteq y \iff x_\sqcap \sqcap y_\sqcap = x \sqcap x \sqcap y \sqcap y = x \sqcap x = x_\sqcap \quad \text{and} \quad x_\sqcup \sqcup y_\sqcup = y_\sqcup.$$

As $\sqcap$ (resp. $\sqcup$) is the meet (resp. join) operation in the Boolean algebra $D_\sqcap$ (resp. $D_\sqcup$) we get

$$x \sqsubseteq y \iff x_\sqcap \leq y_\sqcap \quad \text{and} \quad x_\sqcup \leq y_\sqcup,$$

where $\leq$ is the induced order in the corresponding Boolean algebra.

[†]In [6] the formulae (13) and (13') were not considered.

A double Boolean algebra is called *regular* if the relation $\sqsubseteq$ is an order relation. Maybe abstracting the protoconcept algebras by relational structures $(D, \sqcap, \sqcup, ^{\lhd}, ^{\rhd}, \leq, \top, \bot)$ such that $\leq$ is explicit defined on $D_\sqcap$ and $D_\sqcup$ will shed another light on them.

Lemma 3.1. *For a double Boolean algebra D and $x, y, a \in D$ we have:*

(i) $x \sqcap y \sqsubseteq x, y \sqsubseteq x \sqcup y$,

(ii) $x \sqsubseteq y$ *implies* $x \sqcap a \sqsubseteq y \sqcap a$ *and* $x \sqcup a \sqsubseteq y \sqcup a$.

Proof. For (i) we have

$$\left.\begin{aligned} x \sqcap (x \sqcup y) &= x \sqcap x \\ x \sqcup (x \sqcup y) &= (x \sqcup y) \sqcup (x \sqcup y) \end{aligned}\right\} \Longrightarrow x \sqsubseteq x \sqcup y$$

and

$$\left.\begin{aligned} x \sqcup (x \sqcap y) &= x \sqcup x \\ x \sqcap (x \sqcap y) &= (x \sqcap y) \sqcap (x \sqcap y) \end{aligned}\right\} \Longrightarrow x \sqsupseteq x \sqcap y.$$

For (ii), let $x \sqsubseteq y$. We have $x \sqcap y = x \sqcap x$ and $x \sqcup y = y \sqcup y$.

$$(x \sqcap a) \sqcap (y \sqcap a) = (x \sqcap y) \sqcap a = (x \sqcap x) \sqcap a = (x \sqcap a) \sqcap (x \sqcap a)$$

and

$$(x \sqcap a) \sqcup (y \sqcap a) = (x \sqcap x \sqcap a) \sqcup (y \sqcap a)$$

$$= (x \sqcap y \sqcap a) \sqcup (y \sqcap a) = (y \sqcap a) \sqcup (y \sqcap a)$$

since by (i), it holds $x \sqcap y \sqcap a \sqsubseteq y \sqcap a$. The remaining assertion is proved similarly. ∎

We can deduce that $(a \sqcap x) \sqcup (b \sqcap x) \sqsubseteq (a \sqcup b) \sqcap x$. We will call a double Boolean algebra *distributive* if the equalities

$$(a \sqcap x) \sqcup (b \sqcap x) = (a \sqcup b) \sqcap x \text{ and } (a \sqcup x) \sqcap (b \sqcup x) = (a \sqcap b) \sqcup x$$

hold.

Remark 3.1. If a protoconcept algebra is distributive, then $x \sqcap x$ and $x \sqcup x$ are all concepts for all x.

Are protoconcept algebras distributive and regular double Boolean algebras?

Definition 3.1. Let D be a double Boolean algebra. A nonempty subset F of D is called a *filter* if it satisfies

$$x, y \in F \implies x \sqcap y \in F \quad \text{and} \quad x \in F,\ y \in D,\ x \sqsubseteq y \implies y \in F.$$

Dually an *ideal* of D is a nonempty subset I of D satisfying

$$x, y \in I \implies x \sqcup y \in I \quad \text{and} \quad x \in I,\ y \in D,\ x \sqsupseteq y \implies y \in I.$$

If $(F_k)_{k \in K}$ is a family of filter of a double Boolean algebra D then $F := \bigcap\{F_k \mid k \in K\}$ is nonempty since all F_k contain $\top$. Moreover

$$x, y \in F \implies x, y \in F_k\ \forall_{k \in K} \implies x \sqcap y \in F_k\ \forall_{k \in K} \implies x \sqcap y \in F$$

and

$$x \in F, y \in D, x \sqsubseteq y \implies y \in D,\ x \sqsubseteq y, x \in F_k \forall_{k \in K} \implies y \in F_k \forall_{k \in K} \implies y \in F.$$

