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# REMARKS TO GŁAZEK'S RESULTS ON $n$-ARY GROUPS

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**Abstract**

This is a survey of the results obtained by K. Głazek and his co-workers. We restrict our attention to the problems of axiomatizations of $n$-ary groups, classes of $n$-ary groups, properties of skew elements and homomorphisms induced by skew elements, constructions of covering groups, classifications and representations of $n$-ary groups. Some new results are added too.

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## 1. Introduction

Ternary and $n$-ary generalizations of algebraic structures are the most natural ways for further development and deeper understanding of their fundamental properties. First ternary algebraic operations were introduced already in the XIXth century by A. Cayley. As a development of Cayley's ideas they were considered $n$-ary generalization of matrices and their determinants and general theory of $n$-ary algebras [5, 48], $n$-group rings [62] and algebras [61]. For some physical applications in Nambu mechanics, supersymmetry, Yang-Baxter equation, etc. see e.g. [58].

On the other hand, Hopf algebras and their ternary generalizations play the basic role in the quantum group theory. On one of the L. Gluskin's seminars (in '60s of the past century) B. Gleichgewicht, a friend of K. Głazek, had familiarized himself with the theory of $n$-ary systems. It was him who brought the idea of researching such structures to Wrocław where at that time a group of algebraists lead by E. Marczewski was active. Some time later (in '70s and '80s) a group of mathematicians interested in $n$-ary systems gathered around the Głazek's and Gleichgewicht's algebraic seminar at the Institute of Mathematics of Wrocław University. Constructive discussions on this seminar resulted later in many articles of such authors like (in alphabetic order): W.A. Dudek, B. Gleichgewicht, J. Michalski, I. Sierocki, M.B. Wanke-Jakubowska and M.E. Wanke-Jerie. The first bibliography of $n$-groups and some group-like $n$-ary systems [28] prepared by K. Głazek in 1983 was based on the work of this seminar.

Below we present a short survey of the results of K. Głazek's and his co-workers. We also present few theorems of other authors and add several new unpublished results. We restrict our attention to the problems of axiomatizations of $n$-ary groups, classes of $n$-ary groups, properties of skew elements and homomorphisms induced by skew elements, constructions of covering groups, classifications and representations of $n$-ary groups. We finish our survey with results on independent sets of $n$-ary groups contained in the article [16], which is probably the last Glazek's article.

## 2. Preliminaries

The non-empty set $G$ together with an $n$-ary operation $f : G^n \to G$ is called an *$n$-ary groupoid* or an *$n$-ary operative* and is denoted by $(G; f)$.

According to the general convention used in the theory of such groupoids the sequence of elements $x_i, x_{i+1}, \ldots, x_j$ is denoted by $x_i^j$. In the case $j < i$ this symbol is empty. If $x_{i+1} = x_{i+2} = \ldots = x_{i+t} = x$, then instead of $x_{i+1}^{i+t}$ we write $\overset{(t)}{x}$. In this convention $f(x_1, \ldots, x_n) = f(x_1^n)$ and

$$f(x_1, \ldots, x_i, \underbrace{x, \ldots, x}_{t}, x_{i+t+1}, \ldots, x_n) = f\left(x_1^i, \overset{(t)}{x}, x_{i+t+1}^n\right).$$

If $m = k(n-1)+1$, then the m-ary operation $g$ of the form

$$g\left(x_1^{k(n-1)+1}\right) = \underbrace{f(f(...f(f}_{k}(x_1^n), x_{n+1}^{2n-1}), ...), x_{(k-1)(n-1)+2}^{k(n-1)+1}\Big)$$

is denoted by $f_{(k)}$ and is called the *long product* of $f$ or an $m$-ary operation *derived* from $f$. In certain situations, when the arity of $g$ does not play a crucial role, or when it will differ depending on additional assumptions, we write $f_{(.)}$, to mean $f_{(k)}$ for some $k = 1, 2, ....$

An $n$-ary groupoid $(G; f)$ is called $(i, j)$-*associative* or an $(i, j)$-*associative* if

$$f\left(x_1^{i-1}, f\left(x_i^{n+i-1}\right), x_{n+i}^{2n-1}\right) = f\left(x_1^{j-1}, f\left(x_j^{n+j-1}\right), x_{n+j}^{2n-1}\right) \tag{1}$$

holds for all $x_1, \ldots, x_{2n-1} \in G$. If this identity holds for all $1 \leqslant i < j \leqslant n$, then we say that the operation $f$ is *associative* and $(G; f)$ is called an *$n$-ary semigroup* or, in the Gluskin's terminology, an *$n$-ary associative* (cf. [41]). In the binary case (i.e., for $n = 2$) it is an arbitrary semigroup.

