Discussiones Mathematicae
General Algebra and Applications 27 (2007) 245–262

Nd-SOLID VARIETIES

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To the memory of Professor Kazimierz Głazek

Abstract

A non-deterministic hypersubstitution maps any operation symbol of a tree language of type τ to a set of trees of the same type, i.e. to a tree language. Non-deterministic hypersubstitutions can be extended to mappings which map tree languages to tree languages preserving the arities. We define the application of a non-deterministic hypersubstitution to an algebra of type τ and obtain a class of derived algebras. Non-deterministic hypersubstitutions can also be applied to equations of type τ. Formally, we obtain two closure operators which turn out to form a conjugate pair of completely additive closure operators. This allows us to use the theory of conjugate pairs of additive closure operators for a characterization of M-solid non-deterministic varieties of algebras. As an application we consider M-solid non-deterministic varieties of semigroups.

Keywords: Non-deterministic hypersubstitution, conjugate pair of additive closure operators, M-solid non-deterministic variety.

2000 Mathematics Subject Classification: 08A35, 08A40, 08A70.

1. Introduction

Let $(f_i)_{i\in I}$ be an indexed set of operation symbols where f_i is n_i-ary, let $X := \{x_1,\ldots,x_n,\ldots\}$ be a countably infinite set of variables and for each $n \geq 1$ let $X_n := \{x_1,\ldots,x_n\}$ be a finite set of variables. We denote by $W_\tau(X)$ and $W_\tau(X_n)$, respectively the sets of all terms of a finite type $\tau = (n_i)_{i\in I}$ and of all n-ary terms of type τ. We use the well-known Galois connection Id-Mod between sets of identities and classes of algebras of a given type. For any set Σ of identities we denote by ModΣ the model class of all algebras of type τ which satisfy all identities of Σ; and for any class K of algebras of the same type we denote by IdK the set of all identities satisfied by all algebras in K. Classes of the form ModΣ are called varieties of algebras of type τ. If $\mathcal{A}$ satisfies the equation $s \approx t$ as an identity, we write $\mathcal{A} \models s \approx t$ and if the class K of algebras of type τ satisfies $s \approx t$, we write $K \models s \approx t$. If $\Sigma \subseteq W_\tau(X)^2$ is a set of equations, then $K \models \Sigma$ means that every equation from Σ is satisfied by every algebra from K. Any subset of $W_\tau(X)$, i.e. any element of the power set $\mathcal{P}(W_\tau(X))$ or of $\mathcal{P}(W_\tau(X_n))$ is called a tree language. Our restriction to a finite type is motivated by applications of tree languages in computer science. For tree languages one may define the following superposition operations

$$\hat{S}^n_m : \mathcal{P}(W_\tau(X_n)) \times \mathcal{P}(W_\tau(X_m))^n \to \mathcal{P}(W_\tau(X_m))$$

inductively by the following steps:

Definition 1.1. Let $m, n \in \mathbb{N}^+(:= \mathbb{N} \setminus \{0\})$ and let $B \in \mathcal{P}(W_\tau(X_n))$ and $B_1,\ldots,B_n \in \mathcal{P}(W_\tau(X_m))$ such that $B, B_1,\ldots,B_n$ are non-empty.

(i) If $B = \{x_j\}$ for $1 \leq j \leq n$, then $\hat{S}^n_m(\{x_j\}, B_1,\ldots,B_n) := B_j$.

(ii) If $B = \{f_i(t_1,\ldots,t_{n_i})\}$, and if we assume that $\hat{S}^n_m(\{t_j\}, B_1,\ldots,B_n)$ for $1 \leq j \leq n$; are already defined, then $\hat{S}^n_m(\{f_i(t_1,\ldots,t_{n_i})\}, B_1,\ldots,B_n) := \{f_i(r_1,\ldots,r_{n_i}) \mid r_j \in \hat{S}^n_m(\{t_j\}, B_1,\ldots,B_n)$ for $1 \leq j \leq n_i\}$.

(iii) If B is an arbitrary subset of $W_\tau(X_n)$, we define

$$\hat{S}^n_m(B, B_1,\ldots,B_n) := \bigcup_{b\in B} \hat{S}^n_m(\{b\}, B_1,\ldots,B_n).$$

If one of the sets $B, B_1, \ldots, B_n$ is empty, we define $\hat{S}^n_m(B, B_1, \ldots, B_n) := \emptyset$. Then we may consider the heterogeneous algebra

$$\mathcal{P} - clone\ \tau := ((\mathcal{P}(W_\tau(X_n)))_{n\in\mathbb{N}^+}; (\hat{S}^n_m)_{m,n\in\mathbb{N}^+}, (\{x_i\})_{i\leq n, n\in\mathbb{N}^+})$$

which is called the power clone of τ ([?]). We mention that $\mathcal{P} - clone\ \tau$ satisfies the well-known clone axioms (C1), (C2), (C3) (see e.g. [?, ?]). If $\mathcal{P}_{fin}(W_\tau(X_n))$ is the set of all finite subsets of $W_\tau(X_n)$, then

$$\mathcal{P}_{fin} - clone\ \tau := ((\mathcal{P}_{fin}(W_\tau(X_n)))_{n\in\mathbb{N}^+}; (\hat{S}^n_m)_{n\in\mathbb{N}^+}, (\{x_i\})_{i\leq n, n\in\mathbb{N}^+})$$

is a subalgebra of $\mathcal{P} - clone\ \tau$ ([?]).

We mention also that there is a one-based version of $\mathcal{P} - clone\ \tau$, the algebra $\mathcal{P}_n - clone\ \tau_n := (\mathcal{P}(W_{\tau_n}(X_n)); \hat{S}^n, \{x_1\}, \ldots, \{x_n\})$ where τ_n is a finite type consisting of n-ary operation symbols only and where $\hat{S}^n := \hat{S}^n_n$. $\mathcal{P}_n - clone\ \tau_n$ is an example of a unitary Menger algebra of rank n (see e.g [?]).

Similar structures can be obtained if one defines a superposition for sets of operations. Let $O^{(n)}(A)$ be the set of all n-ary operations ($n \geq 1$) defined on the set A and let $O(A) := \bigcup_{n\geq 1} O^{(n)}(A)$ be the set of all operations defined on A. Let $e^{n,A}_i$ be an n-ary projection defined on A, i.e., $e^{n,A}_i(a_1, \ldots, a_n) := a_i$ for $1 \leq i \leq n$, and let $\mathcal{P}(O^{(n)}(A))$ be the power set of $O^{(n)}(A)$.

Definition 1.2. Let $m, n \in \mathbb{N}^+$ and $B \in \mathcal{P}(O^{(n)}(A)), B_1, \ldots, B_n \in \mathcal{P}(O^{(m)}(A))$ such that $B, B_1, \ldots, B_n$ are non-empty.

(i) If $B = \{e^{n,A}_j\}$ for $1 \leq j \leq n$, then $\hat{S}^{n,A}_m(\{e^{n,A}_j\}, B_1, \ldots, B_n) := B_j$.