Therefore the set of all filters (resp. ideals) of D form a closure system. We denote by $\mathrm{Filter}\langle X\rangle$ (resp. $\mathrm{Ideal}\langle X\rangle$) the filter (resp. ideal) generated by X. For example the principal filter (resp. ideal) generated by x is:

$$\mathrm{Filter}\langle\{a\}\rangle = \{x \in D \mid a \sqcap a \sqsubseteq x\}$$

$$(\text{resp. } \mathrm{Ideal}\langle\{a\}\rangle = \{x \in D \mid a \sqcup a \sqsupseteq x\}).$$

Lemma 3.2. *Let F be a filter and I an ideal of D. For an element $w \in D$ we have:*

$$\mathrm{Filter}\langle F \cup \{w\}\rangle = \{x \in D \mid v \sqcap w \sqsubseteq x \textit{ for some } v \in F\}$$

and

$$\mathrm{Ideal}\langle I \cup \{w\}\rangle = \{x \in D \mid v \sqcup w \sqsupseteq x \textit{ for some } v \in I\}.$$

Proof. We are going to prove that

$$H := \{x \in D \mid v \sqcap w \sqsubseteq x \text{ for some } v \in F\}$$

is the smallest filter containing $F \cup \{w\}$. Note that $\top \in F$ and $w \sqcap x \sqsubseteq x, w$. Thus H contains $F \cup \{w\}$. For $x \in H$ and $y \in D$ with $x \sqsubseteq y$, there is $v \in F$ such that $v \sqcap w \sqsubseteq x$, and by then $v \sqcap w \sqsubseteq y$. Thus $y \in H$. Now, let x and y in H. There are a and b in F such that $a \sqcap w \sqsubseteq x$ and $b \sqcap w \sqsubseteq y$. By Lemma 3.1 we get $a \sqcap b \sqcap w = (a \sqcap w) \sqcap (b \sqcap w) \sqsubseteq x \sqcap y$ with $a \sqcap b \in F$. Thus $x \sqcap y$ is in H. This proves that H is a filter. If G is another filter containing $F \cup \{w\}$, then G contains H. Thus $H = \mathrm{Filter}\langle F \cup \{w\}\rangle$. ∎

Definition 3.2. Let D be a double Boolean algebra. A filter F is called proper if $F \neq D$, and *primary* if it is proper and satisfies $x \in F$ or $x^{\lhd} \in F$ for all $x \in D$. Dually are defined *primary ideals.* $\mathcal{F}_{pr}(D)$ denotes the set of primary filters and $\mathcal{I}_{pr}(D)$ the set of primary ideals of D.

Theorem 3.3 (Prime ideal theorem). *Let D be a double Boolean algebra, F a filter and I an ideal such that $F \cap I = \emptyset$. There exists a primary filter G and a primary ideal J with $F \subseteq G$, $I \subseteq J$ and $G \cap J = \emptyset$.*

Proof. We set

$$\mathcal{F}_I := \{H \text{ filter } \mid H \cap I = \emptyset \text{ and } F \subseteq H\}.$$

$\mathcal{F}_I$ contains F. The poset $(\mathcal{F}_I, \subseteq)$ satisfies the conditions of the Zorn's lemma. Therefore $(\mathcal{F}_I, \subseteq)$ has maximal elements. Let G be maximal in $(\mathcal{F}_I, \subseteq)$. We claim that G is a primary filter. Otherwise there would exist an element $w \in D$ such that $w \notin G$ and $w^{\lhd} \notin G$. In this case, G would be a proper subset of Filter$\langle G \cup \{w\}\rangle$ and of Filter$\langle G \cup \{w^{\lhd}\}\rangle$. From the maximality of G in $(\mathcal{F}_I, \subseteq)$, we would have

$$\text{Filter}\langle G \cup \{w\}\rangle \cap I \neq \emptyset \neq \text{Filter}\langle G \cup \{w^{\lhd}\}\rangle.$$

Thus there would be elements $a, b \in I$ and $v_1, v_2 \in G$ such that $v_1 \sqcap w \sqsubseteq a$ and $v_2 \sqcap w^{\lhd} \sqsubseteq b$. It would follow that

$$v_1 \sqcap v_2 \sqcap w \sqsubseteq a \sqcap v_2 \sqsubseteq a \quad \text{and} \quad v_1 \sqcap v_2 \sqcap w^{\lhd} \sqsubseteq b \sqcap v_1 \sqsubseteq b.$$

Thus

$$(v_1 \sqcap v_2) \sqcap (v_1 \sqcap v_2) \sqsubseteq (v_1 \sqcap v_2 \sqcap w) \sqcup (v_1 \sqcap v_2 \sqcap w^{\lhd}) \sqsubseteq a \sqcup b \quad \text{by (13)}.$$