If for all $x_0, x_1, \ldots, x_n \in G$ and fixed $i \in \{1, \ldots, n\}$ there exists an element $z \in G$ such that

$$f\left(x_1^{i-1}, z, x_{i+1}^{n}\right) = x_0 \tag{2}$$

then we say that this equation is *$i$-solvable* or *solvable at the place $i$*. If this solution is unique, then we say that (2) is *uniquely $i$-solvable*.

An $n$-ary groupoid $(G; f)$ uniquely solvable for all $i = 1, \ldots, n$ is called an *$n$-ary quasigroup*. An associative $n$-ary quasigroup is called an *$n$-ary group*. Note that for $n = 2$ it is an arbitrary group.

The idea of investigations of such groups seems to be going back to E. Kasner's lecture [43] at the fifty-third annual meeting of the American Association for the Advancement of Science in 1904. But the first paper containing the first important results was written (under inspiration of Emmy Noether) by W. Dörnte in 1928 (cf. [7]). In this paper Dörnte observed that any $n$-ary groupoid $(G; f)$ of the form $f(x_1^n) = x_1 \circ x_2 \circ \ldots \circ x_n$, where $(G, \circ)$ is a group, is an $n$-ary group but for every $n > 2$ there are $n$-ary groups which are not of this form. $n$-Ary groups of the first form are called *reducible* or *derived* from the group $(G, \circ)$, the second - *irreducible*.

Moreover in some $n$-ary groups there exists an element $e$ (called an *$n$-ary identity* or a *neutral element*) such that

$$f\left(\overset{(i-1)}{e}, x, \overset{(n-i)}{e}\right) = x$$

holds for all $x \in G$ and for all $i = 1, \ldots, n$. It is interesting that $n$-ary groups containing neutral element are reducible (cf. [7]). Irreducible $n$-ary groups do not contains such elements. On the other hand, there are $n$-ary groups with two, three and more neutral elements. The set $Z_n = \{0, 1, \ldots, n-1\}$ with the operation $f(x_1^{n+1}) = (x_1 + x_2 + \ldots + x_{n+1})(\mathrm{mod}\, n)$ is an example of an $n$-ary group in which all elements are neutral. The set of all neutral elements of a given $n$-ary group (if it is non-empty) forms an $n$-ary subgroup (cf. [14] or [26]).

It is worth to note that in the definition of an $n$-ary group, under the assumption of the associativity of $f$, it suffices to postulate the existence of a solution of (2) at the places $i = 1$ and $i = n$ or at one place $i$ other than 1 and $n$. Then one can prove uniqueness of the solution of (2) for all $i = 1, \ldots, n$ (cf. [49], p. 213[17]).

On the other hand, Sokolov proved in [54] that in the case of $n$-ary quasigroups (i.e., in the case of the existence of a unique solution of (2) for any place $i = 1, \ldots, n$) it is sufficient to postulate the $(j, j+1)$-associativity for some fixed $j = 1, \ldots, n-1$. Basing on the Sokolov's method W.A. Dudek, K. Głazek and B. Gleichgewicht proved in 1997 the following proposition (for details see [17]).

**Proposition 1** (W.A. Dudek, K. Głazek, B. Gleichgewicht, 1997)**.** *An $n$-ary groupoid $(G; f)$ is an $n$-ary group if and only if (at least) one of the following conditions is satisfied:*

(a) *the $(1,2)$-associative law holds and the equation (2) is solvable for $i = n$ and uniquely solvable for $i = 1$,*

(b) *the $(n-1, n)$-associative law holds and the equation (2) is solvable for $i = 1$ and uniquely solvable for $i = n$,*

(c) *the $(i, i+1)$-associative law holds for some $i \in \{2, ..., n-2\}$ and the equation (2) is uniquely solvable for $i$ and some $j > i$.*

This result was generalized by W.A. Dudek and I. Groździńska (cf. [18]) and independently by N. Celakoski (cf. [6]) in the following way:

**Proposition 2** (N. Celakoski, 1977, W.A. Dudek, I. Groździńska, 1979)**.** *An $n$-ary semigroup $(G; f)$ is an $n$-ary group if and only if for some $1 \leqslant k \leqslant n-2$ and all $a_1^k \in G$ there are elements $x_{k+1}^{n-1}$, $y_{k+1}^{n-1} \in G$ such that*

$$f\left(a_1^k, x_{k+1}^{n-1}, b\right) = f\left(b, y_{k+1}^{n-1}, a_1^k\right) = b$$

*for all $b \in G$.*

The above two propositions and methods used in the proofs gave the impulse to further study the axiomatics of $n$-ary groups (cf. [50, 51, 56, 57] and many others). From different results obtained by various authors we select one simple characterization proved in [23].