(ii) If $B = \{f^A_i(t^A_1, \ldots, t^A_{n_i})\}$ with $f^A_i \in O^{(n_i)}(A), t^A_j \in O^{(n)}(A)$ and assume that $\hat{S}^{n,A}_m(\{t^A_j\}, B_1, \ldots, B_n)$ for $1 \leq j \leq n_i$ are already defined, then $\hat{S}^{n,A}_m(\{f^A_i(t^A_1, \ldots, t^A_{n_i})\}, B_1, \ldots, B_n) :=$ $\{f^A_i(r^A_1, \ldots, r^A_{n_i}) \mid r^A_j \in \hat{S}^{n,A}_m(\{t^A_j\}, B_1, \ldots, B_n), 1 \leq j \leq n_i\}$.

(iii) If $B \in \mathcal{P}(O^{(n)}(A))$ is arbitrary, then we define

$$\hat{S}^{n,A}_m(B, B_1, \ldots, B_n) := \bigcup_{b\in B} \hat{S}^{n,A}_m(\{b\}, B_1, \ldots, B_n).$$

If one of the sets $B, B_1, \ldots, B_n$ is empty, then we define $\hat{S}_m^{n,A}(B, B_1, \ldots, B_n)$ $:= \emptyset$. In this case we consider the heterogeneous algebra

$$\mathcal{P}_A - clone := ((\mathcal{P}(O^{(n)}(A)))_{n\in\mathbb{N}^+}; (\hat{S}_m^{n,A})_{m,n\in\mathbb{N}^+}, (\{e_i^{n,A}\})_{i\leq n, n\in\mathbb{N}^+}).$$

Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be an algebra of type τ. Then we may consider the subclone $\mathcal{P}_A - clone\mathcal{A}$ of $\mathcal{P}_A - clone$ which is defined as follows.

Definition 1.3. Let $n \in \mathbb{N}^+$ and $B \in \mathcal{P}(W_\tau(X_n))$. Then we define the set B^A of term operations induced on the algebra $\mathcal{A} = (A; (f_i^A)_{i\in I})$ as follows:

(i) If $B = \{x_j\}$ for $1 \leq j \leq n$, then $B^\mathcal{A} := \{e_j^{n,\mathcal{A}}\}$.

(ii) If $B = \{f_i(t_1, \ldots, t_{n_i})\}$ then $B^\mathcal{A} = \{f_i^\mathcal{A}(t_1^\mathcal{A}, \ldots, t_{n_i}^\mathcal{A})\}$ where $f_i^\mathcal{A}$ is the fundamental operation of $\mathcal{A}$ coresponding to the operation symbol f_i and where $t_j^\mathcal{A}$ are term operations on $\mathcal{A}$ which are induced in the usual way by the t_j's.

(iii) If B is an arbitrary non-empty subset of $W_\tau(X_n)$, then we define $B^\mathcal{A} := \bigcup_{b\in B}\{b\}^\mathcal{A}$. If the set B is empty, then we define $B^\mathcal{A} := \emptyset$.

Let $\mathcal{P}(W_\tau(X_n))^\mathcal{A}$ be the collection of all sets of n-ary term operations induced by sets of n-ary terms of type τ on the algebra $\mathcal{A} = (A; (f_i^A)_{i\in I})$.

From these definitions we obtain the following

Lemma 1.4. *Let $B \in \mathcal{P}(W_\tau(X_n))$ and let $B_1, \ldots, B_n \in \mathcal{P}(W_\tau(X_m))$. Then*

$$[\hat{S}_m^n(B, B_1, \ldots, B_n)]^\mathcal{A} = \hat{S}_m^{n,\mathcal{A}}(B^\mathcal{A}, B_1^\mathcal{A}, \ldots, B_n^\mathcal{A}).$$

Proof. If one of the sets $B, B_1, \ldots, B_n$ is empty, then one of the sets $B^\mathcal{A}, B_1^\mathcal{A}, \ldots, B_n^\mathcal{A}$ is also empty. Thus

$$[\hat{S}_m^n(B, B_1, \ldots, B_n)]^\mathcal{A} = \emptyset^\mathcal{A} = \emptyset = \hat{S}_m^{n,\mathcal{A}}(B^\mathcal{A}, B_1^\mathcal{A}, \ldots, B_n^\mathcal{A}).$$

Assume now that all of $B, B_1, \ldots, B_n$ are different from the empty set. At first we show by induction on the complexity of the term t that for one-element sets $B = \{t\}$ our equation is satisfied.

For $t = x_i$ with $1 \leq i \leq n$, we have $B^{\mathcal{A}} = \{x_i\}^{\mathcal{A}} = \{e_i^{n,\mathcal{A}}\}$ and

$$\begin{aligned}
[\hat{S}_m^n(B, B_1, \ldots, B_n)]^{\mathcal{A}} &= [\hat{S}_m^n(\{x_i\}, B_1, \ldots, B_n)]^{\mathcal{A}} \\
&= B_i^{\mathcal{A}} \\
&= \hat{S}_m^{n,\mathcal{A}}(\{e_i^{n,\mathcal{A}}\}, B_1^{\mathcal{A}}, \ldots, B_n^{\mathcal{A}}) \\
&= \hat{S}_m^{n,\mathcal{A}}(\{x_i\}^{\mathcal{A}}, B_1^{\mathcal{A}}, \ldots, B_n^{\mathcal{A}}) \\
&= \hat{S}_m^{n,\mathcal{A}}(B^{\mathcal{A}}, B_1^{\mathcal{A}}, \ldots, B_n^{\mathcal{A}}).
\end{aligned}$$

Let now $t = f_i(t_1, \ldots, t_{n_i})$ and assume that for all $1 \leq k \leq n_i$,

$$[\hat{S}_m^n(\{t_k\}, B_1, \ldots, B_n)]^{\mathcal{A}} = \hat{S}_m^{n,\mathcal{A}}(\{t_k\}^{\mathcal{A}}, B_1^{\mathcal{A}}, \ldots, B_n^{\mathcal{A}}).$$

Then

$$\begin{aligned}
&[\hat{S}_m^n(\{f_i(t_1, \ldots, t_{n_i})\}, B_1, \ldots, B_n)]^{\mathcal{A}} \\
&\quad = \{f_i(r_1, \ldots, r_{n_i}) \mid r_k \in \hat{S}_m^n(\{t_k\}, B_1, \ldots, B_n), 1 \leq k \leq n_i\}^{\mathcal{A}} \\
&\quad = \{f_i^{\mathcal{A}}(r_1^{\mathcal{A}}, \ldots, r_{n_i}^{\mathcal{A}}) \mid r_k \in \hat{S}_m^n(\{t_k\}, B_1, \ldots, B_n), 1 \leq k \leq n_i\} \\
&\quad = \{f_i^{\mathcal{A}}(r_1^{\mathcal{A}}, \ldots, r_{n_i}^{\mathcal{A}}) \mid r_k^{\mathcal{A}} \in \hat{S}_m^n(\{t_k\}, B_1, \ldots, B_n)^{\mathcal{A}}, 1 \leq k \leq n_i\} \\
&\quad = \{f_i^{\mathcal{A}}(r_1^{\mathcal{A}}, \ldots, r_{n_i}^{\mathcal{A}}) \mid r_k^{\mathcal{A}} \in \hat{S}_m^{n,\mathcal{A}}(\{t_k\}^{\mathcal{A}}, B_1^{\mathcal{A}}, \ldots, B_n^{\mathcal{A}}), 1 \leq k \leq n_i\} \\
&\quad = \hat{S}_m^{n,\mathcal{A}}(\{f_i^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_{n_i}^{\mathcal{A}})\}, B_1^{\mathcal{A}}, \ldots, B_n^{\mathcal{A}}) \\
&\quad = \hat{S}_m^{n,\mathcal{A}}(\{f_i(t_1, \ldots, t_{n_i})\}^{\mathcal{A}}, B_1^{\mathcal{A}}, \ldots, B_n^{\mathcal{A}}).
\end{aligned}$$