This would lead to $G \ni v_1 \sqcap v_2 \sqsubseteq a \sqcup b \in I$ which would be a contradiction with $G \cap I = \emptyset$. Thus G is a primary filter. The existence of J is proved similarly using the family

$$\mathcal{I}_I := \{S \text{ ideal } \mid G \cap S = \emptyset \text{ and } I \subseteq S\}.$$

■

Corollary 3.4. *For $x \sqcap x \not\sqsubseteq y \sqcup y$ in D there is a primary filter G with $x \in G$ and $y \notin G$.*

Proof. If $x \sqcap x \not\sqsubseteq y \sqcup y$ then Filter$\langle\{x\}\rangle \cap$ Ideal$\langle\{y\}\rangle = \emptyset$. By the prime ideal theorem, there is a primary filter F containing x and a primary ideal I containing y such that $F \cap I = \emptyset$. ■

How can we separate $x \sqcup x$ and $x \sqcap x$ from x?

Remark 3.5.

(i) $x \sqcap \bot = x \sqcap (x \sqcap x^{\lhd}) = (x \sqcap x) \sqcap x^{\lhd} = x \sqcap x^{\lhd} = \bot$. Dually $x \sqcup \top = \top$.

(ii) $x \sqcap \top = x \sqcap (x \sqcup x^{\rhd}) = x \sqcap x$. Dually $x \sqcup \bot = x \sqcup x$.

(iii) In the context (G, M, I) we have $\top \sqcap \top = \bot \sqcup \bot \iff I = G \times M$. Such a context has exactly one concept. Its algebras of preconcepts, of protoconcepts and of semi-concepts are identical and is the vertical sum of two Boolean algebras: $\mathfrak{P}(G, M, I)_{\sqcap} \oplus \mathfrak{P}(G, M, I)_{\sqcup}$.

Definition 3.3. We call a double Boolean algebra *trivial* iff $\top \sqcap \top = \bot \sqcup \bot$.

The three element chain $\{\bot, a, \top\}$ with $\bot \leq a \leq \top$, $\bot \sqcup \bot = a = \top \sqcap \top$ and $a \sqcap a = a = a \sqcup a$ is a trivial double Boolean algebra. The following result ensures the existence of primary filters and primary ideals.

Corollary 3.6. *Each nontrivial double Boolean algebra has primary filters and primary ideals.*

Proof. It is enough to prove that if $\top \sqcap \top \neq \bot \sqcup \bot$ then $\top \sqcap \top \not\sqsubseteq \bot \sqcup \bot$. In fact, $\top \sqcap \top \sqsubseteq \bot \sqcup \bot$ implies $(\top \sqcap \top) \sqcap (\bot \sqcup \bot) = \top \sqcap \top$ and $(\top \sqcap \top) \sqcup (\bot \sqcup \bot) = \bot \sqcup \bot$. Therefore $\top \sqcap \top = \top \sqcap (\bot \sqcup \bot) = (\bot \sqcup \bot) \sqcap (\bot \sqcup \bot) = (\bot \sqcap \bot) \sqcup (\bot \sqcap \bot) = \bot \sqcup \bot$. ■

4. Conclusion

This work in progress should be considered as the author's reading note of [2] and [6] by Wille and coworkers. Of course the prime ideal theorem (Theorem 3.3) is a new an important result. The next step would be to look after some applications (for example a concrete representation or a duality theorem for double Boolean algebras). This will be carried out in future works.

References

[1] G. Boole, An investigation into the Laws of Thought on which are founded the Mathematical Theories of Logic and Probabilities, Macmillan 1854, reprinted by Dover Publ. New York 1958.

[2] C. Herrmann, P. Luksch, M. Skorsky and R. Wille, *Algebras of semiconcepts and double Boolean algebras*, J. Heyn Klagenfurt, Contributions to General Algebra **13** (2001), 175–188.

[3] B. Ganter and R. Wille, Formal Concept Analysis. Mathematical Foundations, Springer 1999.

[4] L. Kwuida, Dicomplemented Lattices. A Contextual Generalization of Boolean Algebras, Shaker Verlag 2004.

[5] R. Wille, *Restructuring lattice theory: an approach based on hierarchies of concepts*, in: I. Rival (Ed.) Ordered Sets Reidel (1982), 445–470.

[6] R. Wille, *Boolean Concept Logic*, LNAI **1867** Springer (2000), 317–331.

[7] R. Wille, *Boolean Judgement Logic*, LNAI **2120** Springer (2001), 115–128.

[8] R. Wille, *Preconcept algebras and generalized double Boolean algebras*, LNAI **2961** Springer (2004), 1–13.

Received 24 July 2006
Revised 13 December 2006