**Proposition 3** (A.M. Gal'mak, 1995)**.** *An $n$-ary semigroup $(G; f)$ is an $n$-ary group if and only if for some $1 \leqslant i, j \leqslant n-1$ and all $a, b \in G$ there are $x, y \in G$ such that*

$$f\left(x, \overset{(i-1)}{b}, \overset{(n-i)}{a}\right) = f\left(\overset{(n-j)}{a}, \overset{(j-1)}{b}, y\right) = b.$$

Note that in some papers there were investigated so-called *infinitary* ($n = \infty$) semigroups and quasigroups, i.e., groupoids $(G; f)$, where for all natural $i, j$ the operation $f : G^\infty \to G$ satisfies the identity

$$f\left(x_1^{i-1}, f\left(x_i^\infty\right), y_1^\infty\right) = f\left(x_1^{j-1}, f\left(x_j^\infty\right), y_1^\infty\right)$$

and the equation $f(x_1^{k-1}, z_k, x_{k+1}^\infty) = x_0$ has a unique solution $z_k$ at any place $k$.

From the general results obtained in [2] and [44] one can deduce that infinitary groups have only one element. Below we present a simple proof of this fact.

If $(G; f)$ is an infinitary group, then, according to the definition, for any

$y, z \in G$ and $u = f(\overset{(\infty)}{y})$ there exists $x \in G$ such that $z = f(u, y, x, \overset{(\infty)}{y})$.

Thus

$$f\left(z, \overset{(\infty)}{y}\right)$$

$$= f\left(f\left(u, y, x, \overset{(\infty)}{y}\right), \overset{(\infty)}{y}\right) = f\left(u, y, f\left(x, \overset{(\infty)}{y}\right), \overset{(\infty)}{y}\right)$$

$$= f\left(f\left(\overset{(\infty)}{y}\right), y, f\left(x, \overset{(\infty)}{y}\right), \overset{(\infty)}{y}\right) = f\left(y, f\left(\overset{(\infty)}{y}\right), y, f\left(x, \overset{(\infty)}{y}\right), \overset{(\infty)}{y}\right)$$

$$= f\left(y, u, y, f\left(x, \overset{(\infty)}{y}\right), \overset{(\infty)}{y}\right) = f\left(y, f\left(u, y, x, \overset{(\infty)}{y}\right), \overset{(\infty)}{y}\right) = f\left(y, z, \overset{(\infty)}{y}\right),$$

i.e., for all $y, z \in G$ we have

$$f\left(z, \overset{(\infty)}{y}\right) = f\left(y, z, \overset{(\infty)}{y}\right).$$

Using this identity and the fact that for all $x, y \in G$ there exists $z \in G$ such that $x = f(z, \overset{(\infty)}{y})$, we obtain

$$f\left(\overset{(\infty)}{x}\right) = f\left(x, f\left(z, \overset{(\infty)}{y}\right), \overset{(\infty)}{x}\right) = f\left(x, f\left(y, z, \overset{(\infty)}{y}\right), \overset{(\infty)}{x}\right)$$

$$= f\left(x, y, f\left(z, \overset{(\infty)}{y}\right), \overset{(\infty)}{x}\right) = f\left(x, y, \overset{(\infty)}{x}\right),$$

which, by the uniqueness of the solution at the second place, implies $x = y$. So, $G$ has only one element.

To avoid repetitions in the sequel we consider only the case when $n$ is a natural number higher than 2, but a part of our results is also true for $n = 2$.

## 3. VARIETIES OF $n$-ARY GROUPS

Directly from the definition of an $n$-ary group $(G; f)$ we can see that for every $x \in G$ there exists only one $z \in G$ satisfying the equation

$$f\left(\overset{(n-1)}{x}, z\right) = x. \tag{3}$$

This element is called *skew* to $x$ and is denoted by $\overline{x}$. In a ternary group $(n = 3)$ derived from the binary group $(G, \circ)$ the skew element coincides with the inverse element in $(G, \circ)$. Thus, in some sense, the skew element is a generalization of the inverse element in binary groups. This suggests that for $n \geqslant 3$ any $n$-ary group $(G; f)$ can be considered as an algebra $(G; f, ^-)$ with two operations: one $n$-ary $f : G^n \to G$ and one unary $^-\colon x \to \overline{x}$.