If B is a set of terms consisting of more than one element, then we have

$$[\hat{S}^n_m(B, B_1, \ldots, B_n)]^{\mathcal{A}} = \left[\hat{S}^n_m(\bigcup_{b\in B}\{b\}, B_1, \ldots, B_n)\right]^{\mathcal{A}}$$

$$= \left[\bigcup_{b\in B}\hat{S}^n_m(\{b\}, B_1, \ldots, B_n)\right]^{\mathcal{A}}$$

$$= \bigcup_{b\in B}[\hat{S}^n_m(\{b\}, B_1, \ldots, B_n)]^{\mathcal{A}}$$

$$= \bigcup_{b\in B}\hat{S}^{n,\mathcal{A}}_m(\{b\}^{\mathcal{A}}, B_1^{\mathcal{A}}, \ldots, B_n^{\mathcal{A}})$$

$$= \hat{S}^{n,\mathcal{A}}_m\left(\bigcup_{b\in B}\{b\}^{\mathcal{A}}, B_1^{\mathcal{A}}, \ldots, B_n^{\mathcal{A}}\right)$$

$$= \hat{S}^{n,\mathcal{A}}_m(B^{\mathcal{A}}, B_1^{\mathcal{A}}, \ldots, B_n^{\mathcal{A}}).$$

■

Proposition 1.5.

$$\mathcal{P}_A - clone\mathcal{A} = ((\mathcal{P}(W_\tau(X_n))^{\mathcal{A}})_{n\in\mathbb{N}^+}; (\hat{S}^{n,A}_m)_{m,n\in\mathbb{N}^+}, (\{e_i^{n,A}\})_{i\leq n, n\in\mathbb{N}^+})$$

is a subalgebra of $\mathcal{P}_A - clone$.

Proof. Let $B^{\mathcal{A}} \in \mathcal{P}(W_\tau(X_n))^{\mathcal{A}}$ and let $B_1^{\mathcal{A}}, \ldots, B_n^{\mathcal{A}} \in \mathcal{P}(W_\tau(X_m))^{\mathcal{A}}$, then $B \in \mathcal{P}(W_\tau(X_n))$ and $B_1, \ldots, B_n \in \mathcal{P}(W_\tau(X_m))$.

From Lemma 1.4 we have that

$$\hat{S}^{n,A}_m(B^{\mathcal{A}}, B_1^{\mathcal{A}}, \ldots, B_n^{\mathcal{A}}) = [\hat{S}^n_m(B, B_1, \ldots, B_n)]^{\mathcal{A}} \in \mathcal{P}(W_\tau(X_m))^{\mathcal{A}}.$$

■

If $T^{(n)}(\mathcal{A})$ is the set of all derived n-ary operations of the algebra $\mathcal{A} = (A; (f_i^A)_{i\in I})$, then we can also consider the algebra $\mathcal{P}(\mathcal{T}(\mathcal{A})) := ((\mathcal{P}(T^{(n)}(\mathcal{A})))_{n\in\mathbb{N}^+}; (\hat{S}^{n,A}_m)_{n,m\in\mathbb{N}^+}, (\{e_i^{n,A}\})_{i\leq n,n\in\mathbb{N}^+})$. It is not difficult to prove that $\mathcal{P}_A - clone\mathcal{A} = \mathcal{P}(\mathcal{T}(\mathcal{A}))$.

Any mapping $\sigma : \{f_i \mid i \in I\} \to \mathcal{P}(W_\tau(X))$ with $\sigma(f_i) \subseteq W_\tau(X_{n_i})$, for $i \in I$, is called a non-deterministic hypersubstitution (for short nd-hypersubstitution) of type τ. We denote by $Hyp^{nd}(\tau)$ the set of all non-deterministic hypersubstitutions of type τ. Every nd-hypersubstitution can be extended in the following inductive way to a mapping $\hat{\sigma} : \mathcal{P}(W_\tau(X)) \to \mathcal{P}(W_\tau(X))$.

(i) $\hat{\sigma}[\emptyset] := \emptyset$.

(ii) $\hat{\sigma}[\{x_i\}] := \{x_i\}$ for every variable $x_i \in X$.

(iii) $\hat{\sigma}[\{f_i(t_1, \ldots, t_{n_i})\}] := \hat{S}^{n_i}_n(\sigma(f_i), \hat{\sigma}[\{t_1\}], \ldots, \hat{\sigma}[\{t_{n_i}\}])$ if we inductively assume that $\hat{\sigma}[\{t_k\}], 1 \leq k \leq n_i$, are already defined.

(iv) $\hat{\sigma}[B] := \bigcup\{\hat{\sigma}[\{b\}] \mid b \in B\}$ for $B \subseteq W_\tau(X)$.

In the sequel instead of $\hat{\sigma}[\{t\}]$ for a term $t \in W_\tau(X)$ we will simply write $\hat{\sigma}[t]$.

In [**?**] was proved that for every nd-hypersubstitution σ the mapping $\hat{\sigma}$ is an endomorphism of $\mathcal{P}-clone\ \tau$. We recall also that the set $Hyp^{nd}(\tau)$ forms a monoid with respect to the operation $\circ_{nd}$ defined by $\sigma_1 \circ_{nd} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ and the identity element $\sigma_{pid} : f_i \mapsto \{f_i(x_1, \ldots, x_{n_i})\}$ for every $i \in I$.

In the next section we apply nd-hypersubstitutions to equations and to algebras.