In ternary groups, as it was proved by W. Dörnte (cf. [7]), we have $\overline{f(x, y, z)} = f(\overline{z}, \overline{y}, \overline{x})$ and $\overline{\overline{x}} = x$, but for $n > 3$ it is not true. For $n > 3$ there are $n$-ary groups in which one fixed element is skew to all elements (cf. [12]) and $n$-ary groups in which any element is skew to itself. Then, of course, $f(\overset{(n)}{x}) = x$ for every $x \in G$. Such $n$-ary groups are called *idempotent*.

Nevertheless, the concept of skew elements plays a crucial role in the theory of $n$-ary groups. Namely, as W. Dörnte proved, in any $n$-ary group $(G; f)$ for all $x, y \in G$, $2 \leqslant i, j \leqslant n$ and $1 \leqslant k \leqslant n$ we have

$$f\left(\overset{(i-2)}{x}, \overline{x}, \overset{(n-i)}{x}, y\right) = y, \tag{4}$$

$$f\left(y, \overset{(n-j)}{x}, \overline{x}, \overset{(j-2)}{x}\right) = y. \tag{5}$$

These two identities, called now *Dörnte's identities*, are used by many authors in description of a class of $n$-ary groups.

**Theorem 4** (B. Gleichgewicht, K. Głazek, 1967)**.** *An algebra $(G; f, ^-)$ with one associative $n$-ary $(n > 2)$ operation $f$ and one unary operation $^-\colon x \mapsto \overline{x}$ is an $n$-ary group if and only if the conditions* (4) *and* (5) *are satisfied for all $x, y \in G$ and $i = j = 2, 3$.*

It is the first important characterization of the variety of $n$-ary groups. For example, basing on this theorem it is not difficult to see that the function

$$h(x,y,z) = f\left(x, \overline{y}, \overset{(n-3)}{x}, z\right)$$

is the so-called Mal'cev operation. So, the class of all $n$-groups (for any fixed $n > 2$) is a Mal'cev variety, all congruences of a given $n$-ary group commutes and the lattice of all congruences of a fixed $n$-ary group is modular. Moreover, the generalized Zassenhaus Lemma and the generalized Schreier and Hölder-Jordan Theorems hold in any $n$-ary group (cf. [32]). Schreier varieties of $n$-ary groups are described by V. A. Artamonov (see [1]).

Unfortunately, the above system of identities defining the variety of $n$-ary groups is not independent. The first independent system of identities selecting the variety of $n$-ary groups from the variety of $n$-ary semigroups was given by K. Głazek and his coauthors ten years later (cf. [17]). Below we present the minimal system of identities defining this variety. It is the main result of [9].

**Theorem 5** (W.A. Dudek, 1980)**.** *The class of $n$-ary groups coincides with the variety of $n$-ary groupoids $(G; f, ^-)$ with an unary operation $^-\colon x \to \overline{x}$ for some fixed $i, j \in \{2, \dots, n\}$ satisfying the identities* (4), (5) *and the identity*

$$f\Big(f(x_1^n), x_{n+1}^{2n-1}\Big) = f\Big(x_1, f(x_2^{n+1}), x_{n+2}^{2n-1}\Big).$$

Theorem 5 is valid for $n > 2$, but this theorem can be extended to the case $n = 2$. Namely, let $\widehat{\ }\colon x \to \widehat{x}$ be an unary operation, where $\widehat{x}$ is a solution of the equation

$$f_{(2)}\left(\overset{(2n-2)}{x}, \widehat{x}\right) = x.$$

Then using the same method as in the proof of Theorem 2 in [17] we can prove the following result announced in [9].

**Theorem 6** (W.A. Dudek, 1980)**.** *The class of $n$-ary ($n \geqslant 2$) groups coincides with the variety of algebras $(G; f, \widehat{\ })$ with one associative $n$-ary operation $f$ and one unary operation $\widehat{\ }\colon x \to \widehat{x}$ satisfying for some fixed $i, j \in \{2, \dots, n\}$ the following identities*

$$f_{(2)}\left(y, \overset{(i-2)}{x}, \widehat{x}, \overset{(2n-i-1)}{x}\right) = y = f_{(2)}\left(\overset{(2n-1-j)}{x}, \widehat{x}, \overset{(j-2)}{x}, y\right).$$

Other systems of identities defining the variety of $n$-ary groups one can find in [13] and [57].

## 4. SKEW ELEMENTS

An *$n$-ary power* of $x$ in an $n$-ary group $(G; f)$ is defined in the following way: $x^{<0>} = x$ and $x^{<k+1>} = f(\overset{(n-1)}{x}, x^{<k>})$ for all $k > 0$. $x^{<-k>}$ is an element $z$ such that $f(x^{<k-1>}, \overset{(n-2)}{x}, z) = x^{<0>} = x$. Then $\overline{x} = x^{<-1>}$, $\overline{\overline{x}} = x^{<n-3>}$ and

$$f\left(x^{<k_1>}, \dots, x^{<k_n>}\right) = x^{<k_1+\dots+k_n+1>} \tag{6}$$

$$\left(x^{<k>}\right)^{<t>} = x^{<kt(n-1)+k+t>}. \tag{7}$$

Now, putting $\overline{x}^{(0)} = x$ and denoting by $\overline{x}^{(s+1)}$ the skew element to $\overline{x}^{(s)}$, we obtain the sequence of elements: $x$, $\overline{x}^{(1)}, \overline{x}^{(2)}, \overline{x}^{(3)}, \overline{x}^{(4)}$ and so on. In a 4-ary group derived from the additive group $\mathbb{Z}_8$ we have $\overline{x} \equiv 6x (\mathrm{mod}\, 8)$, $\overline{\overline{x}} \equiv 4x (\mathrm{mod}\, 8)$ and $\overline{x}^{(s)} = 0$ for every $s > 2$, but in an $n$-ary group derived from the additive group of integers $\overline{x}^{(s)} \neq \overline{x}^{(t)}$ for all $s \neq t$. Any subgroup containing $x$ contains also $\overline{x}$ and all $\overline{x}^{(s)}$. The order of the smallest subgroup containing $x$ is called the *$n$-ary order of $x$* and is denoted by $\mathrm{ord}_n(x)$. It is the smallest positive integer $k$ such that $x^{<k>} = x$ (cf. [49]). Obviously

$$\mathrm{ord}_n(x) \geqslant \mathrm{ord}_n\left(\overline{x}\right) \geqslant \mathrm{ord}_n\left(\overline{x}^{(2)}\right) \geqslant \mathrm{ord}_n\left(\overline{x}^{(3)}\right) \geqslant \dots$$

In fact $\mathrm{ord}_n(\overline{x})$ is a divisor of $\mathrm{ord}_n(x)$.

In connection with this K. Głazek posed in 1978 the following question:

**Question 1.** *When $ord_n(x) = \mathrm{ord}_n(\overline{x})$ ?*

The first partial answer was given in [59]: *If an $n$-ary group $(G; f)$ has a finite order relatively prime to $n-2$, then $ord_n(x) = \mathrm{ord}_n(\overline{x})$ for all $x \in G$.* The full answer was found two years later (cf. [8]): *in the case when $ord_n(x)$ is finite, $ord_n(x) = \mathrm{ord}_n(\overline{x})$ if and only if $ord_n(x)$ and $n-2$ are relatively prime.*

Examples of infinite $n$-ary groups in which all elements have the same $n$-ary order $k > 1$ are given in [10]. Such groups are a set-theoretic union of disjoint isomorphic subgroups of order $k$. The skew element is idempotent, i.e., $\mathrm{ord}_n(\overline{x}) = 1$, if and only if $\mathrm{ord}_n(x)$ is a divisor of $n-2$ (cf. [8]). If an $n$-ary group has a finite order $g$ and every prime divisor of $g$ is a divisor of $n-2$, then for every element $x$ of this group there exists a natural number $t$ such that $\mathrm{ord}_n(\overline{x}^{(t)}) = 1$.

**Theorem 7** (I.M. Dudek, W.A. Dudek, 1981)**.** *Let* $ord_n(x) = p_1^{\beta_1} p_2^{\beta_2} \dots p_m^{\beta_m}$, *where* $p_1, p_2 \dots, p_m$ *are primes. Then* $\lim_{t\to\infty} \mathrm{ord}_n(\overline{x}^{(t)}) = 1$ *or* $\lim_{t\to\infty} \mathrm{ord}_n(\overline{x}^{(t)}) = p_1^{\beta_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, *where primes* $p_1, p_2 \dots, p_k$, $k \leqslant m$, *are not divisors of* $n-2$.

In [10] it is proved that $\overline{x}^{(m)} = x^{<S_m>}$, where $S_m = \frac{(2-n)^m - 1}{n-1}$. So, $\overline{x}^{(m)} = x$ if and only if $\mathrm{ord}_n(x)$ divides $S_m$. The natural question is: *for which $n$-ary groups there exists fixed $m \in \mathbb{N}$ such that $\overline{x}^{(m)} = \overline{y}^{(m)}$ for all $x, y \in G$* (cf. [11]). For $m = 1$ the full answer is given in [12]. For $m > 1$ we have only a partial answer. Namely, an $n$-ary group $(G; f)$ in which $\overline{x}^{(m)} = \overline{y}^{(m)}$ holds for all $x, y \in G$ is torsion free and its exponent is a divisor of $S_m^2 - S_m$ (cf. [53]).