2. The Conjugate Pair $\left(\chi^{\mathcal{A}}_{nd}, \chi^{E}_{nd}\right)$

If $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i\in I})$ is an algebra of type τ and if $\sigma \in Hyp^{nd}(\tau)$ is an nd-hypersubstitution, then we define

$$\sigma(\mathcal{A}) := \{(A; (l_i^{\mathcal{A}})_{i\in I}) \mid l_i \in \sigma(f_i)\}.$$

The set $\sigma(\mathcal{A})$ is called the set of derived algebras. Since for every sequence $(l_i)_{i\in I}$ of terms there is a hypersubstitution mapping f_i to l_i we can write $\sigma(\mathcal{A})$ also in the form $\sigma(\mathcal{A}) = \{\rho(\mathcal{A}) \mid \rho \in Hyp(\tau) \text{ with } \rho(f_i) \in \sigma(f_i) \text{ for } i \in I\}$. For a class K of algebras of type τ we define

$$\sigma(K) := \bigcup_{\mathcal{A}\in K} \sigma(\mathcal{A}).$$

If $M \subseteq Hyp^{nd}(\tau)$ is the universe of a submonoid of $Hyp^{nd}(\tau)$, then we define $\chi^{A}_{M-nd}[K] := \bigcup_{\sigma\in M} \sigma(K)$. For $M = Hyp^{nd}(\tau)$ we will simply write χ^{A}_{nd}. We notice that $\chi^{A}_{M-nd}[K]$ consists of algebras of the same type. For a set $\mathcal{K} \in \mathcal{P}(\mathcal{P}(Alg(\tau)))$ of sets of algebras of type τ and a monoid M

of nd-hypersubstitutions we define $\chi^A_{M-nd}[\mathcal{K}] := \{\sigma(K) \mid K \in \mathcal{K}, \sigma \in M\}$. For $B_1, B_2 \in \mathcal{P}(W_\tau(X))$ we define equations $B_1 \approx B_2$. If $\Sigma \in \mathcal{P}(\mathcal{P}(W_\tau(X)) \times \mathcal{P}(W_\tau(X)))$ and $\sigma \in M \subseteq Hyp^{nd}(\tau)$ we define

$$\hat{\sigma}[\Sigma] := \{\hat{\sigma}[B_1] \approx \hat{\sigma}[B_2] \mid B_1 \approx B_2 \in \Sigma\}$$

and

$$\chi^E_{M-nd}[\Sigma] := \{\hat{\sigma}[B_1] \approx \hat{\sigma}[B_2] \mid B_1 \approx B_2 \in \Sigma, \sigma \in M\}.$$

For $M = Hyp^{nd}(\tau)$ we will use simply the notation χ^E_{nd}.

We want to prove that there is a close connection between both operators. Instead of $\chi^A_{M-nd}[\{\{A\}\}]$ we will write $\chi^A_{M-nd}[A]$. For $K \subseteq \chi^A_{M-nd}[A]$ and for a set $B \subseteq W_\tau(X)$ of terms we define the set B^K of induced term operations. For the set $\sigma(\mathcal{A})$ of derived algebras and for a set $B \in \mathcal{P}(W_\tau(X_n))$ of n-ary terms we define the set $B^{\sigma(\mathcal{A})}$ of term operations induced by the set $\sigma(\mathcal{A})$ of derived algebras as follows

Definition 2.1. Let $n \in \mathbb{N}^+$ and $B \in \mathcal{P}(W_\tau(X_n))$, let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be an algebra of type τ, let $\sigma \in Hyp^{nd}(\tau)$ be an nd-hypersubstitution and let $\sigma(\mathcal{A}) = \{(A; (l_i^A)_{i\in I}) \mid l_i \in \sigma(f_i)\}$ be the set of derived algebras. Then we define the set $B^{\sigma(\mathcal{A})}$ of term operations induced by the set $\sigma(\mathcal{A})$ of derived algebras as follows:

(i) If $B := \{x_j\}$ for $1 \leq j \leq n$, then $B^{\sigma(\mathcal{A})} := \{e_j^{n,\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma(\mathcal{A})\} = \{e_j^{n,A}\}$.

(ii) If $B = \{f_i(t_1, \ldots, t_{n_i})\}$ then

$$\begin{aligned} B^{\sigma(\mathcal{A})} &:= \{\hat{S}_n^{n_i,\mathcal{A}}(\{f_i^{\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma(\mathcal{A})\}, \{t_1\}^{\sigma(\mathcal{A})}, \ldots, \{t_{n_i}\}^{\sigma(\mathcal{A})})\} \\ &= \bigcup_{\rho(\mathcal{A})\in\sigma(\mathcal{A})} \{\hat{S}_n^{n_i,\mathcal{A}}(\{f_i^{\rho(\mathcal{A})}\}, \{t_1\}^{\sigma(\mathcal{A})}, \ldots, \{t_{n_i}\}^{\sigma(\mathcal{A})})\} \\ &= \bigcup_{\rho(\mathcal{A})\in\sigma(\mathcal{A})} \{f_i^{\rho(\mathcal{A})}(r_1, \ldots, r_{n_i}) \mid r_k \in \{t_k\}^{\sigma(\mathcal{A})},\ \text{for } 1 \leq k \leq n_i\} \end{aligned}$$

where $f_i^{\rho(\mathcal{A})}$ denotes the fundamental operation of the algebra $\rho(\mathcal{A})$ belonging to the operation symbol f_i and assume that $\{t_k\}^{\sigma(\mathcal{A})}, 1 \leq k \leq n_i$, are already defined.

(iii) If B is an arbitrary non-empty subset of $W_\tau(X_n)$, then we define $B^{\sigma(\mathcal{A})} := \bigcup_{b \in B} \{b\}^{\sigma(\mathcal{A})}$. If the set B is empty, then we define $B^{\sigma(\mathcal{A})} := \emptyset$.

For any term $t \in W_\tau(X_n)$ and a class G of algebras of type τ we define

$$t^G := \{t\}^G := \{t^{\mathcal{A}} \mid \mathcal{A} \in G\}.$$

Definition 2.2. Let $\mathcal{A}$ be an algebra of type τ and let $K \subseteq \chi^A_{M-nd}[\mathcal{A}]$ and let $n \geq 1$ be an integer. Then we define

(i) If $B = \{x_j\}$ for $1 \leq j \leq n$, then $B^K = \{e_j^{n,A}\} \subseteq \mathcal{T}^{(n)}(\mathcal{A})$.

(ii) If $B = \{f_i(t_1, \ldots, t_n)\}$ and let $B_j = t_j^K \subseteq \mathcal{T}^{(n)}(\mathcal{A})$ for $1 \leq j \leq n_i$ are already known, then

$$\begin{aligned} B^K :=& \{\hat{S}_n^{n_i,A}(S, B_1, \ldots, B_{n_i}) \mid S = \{\rho(f_i)^{\mathcal{A}} \mid \rho \in Hyp(\tau), \rho(\mathcal{A}) \in K\} \\ &\subseteq \mathcal{T}^{(n_i)}(\mathcal{A})\}. \end{aligned}$$

Finally for an arbitrary nonempty set $B \in \mathcal{P}(W_\tau(X))$ we set $B^K := \bigcup_{b \in B} \{b\}^K$ and for the empty set B we let $B^K := \emptyset$.

Definition 2.2 contains Definition 2.1 as a special case since for every $\sigma \in Hyp^{nd}(\tau)$ we have $\sigma(\mathcal{A}) \subseteq \chi^A_{M-nd}[A]$. We have also $\{\mathcal{A}\} \subseteq \chi^A_{M-nd}[A]$ and $\{\rho(\mathcal{A})\} \subseteq \chi^A_{M-nd}[A]$ for a hypersubstitution $\rho \in Hyp(\tau)$ and it is easy to see that for a single term $s \in W_\tau(X_n)$ we have $\{\hat{\rho}[s]\}^{\{\mathcal{A}\}} = \hat{\rho}[s]^{\mathcal{A}} = s^{\rho(\mathcal{A})} = \{s\}^{\{\rho(\mathcal{A})\}}$.