An $n$-ary group is said to be *semiabelian* if it satisfies the identity

$$f\Big(x_1^n\Big) = f\Big(x_n, x_2^{n-1}, x_1\Big).$$

The class of all semiabelian $n$-ary groups coincides with the class of *medial* (*entropic*) $n$-ary groups, i.e., $n$-ary groups satisfying the identity

$$f\Big(f\big(x_{11}^{1n}\big), f\big(x_{21}^{2n}\big), \dots, f\big(x_{n1}^{nn}\big)\Big) = f\Big(f\big(x_{11}^{n1}\big), f\big(x_{12}^{n2}\big), \dots, f\big(x_{1n}^{nn}\big)\Big).$$

This means that the matrix $[x_{ij}]_{n\times n}$ can be read by rows or by columns.

Some authors use also the term *abelian* instead of *semiabelian* and consider such $n$-ary groups as a special case of the so-called abelian (commutative) general algebras. This implies that every $n$-ary subgroup of a semiabelian $n$-ary group is a block of some congruence of this group and the lattice of all $n$-ary subgroups of this group is modular.

An $n$-ary group $(G; f)$ is semiabelian if and only if there exists an element $a \in G$ such that

$$f\left(x, \overset{(n-2)}{a}, y\right) = f\left(y, \overset{(n-2)}{a}, x\right)$$

for all $x, y \in G$ (cf. [9]). This means that for $n = 2$, a semiabelian $n$-ary (i.e., binary) group is commutative. For $n > 2$ a commutative $n$-ary group is defined as a group in which $f(x_1^n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for any permutation $\sigma$ of $\{1, 2, \ldots, n\}$. The class of commutative $n$-ary groups is described by J. Timm (cf. [55]).

K. Głazek and B. Gleichgewicht observed in [32] that in any semiabelian $n$-ary group we have

$$\overline{f(x_1^n)} = f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n), \tag{8}$$

which means that in these $n$-groups the operation $^-\colon x \to \overline{x}$ is an endomorphism. Also $h^s(x) = \overline{x}^{(s)}$ is an endomorphism. The converse is not true. So, in semiabelian $n$-ary groups

$$G^{(s)} = \{\overline{x}^{(s)} | \, x \in G\}$$

is an $n$-ary subgroup for every natural $s$. (It is an $n$-ary subgroup in any $n$-ary group satisfying (8), not necessary semiabelian.) Obviously $G \supset G^{(1)} \supset G^{(2)} \ldots$ It is clear that for any finite $n$-ary group $(G; f)$ there exists $t \in \mathbb{N}$ such that $G^{(s)} = G^{(t)}$ for $s \geqslant t$. On the other hand, an $n$-ary group derived from the additive group of integers is a simple example of $n$-ary group in which $G^{(s)} \neq G^{(t)}$ for all $s \neq t$ ($G^{(s)}$ contains all integers divided by $(n-2)^s$. The question (Problem 5 in [11]) on the characterization of classes of $n$-ary groups satisfying the descending chain condition for $G^{(s)}$ is open.

Note that in some $n$-ary groups $(G; f)$ the operation $^-\colon x \to \overline{x}$ induces a cyclic subgroup in the group of all automorphisms of $(G; f)$ (cf. [11]). Moreover this subgroup is invariant in the group $Aut(G; f)$ and in the group of all splitting-automorphisms in the sense of Płonka (cf. [47]), i.e., automorphisms satisfying the identity $h(f(x_1^n)) = f(x_1^{i-1}, h(x_i), x_{i+1}^n)$.

The natural question is:

**Question 2.** *When the operation* $^-\colon x \to \overline{x}$ *is an endomorphism?*

The first answer was given in [14]:

**Proposition 8** (W.A. Dudek, 2001)**.** *The operation* $^-\colon x \to \overline{x}$ *is an endomorphism of an $n$-ary group $(G; f)$ if and only if there exists an element $a \in G$ such that*

(i) $\overline{f(x,a,\dots,a,y)} = f(\overline{x},\overline{a},\dots,\overline{a},\overline{y})$,

(ii) $\overline{f(\overline{a},x,a,\dots,a)} = f(\overline{\overline{a}},\overline{x},\overline{a},\dots,\overline{a})$,

(iii) $\overline{f(\overline{a},\overline{a},\dots,\overline{a})} = f(\overline{\overline{a}},\overline{\overline{a}},\dots,\overline{\overline{a}})$

*for all* $x, y \in G$.

The last condition can be omitted. Indeed, using (6), (7) and the fact that $\overline{a} = a^{<-1>}$ and $\overline{\overline{a}} = a^{<n-3>}$ for every $a \in G$, it is not difficult to see that the left and right side of (iii) are equal to $a^{<n^2-3n+1>}$. So, (iii) is valid in any $n$-ary group.

**Corollary 9.** *The operation* $^- : x \to \overline{x}$ *is an endomorphism of an* $n$*-ary group* $(G; f)$ *if and only if the equations*

(i) $\overline{f(x,a,\dots,a,y)} = f(\overline{x},\overline{a},\dots,\overline{a},\overline{y})$,

(ii) $\overline{f(\overline{a},x,a,\dots,a)} = f(\overline{\overline{a}},\overline{x},\overline{a},\dots,\overline{a})$,

*hold for all* $x, y \in G$ *and some fixed* $a \in G$.

Another answer is given in [53] and [52]:

**Proposition 10** (F.M. Sokhatsky, 2003)**.** *The operation* $^- : x \to \overline{x}$ *is an endomorphism of an* $n$*-ary group* $(G; f)$ *if and only if the following two identities are satisfied:*

$$f\Big(\overset{(n-1)}{u}, f\Big(\overset{(n-2)}{x}, u, u\Big)\Big) = f\Big(f\Big(\overset{(n-2)}{x}, u, u\Big), \overset{(n-1)}{u}\Big),$$

$$f\Big(f\Big(x, \overset{(n-2)}{u}, y\Big), \dots, f\Big(x, \overset{(n-2)}{u}, y\Big), u, u\Big) =$$

$$= f\Big(\overset{(n-2)}{y}, f\Big(u, f\Big(x, \overset{(n-1)}{u}\Big), \dots, f\Big(x, \overset{(n-1)}{u}\Big), x, u\Big), u\Big).$$

**Theorem 11** (N.A. Shchuchkin, 2006)**.** *For odd $k$, the operation $^{-(k)} : x \to \overline{x}^{(k)}$ is an endomorphism of an $n$-ary group $(G; f)$ if and only if the identity:*

$$f_{(n-1)}\left(x_1, \overset{((n-2)^k)}{x_2}, \overset{((n-2)^k)}{x_3}, \ldots, \overset{((n-2)^k)}{x_{n+1}}, x_{n+2}\right) =$$

$$= f(x_1, \underbrace{f(x_{n+1}, x_n, ..., x_2), \ldots, f(x_{n+1}, x_n, ..., x_2)}_{(n-2)^k \quad times}, x_{n+2}).$$

*is satisfied.*

**Theorem 12** (N.A. Shchuchkin, 2006)**.** *For even $k$, the operation $^{-(k)} : x \to \overline{x}^{(k)}$ is an endomorphism of an $n$-ary group $(G; f)$ if and only if the identity:*

$$f_{(\cdot)}(\underbrace{f(x_1^n), \ldots, f(x_1^n)}_{(n-2)^k}) = f_{(\cdot)}\left(\overset{((n-2)^k)}{x_1}, \overset{((n-2)^k)}{x_2}, \ldots, \overset{((n-2)^k)}{x_n}\right)$$

*is satisfied.*

## 5. Hosszú-Gluskin algebras

Let $(G; f)$ be an $n$-ary group. Fixing in $f(x_1^n)$ some $k < n$ elements we obtain a new $(n-k)$-ary operation which in general is not associative. It is associative only in the case when these fixed elements are located in some special places (cf. [19]).

Binary operations of the form $x * y = f(x, a_2^{n-1}, y)$, where elements $a_2, \ldots, a_{n-1} \in G$ are fixed play a very important role. It is not difficult to see that $(G; *)$ is a group. Fixing different elements $a_2, \ldots, a_{n-1}$ we obtain different groups. Since all these groups are isomorphic (cf. [19]) we can consider only one group $(G, \circ)$, where $x \circ y = f(x, \overset{(n-2)}{a}, y)$. This group is denoted by $ret_a(G; f)$ and is called a *binary retract* of $(G; f)$. The identity of this group is $\overline{a}$. The inverse element to $x$ has the form

$$x^{-1} = f\left(\overline{a}, \overset{(n-3)}{x}, \overline{x}, \overline{a}\right).$$

An $n$-ary group $(G; f)$ is semiabelian only in the case when its binary retract $ret_a(G; f)$ is commutative.