Now we prove:

Lemma 2.3. *Let $B \in \mathcal{P}(W_\tau(X_n))$ be an arbitrary set of n-ary terms of type τ, let $\mathcal{A} = (A; (f_i^A)_{i \in I})$ be an algebra of type τ and let σ be an nd-hypersubstitution of type τ. Then $\hat{\sigma}[B]^{\mathcal{A}} = B^{\sigma(\mathcal{A})}$.*

Proof. If B is empty, then all is clear. If B is nonempty we will give a proof by induction on the complexity of the terms from the set B.

If $B = \{x_j\}$ for $1 \leq j \leq n$, then $\hat{\sigma}[B]^{\mathcal{A}} = \{x_j\}^{\mathcal{A}} = \{e_j^{n,A}\}$ by the definition of σ and by Definition 1.3. Further, by Definition 2.1 we have

$$B^{\sigma(\mathcal{A})} = \{x_j\}^{\sigma(\mathcal{A})} = \left\{e_j^{n,\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma(\mathcal{A})\right\} = \left\{e_j^{n,A}\right\}$$

since all algebras $\rho(\mathcal{A})$ have the same universe. Therefore $\hat{\sigma}[B]^{\mathcal{A}} = B^{\sigma(\mathcal{A})}$ for $B = \{x_j\}$ for $1 \leq j \leq n$.

Now let $B = \{f_i(t_1, \ldots, t_{n_i})\}$ and assume that $\hat{\sigma}[\{t_k\}]^{\mathcal{A}} = \{t_k\}^{\sigma(\mathcal{A})}$ for $1 \leq k \leq n_i$. Then

$$\begin{aligned}
\hat{\sigma}[B]^{\mathcal{A}} &= \hat{\sigma}[\{f_i(t_1, \ldots, t_{n_i})\}]^{\mathcal{A}} \\
&= \left[\hat{S}_n^{n_i}(\sigma(f_i), \hat{\sigma}[\{t_1\}], \ldots, \hat{\sigma}[\{t_{n_i}\}])\right]^{\mathcal{A}} \\
&= \hat{S}_n^{n_i,\mathcal{A}}\left(\sigma(f_i)^{\mathcal{A}}, \hat{\sigma}[\{t_1\}]^{\mathcal{A}}, \ldots, \hat{\sigma}[\{t_{n_i}\}]^{\mathcal{A}}\right) \\
&= \hat{S}_n^{n_i,\mathcal{A}}\left(\{l_i \mid l_i \in \sigma(f_i)\}^{\mathcal{A}}, \hat{\sigma}[\{t_1\}]^{\mathcal{A}}, \ldots, \hat{\sigma}[\{t_{n_i}\}]^{\mathcal{A}}\right) \\
&= \bigcup_{l_i \in \sigma(f_i)} \hat{S}_n^{n_i,\mathcal{A}}\left(\{l_i^{\mathcal{A}}\}, \hat{\sigma}[\{t_1\}]^{\mathcal{A}}, \ldots, \hat{\sigma}[\{t_{n_i}\}]^{\mathcal{A}}\right) \\
&= \bigcup_{l_i \in \sigma(f_i)} \hat{S}_n^{n_i,\mathcal{A}}\left(\{l_i^{\mathcal{A}}\}, \{t_1\}^{\sigma(\mathcal{A})}, \ldots, \{t_{n_i}\}^{\sigma(\mathcal{A})}\right) \\
&= \hat{S}_n^{n_i,\mathcal{A}}\left(\{f_i^{\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma(\mathcal{A})\}, \{t_1\}^{\sigma(\mathcal{A})}, \ldots, \{t_{n_i}\}^{\sigma(\mathcal{A})}\right) \\
&= \{f_i(t_1, \ldots, t_{n_i})\}^{\sigma(\mathcal{A})} \\
&= B^{\sigma(\mathcal{A})}.
\end{aligned}$$

If B is a set of terms consisting of more than one element, then we have

$$\hat{\sigma}[B]^{\mathcal{A}} = \left\{ \bigcup_{b \in B} \hat{\sigma}[\{b\}] \right\}^{\mathcal{A}} = \bigcup_{b \in B} \hat{\sigma}[\{b\}]^{\mathcal{A}} = \bigcup_{b \in B} \{b\}^{\sigma(\mathcal{A})} = B^{\sigma(\mathcal{A})}.$$

■

From Lemma 2.3 we obtain the "conjugate pair property" for the pair $(\chi^{A}_{M-nd}, \chi^{E}_{M-nd})$ of operators. We use the notation $\mathcal{A} \models s \approx t$ if the algebra $\mathcal{A}$ of type τ satisfies the equation $s \approx t$ of type τ as an identity and $K \models s \approx t$ if the class K satisfies $s \approx t$. Moreover, we define

Definition 2.4. Let $B_1, B_2 \subseteq W_\tau(X)$ be sets of terms of type τ and assume that $\mathcal{A}$ is an algebra of type τ and that $K \subseteq \chi^{A}_{M-nd}[A]$ for a monoid $\mathcal{M} \subseteq \mathcal{H}yp^{nd}(\tau)$ of non-deterministic hypersubstitution. Then

$$K \models B_1 \approx B_2 \text{ iff } B_1^K = B_2^K.$$

Especially we have $\sigma[\mathcal{A}] \models B_1 \approx B_2$ iff $B_1^{\sigma[\mathcal{A}]} = B_2^{\sigma[\mathcal{A}]}$ and $\{\mathcal{A}\} \models B_1 \approx B_2$ iff $B_1^{\{\mathcal{A}\}} = B_2^{\{\mathcal{A}\}}$ and this means $\mathcal{A} \models B_1 \approx B_2$ iff $B_1^{\mathcal{A}} = B_2^{\mathcal{A}}$.

From Lemma 2.3 we obtain the following conjugate property.

Theorem 2.5. *Let $\mathcal{A}$ be an algebra of type τ, and let $B_1 \approx B_2 \in \mathcal{P}(W\tau(X)) \times \mathcal{P}(W\tau(X))$ and assume that $\sigma \in Hyp^{nd}(\tau)$ be a non-deterministic hypersubstitution of type τ. Then*

$$\sigma(\mathcal{A}) \models B_1 \approx B_2 \Longleftrightarrow \mathcal{A} \models \hat{\sigma}[B_1] \approx \hat{\sigma}[B_2].$$

Proof.

$$\begin{aligned} \sigma(\mathcal{A}) \models B_1 \approx B_2 \quad &\Longleftrightarrow \quad B_1^{\sigma(\mathcal{A})} = B_2^{\sigma(\mathcal{A})} \\ &\Longleftrightarrow \quad \hat{\sigma}[B_1]^{\mathcal{A}} = \hat{\sigma}[B_2]^{\mathcal{A}} \\ &\Longleftrightarrow \quad \mathcal{A} \models \hat{\sigma}[B_1] \approx \hat{\sigma}[B_1]. \end{aligned}$$

■

Let now $M \subseteq Hyp^{nd}(\tau)$ be a monoid of non-deterministic hypersubstitutions. Then we form the set $\bigcup\{\mathcal{P}(\chi^A_{M-nd}[\mathcal{A}]) \mid \mathcal{A} \in Alg(\tau)\}$ and consider $\Sigma \subseteq \mathcal{P}((\mathcal{P}(W_\tau(X)))^2)$ and $\mathcal{K} \subseteq \bigcup\{\mathcal{P}(\chi^A_{M-nd}[\mathcal{A}]) \mid \mathcal{A} \in Alg(\tau)\}$. Definition 2.4 defines a relation between both sets. In the usual way we obtain a Galois connection $(\mathcal{P}Mod; \mathcal{P}Id)$ of non-deterministic models and non-deterministic identities defined by

$$\mathcal{P}Mod\Sigma := \{K \mid K \subseteq \chi^A_{M-nd}[\mathcal{A}] \text{ for some algebra } \mathcal{A} \in Alg(\tau)$$

$$\text{and } \forall B_1 \approx B_2 \in \Sigma(K \models B_1 \approx B_2)\}$$

$$\mathcal{P}Id\mathcal{K} := \{B_1 \approx B_2 \mid B_1 \approx B_2 \in \mathcal{P}(W_\tau(X))^2 \text{ and } \forall K \in \mathcal{K}(K \models B_1 \approx B_2)\}$$

By definition, the operators $\chi^A_{M-nd} : \mathcal{P}(\mathcal{P}(Alg(\tau))) \to \mathcal{P}(\mathcal{P}(Alg(\tau)))$ and $\chi^E_{M-nd} : \mathcal{P}((\mathcal{P}(W_\tau(X)))^2) \to \mathcal{P}((\mathcal{P}(W_\tau(X)))^2)$ are completely additive. This means, for classes $\mathcal{K} \subseteq \mathcal{P}(\mathcal{P}(Alg(\tau)))$ the result of the application of χ^A_{M-nd} to $\mathcal{K}$ is the union of the results obtained by application of χ^A_{M-nd} to the single classes $K \subseteq Alg(\tau) : \chi^A_{M-nd}[\mathcal{K}] = \bigcup_{\sigma\in M,} \bigcup_{K\in\mathcal{K}} \sigma(K)$. In a corresponding way for a set $\Sigma \subseteq \mathcal{P}((\mathcal{P}(W_\tau(X)))^2)$ and a submonoid $M \subseteq Hyp^{nd}(\tau)$ we have $\chi^E_{M-nd}[\Sigma] = \bigcup_{\sigma\in M} \bigcup_{B_1\approx B_2\in\Sigma} \hat{\sigma}[B_1] \approx \hat{\sigma}[B_2]$. Therefore, both operators are monotone, i.e.

$$\mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow \chi^A_{M-nd}[\mathcal{K}_1] \subseteq \chi^A_{M-nd}[\mathcal{K}_2]$$

and

$$\Sigma_1 \subseteq \Sigma_2 \Rightarrow \chi^E_{M-nd}[\Sigma_1] \subseteq \chi^E_{M-nd}[\Sigma_2].$$

Since $\sigma_{pid} \in M$ and $\sigma_{pid}(K) = \{K\}$, the operator χ^A_{M-nd} is extensive, i.e. $\mathcal{K} \subseteq \chi^A_{M-nd}[\mathcal{K}]$ for every class $\mathcal{K} \subseteq \mathcal{P}(\mathcal{P}(Alg(\tau)))$. Since $\hat{\sigma}_{pid}[\{B\}] = \{B\}$ for every $B \in \mathcal{P}(W_\tau(X))$, the operator χ^E_{M-nd} is also extensive. It turns out that both operators, χ^A_{M-nd} and χ^E_{M-nd} are closure operators. Altogether, we have

Theorem 2.6. *The pair* $(\chi^A_{M-nd}, \chi^E_{M-nd})$ *is a conjugate pair of additive closure operators.*

Proof. From Theorem 2.5, there follows $\chi^A_{M-nd}[K] \models B_1 \approx B_2 \Longleftrightarrow K \models \chi^E_{M-nd}[B_1 \approx B_2]$. By the previous remarks it is left to show that the operators χ^A_{M-nd} and χ^E_{M-nd} are idempotent. Extensivity of χ^A_{M-nd} and χ^E_{M-nd}, implies $\chi^A_{M-nd}[\mathcal{K}] \subseteq \chi^A_{M-nd}[\chi^A_{M-nd}[\mathcal{K}]]$ and $\chi^E_{M-nd}[\Sigma] \subseteq \chi^E_{M-nd}[\chi^E_{M-nd}[\Sigma]]$ for $\mathcal{K} \in \mathcal{P}(\bigcup\{\mathcal{P}(\chi^A_{M-nd}[\mathcal{A}]) \mid \mathcal{A} \in Alg(\tau)\})$ and $W \in \mathcal{P}((\mathcal{P}(W_\tau(X)))^2)$. We write $\mathcal{K} \models W$ iff $K \models A \approx B$ for all $K \in \mathcal{K}$ and all $B_1 \approx B_2 \in W$. We have to show that the opposite inclusions are satisfied. Let $\mathcal{B} \in \chi^A_{M-nd}[\chi^A_{M-nd}[\mathcal{K}]]$. Then there are nd-hypersubstitutions $\sigma_1, \sigma_2 \in M$ and an algebra $\mathcal{A} \in \mathcal{K}$ such that

$$\begin{aligned}
\mathcal{B} \in \sigma_1[\sigma_2(\mathcal{A})] &= \sigma_1[\{(A;(l_i^{\mathcal{A}})_{i\in I}) \mid l_i \in \sigma_2(f_i)\}] \\
&= \{\sigma_1(A;(l_i^{\mathcal{A}})_{i\in I}) \mid l_i \in \sigma_2(f_i)\} \\
&= \{\{(A;(h_i^{\mathcal{A}})_{i\in I}) \mid h_i \in \hat{\sigma}_1[l_i]\} \mid l_i \in \sigma_2(f_i)\} \\
&= \{(A;(h_i^{\mathcal{A}})_{i\in I}) \mid h_i \in \hat{\sigma}_1[l_i] \text{ and } l_i \in \sigma_2(f_i)\} \\
&= \{(A;(h_i^{\mathcal{A}})_{i\in I}) \mid h_i \in \hat{\sigma}_1[\sigma_2(f_i)]\} \\
&= \{(A;(h_i^{\mathcal{A}})_{i\in I}) \mid h_i \in (\sigma_1 \circ_{nd} \sigma_2)(f_i)\} \\
&= (\sigma_1 \circ_{nd} \sigma_2)(\mathcal{A}) \in \chi^{\mathcal{A}}_{M-nd}[\mathcal{K}].
\end{aligned}$$

This shows $\chi^A_{M-nd}[\chi^A_{M-nd}[\mathcal{K}]] = \chi^A_{M-nd}[\mathcal{K}]$. Now let $B_1 \approx B_2 \in \chi^E_{M-nd}$ $[\chi^E_{M-nd}[\Sigma]]$. Then there is an equation $U \approx V$ in Σ and an nd-hypersubstitution $\sigma_1, \sigma_2 \in M$ such that $B_1 \approx B_2 \in \hat{\sigma}_1[\sigma_2[U]] \approx \hat{\sigma}_1[\sigma_2[V]]$, i.e. $B_1 \approx B_2 \in (\sigma_1 \circ_{nd} \sigma_2)\hat{}\,[U] \approx (\sigma_1 \circ_{nd} \sigma_2)\hat{}\,[V] \in \chi^E_{M-nd}[U \approx V] \subseteq \chi^E_{M-nd}[\Sigma]$. ∎

3. $M-$Nd-Solid Varieties

A solid variety V admits every mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ which maps $n_i - ary$ operation symbols f_i to $n_i - ary$ terms in the sense that every derived algebra $\sigma(\mathcal{A}) = (A;(\sigma(f_i)^A)_{i\in I})$ belongs to V. Equivalently if $s \approx t$ is an identity in a solid variety V, then $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are also satisfied as identities in V for every hypersubstitution σ. We generalize the definition of a solid variety to M-solid non-deterministic varieties.

Definition 3.1. Let $\mathcal{M} \subseteq \mathcal{H}yp^{nd}(\tau)$ be a monoid of non-deterministic hypersubstitutions of type τ. A variety V of type τ is said to be an M-solid non-deterministic variety, for short an $M-nd$-solid variety, if $\{\{\mathcal{A}\} \mid \mathcal{A} \in V\} \models \{\hat{\sigma}[\{s\}] \approx \hat{\sigma}[\{t\}] \mid s \approx t \in IdV, \sigma \in M\}$. In the case that $\mathcal{M} = \mathcal{H}yp^{nd}(\tau)$ we will speak of a solid non-deterministic variety, for short of an nd-solid variety.

Clearly, the class $Alg(\tau)$ of all algebras of type τ is nd-solid. The trivial variety (consisting only of one-element algebras of type τ) is also nd-solid. The class of all nd-solid varieties of type τ is contained in the class of all solid varieties of this type.

Example 3.2. There is no nontrivial nd-solid variety of semigroups.

Let V be a variety of semigroups. For a proof we consider the nd-hypersubstitutions $\sigma_1, \sigma_2 \in Hyp^{nd}(2)$ defined by $\sigma_1(f) = \{x, xy\}$ and $\sigma_2(f) = \{xy, yx\}$. If V were an nd-solid variety of semigroups, then the application of σ_1 to the associative law gives identities which are satisfied in V. Let $V^* := \{\{\mathcal{A}\} \mid \mathcal{A} \in V\}$, then $V^* \models \{\hat{\sigma}_1[f(x, f(y,z))]\} \approx \{\hat{\sigma}_1[f(f(x,y),z)]\}$ gives $V^* \models \{x, f(x,y), f(x, f(y,z))\} \approx \{x, f(x,y), f(x,z), f(f(x,y),z)\}$. Since every nd-solid variety is solid, this gives especially $V^* \models \{f(x, f(y,z))\} \approx \{f(x,z)\}$. Applying σ_2 to this identity gives $V^* \models \{f(x, f(y,z)), f(x, f(z,y)), f(z, f(y,x)), f(y, f(z,x))\} \approx \{f(x,z), f(z,x)\}$. We use again the fact that every nd-solid variety is solid and the previous identity and obtain $V^* \models \{f(x,z)\} \approx \{f(z,x)\}$ or $V^* \models \{f(x,y)\} \approx \{f(x,z)\}$ or $V^* \models \{f(x,z)\} \approx \{f(y,x)\}$. If we use again the fact that every nd-solid variety must be solid in each of the cases we obtain that V is trivial.

If an identity $s \approx t$ in a variety V is satisfied for all nd-hypersubstitutions we speak of an nd-hyperidentity. More generally we define

Definition 3.3. Let V be a variety of algebras of type τ, let $s \approx t$ be an identity satisfied in V and let $\mathcal{M} \subseteq \mathcal{H}yp^{nd}(\tau)$ be a monoid of non-deterministic hypersubstitutions. Then $s \approx t$ is an $M-nd$ hyperidentity in V if $V^* \models \chi^E_{M-nd}[\{s\} \approx \{t\}]$ where $V^* = \{\{\mathcal{A}\} \mid \mathcal{A} \in V\}$. In this case we write $V \models_{M-nd-hyp} s \approx t$ and for $M = Hyp^{nd}(\tau)$ we will simply write $V \models_{nd-hyp} s \approx t$ and call $s \approx t$ an nd-hyperidentity in V.

The relation $K \models B_1 \approx B_2$ introduced in Definition 2.4 defines the Galois connection $(\mathcal{P}Mod, \mathcal{P}Id)$ with the operations

$$\mathcal{P}Mod : \mathcal{P}((\mathcal{P}(W_\tau(X)))^2) \to \mathcal{P}\left(\bigcup\{\mathcal{P}(\chi^A_{M-nd}[\mathcal{A}]) \mid \mathcal{A} \in Alg(\tau)\}\right),$$

$$\mathcal{P}Id : \mathcal{P}\left(\bigcup\{\mathcal{P}(\chi^A_{M-nd}[\mathcal{A}]) \mid \mathcal{A} \in Alg(\tau)\}\right) \to \mathcal{P}((\mathcal{P}(W_\tau(X)))^2).$$

The relation $\models_{M-nd-hyp}$ defines one more Galois connection

$$(H_{M-nd}\mathcal{P}Mod, H_{M-nd}\mathcal{P}Id)$$

for sets $\Sigma \subseteq \mathcal{P}((\mathcal{P}(W_\tau(X)))^2)$ and classes $\mathcal{K} \subseteq \bigcup\{\mathcal{P}(\chi^A_{M-nd}[\mathcal{A}]) \mid \mathcal{A} \in Alg(\tau)\}$ as follows

$$H_{M-nd}\mathcal{P}Mod : \mathcal{P}((\mathcal{P}(W_\tau(X)))^2) \to \mathcal{P}\left(\bigcup\{\mathcal{P}(\chi^A_{M-nd}[\mathcal{A}]) \mid \mathcal{A} \in Alg(\tau)\}\right),$$

$$H_{M-nd}\mathcal{P}Id : \mathcal{P}\left(\bigcup\{\mathcal{P}(\chi^A_{M-nd}[\mathcal{A}]) \mid \mathcal{A} \in Alg(\tau)\}\right) \to \mathcal{P}((\mathcal{P}(W_\tau(X)))^2).$$

The products $\mathcal{P}Mod\mathcal{P}Id$, $\mathcal{P}Id\mathcal{P}Mod$, $H_{M-nd}\mathcal{P}IdH_{M-nd}\mathcal{P}Mod$, $H_{M-nd}\mathcal{P}ModH_{M-nd}\mathcal{P}Id$ are closure operators and their fixed points are complete lattices. The lattice of all $M-nd$-solid varieties arises if we restrict the operator $H_{M-nd}\mathcal{P}ModH_{M-nd}\mathcal{P}Id$ to classes of the form V^* where V is a variety of algebras of type τ. Moreover we have the conjugate pair $(\chi^A_{M-nd}, \chi^E_{M-nd})$ of additive closure operators. Their fixed points form two more complete lattices. Now we may apply the theory of conjugate pairs of additive closure operators (see e.g. [**?**]) and obtain the following propositions:

Lemma 3.4. *Let $K \subseteq Alg(\tau)$ be a class of algebras and let $\Sigma \subseteq (\mathcal{P}W_\tau(X)^2)$ be a set of equations. Then the following properties hold:*

(i) $H_{M-nd}\mathcal{P}Id(K^*) = \mathcal{P}Id\chi^A_{M-nd}[K^*]$,

(ii) $H_{M-nd}\mathcal{P}Id(K^*) \subseteq \mathcal{P}Id(K^*)$,

(iii) $\chi^E_{M-nd}[H_{M-nd}\mathcal{P}Id(K^*)] = H_{M-nd}\mathcal{P}Id(K^*)$,

(iv) $\chi^A_{M-nd}[\mathcal{P}Mod(H_{M-nd}\mathcal{P}Id(K^*))] = \mathcal{P}Mod(H_{M-nd}\mathcal{P}Id(K^*))$,

(v) $H_{M-nd}\mathcal{P}Id(H_{M-nd}\mathcal{P}Mod(\Sigma)) = \mathcal{P}Id(\mathcal{P}Mod(\chi^E_{M-nd}[\Sigma]))$; and dually (i)′ $H_{M-nd}\mathcal{P}Mod(\Sigma) = \mathcal{P}Mod\chi^{E}_{M-nd}(\Sigma)$,

(ii)′ $H_{M-nd}\mathcal{P}Mod(\Sigma) \subseteq \mathcal{P}Mod(\Sigma)$,

(iii)′ $\chi^{A}_{M-nd}[H_{M-nd}\mathcal{P}Mod(\Sigma)] = H_{M-nd}\mathcal{P}Mod(\Sigma)$,

(iv)′ $\chi^{E}_{M-nd}[\mathcal{P}Id(H_{M-nd}\mathcal{P}Mod(\Sigma))] = \mathcal{P}Id(H_{M-nd}\mathcal{P}Mod(\Sigma))$,

(v)′ $H_{M-nd}\mathcal{P}Mod[H_{M-nd}\mathcal{P}Id(K^*)] = \mathcal{P}Mod(\mathcal{P}Id(\chi^{A}_{M-nd}[K^*]))$.

Using these propositions one obtains the following characterization of $M-nd$-solid varieties.

Theorem 3.5. *Let V be a variety of type τ and let Σ be an equational theory of type τ (i.e. $IdMod(\Sigma) = \Sigma$). Further we assume that $\mathcal{M} \subseteq \mathcal{H}yp^{nd}(\tau)$ is a monoid of non-deterministic hypersubstitutions of type τ.*

Then the following propositions are equivalent:

(i) $H_{M-nd}\mathcal{P}ModH_{M-nd}\mathcal{P}Id(V^*) = V^*$,

(ii) $\chi^{A}_{M-nd}[V^*] = V^*$ (i.e. V^* is $M-nd$ solid),

(iii) $\mathcal{P}Id(V^*) = H_{M-nd}\mathcal{P}Id(V^*)$ (i.e. every identity in V^* is satisfied as a non-deterministic hyperidentity),

(iv) $\chi^{E}_{M-nd}[\mathcal{P}IdV^*] = \mathcal{P}IdV^*$.

4. $M-$Nd-Solid Varieties of Semigroups

We consider some examples of $M-nd$-solid varieties of semigroups and use the following notation for varieties of semigroups;

$B = Mod\{x(yz) \approx (xy)z, x^2 \approx x\}$ – the variety of bands,

$RB = Mod\{x(yz) \approx (xy)z \approx xz, x^2 \approx x\}$ – the variety of rectangular bands

$SL = Mod\{x(yz) \approx (xy)z, x^2 \approx x, xy \approx yx\}$ – the variety of semilattices, bands,

$LZ = Mod\{xy \approx x\}$ – the variety of left-zero bands.

Let $M = \{\sigma_{pid}, \sigma_1, \sigma_2\}$ with $\sigma_1(f) = \{x\}$ and $\sigma_2(f) = \{y\}$. Then M forms a monoid and the multiplication $\circ_{nd}$ is given by the following table:

$\circ_{nd}$	σ_{pid}	σ_1	σ_2
σ_{pid}	σ_{pid}	σ_1	σ_2
σ_1	σ_1	σ_1	σ_2
σ_2	σ_2	σ_1	σ_2

We will prove the following proposition:

Proposition 4.1. *Let $M = \{\sigma_{pid}, \sigma_1, \sigma_2\}$ as defined before. A non-trivial variety V of semigroups is $M - nd$-solid iff $RB \subseteq V$.*

Proof. It is well-known that $IdRB$ is the set of all outermost equations of type $\tau = (2)$, i.e. the set of all equations $s \approx t$ such that the first variables in s and in t and the last variables in s and in t agree. Therefore $RB \subseteq V$ means that all identities in V are outermost and for any $s \approx t \in Id$ we have $\hat{\sigma}_1[s] = \{$ first variable in $s\} = \{$ first variable in $t\} = \hat{\sigma}_1[t]$ and $\hat{\sigma}_2[s] = \{$ last variable in s$\} = \{$ last variable in t$\} = \hat{\sigma}_2[t]$. Clearly $s \approx t$ is closed under σ_{pid}.

Conversely, let V be a nontrivial $M - nd$-solid variety. Then $\sigma_1, \sigma_2 \in M$ requires $RB \subseteq V$. ■

Let $var(B)$ be the set of all variables occurring in the set B of terms. Now let

$$M' = \{\sigma \in Hyp^{nd}(\tau) \mid var(\sigma(f)) = \{x\}\}.$$

Clearly $M' \cup \{\sigma_{pid}\}$ forms a submonoid of $Hyp^{nd}(\tau)$. Then we have

Proposition 4.2. *A non-trivial variety V of semigroups is $M' - nd$-solid iff $LZ \subseteq V \subseteq B$.*

Proof. It is well-known that $IdLZ$ is the set of all equations $s \approx t$ of type $\tau = (2)$ such that the first variable in s is equal to the first variable in t. Because of $var(\sigma(f)) = \{x\}$ the terms in $\hat{\sigma}[s]$ and the terms in $\hat{\sigma}[t]$ can be written as x^r and as x^l for some $r, l \in \mathbb{N}^+$. Since $V \subseteq B$ by the idempotent law all equations of the form $x^r \approx x^l$ are satisfied in V. This shows that V is $M' - nd$-solid.

Conversely, let V be a nontrivial $M' - nd$-solid variety of semigroups. If we apply σ with $\sigma(f) = \{x, x^2\}$ to the identity $f(x,y) \approx f(x,y)$ we obtain $x \approx x^2$, i.e. $V \subseteq B$. If we apply σ' with $\sigma'(f) = \{x\}$ we get $leftmost(s) \approx leftmost(t) \in IdV$ and this means $LZ \subseteq V$. Altogether, we have $LZ \subseteq V \subseteq B$. ■

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Received 30 May 2006
Revised 15 June 2007