The strong connection between $n$-ary groups and their binary retracts was observed for the first time in 1963 by M. Hosszú (cf. [42]). He proved the following theorem:

**Theorem 13.** *An $n$-ary groupoid $(G; f)$, $n > 2$, is an $n$-ary group if and only if*

(i) *on $G$ one can define a binary operation $\cdot$ such that $(G; \cdot)$ is a group,*

(ii) *there exist an automorphism $\varphi$ of $(G; \cdot)$ and $b \in G$ such that $\varphi(b) = b$,*

(iii) $\varphi^{n-1}(x) = b \cdot x \cdot b^{-1}$ *holds for every $x \in G$,*

(iv) $f(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \cdot \varphi^3(x_4) \cdot \ldots \cdot \varphi(x_n)^{n-1} \cdot b$ *for all $x_1, \ldots, x_n \in G$.*

Two years later, this theorem was proved by L. M. Gluskin in more general form (cf. [41]). Now this theorem is known as the Hosszú-Gluskin Theorem. Some important generalization of this theorem one can find in [19, 54] and [57].

The algebra $(G; \cdot, \varphi, b)$ of the type $(2, 1, 0)$, where $(G; \cdot)$ is a (binary) group, $b \in G$ is fixed, $\varphi \in Aut(G; \cdot)$, $\varphi(b) = b$ and $\varphi^{n-1}(x) = b \cdot x \cdot b^{-1}$ for every $x \in G$ is called a *Hosszú-Gluskin algebra* (briefly: an *$HG$-algebra*). We say that an $HG$-algebra $(G; \cdot, \varphi, b)$ is *associated* with an $n$-ary group $(G; f)$ if the the last condition of Theorem 13 is satisfied. In this case we also say that an $n$-ary group $(G; f)$ is *$\langle \varphi, b \rangle$-derived* from the group $(G; \cdot)$. If $\varphi(x) = x$ for every $x \in G$ we say that $(G; f)$ is *$b$-derived* from $(G; \cdot)$. The systematical study of connections between $n$-ary groups $b$-derived from given binary group was initiated by K. Głazek and J. Michalski in [34] and continued by J. Michalski, W.A. Dudek, M. Pop and many others. Connections between $(G; f)$ and $n$-ary groups $\langle \varphi, b \rangle$-derived from $ret_a(G; f)$ are described by W.A. Dudek and J. Michalski in [19, 20, 21]. All commutative $n$-ary groups are $b$-derived from some of their retracts (cf. [55]).

Let $\mathfrak{G} = (G; f, ^-)$ be a semiabelian $n$-ary group. Then the $HG$-algebra associated with $\mathfrak{G}$ has a commutative group operation denoted by $+$.

Let $\mathfrak{H} = (G; +, \varphi, b)$ be associated with $\mathfrak{G}$ and $\mathfrak{G}_a = (G; f, ^-, a)$. Then $\mathfrak{H}$ and $\mathfrak{G}_a$ are term equivalent (cf. [16]) and

$$-y = f\left(\overline{a}, \overset{(n-3)}{y}, \overline{y}, \overline{a}\right),$$

$$x + y = f\left(x, \overset{(n-3)}{(-y)}, \overline{(-y)}, \overline{a}\right),$$

$$\varphi(x) = f\left(\overline{a}, x, \overset{(n-2)}{a}\right),$$

$$\text{and} \quad b = f\left(\overset{(n)}{\overline{a}}\right).$$

To describe all term operations of $\mathfrak{G}_a$ by using the language of $HG$-algebras we denote by $g_i$ the following operation

$$g_i(x) = k_{i1}\varphi^{l_{i1}}(x) + k_{i2}\varphi^{l_{i2}}(x) + \ldots + k_{it}\varphi^{l_{it}}(x), \tag{9}$$

where $t, l_{i1}, \ldots, l_{it}$ are non-negative integers and $k_{i1}, \ldots, k_{it} \in \mathbb{Z}$.

**Theorem 14** (W.A. Dudek, K. Głazek, 2006)**.** *Let $\mathfrak{H} = (G; +, \varphi, b)$ be the $HG$-algebra associated with a semiabelian $n$-ary group $\mathfrak{G}$. Then all unary term operations of $\mathfrak{H}$ (and of $\mathfrak{G}_a$) are of the form*

$$g(x) = g_i(x) + k_g b$$

*for some $g_i$ of the form* (9) *and $k_g \in \mathbb{Z}$.*

**Theorem 15** (W.A. Dudek, K. Głazek, 2006)**.** *Let $\mathfrak{H} = (G; +, \varphi, b)$ be the $HG$-algebra associated with a semiabelian $n$-ary group $\mathfrak{G}$. Then all $m$-ary term operations of $\mathfrak{H}$ (and of $\mathfrak{G}_a$) are of the form*

$$F(x_1, \ldots, x_m) = \sum_{i=1}^{m} g_i(x_i) + k_F b \tag{10}$$

*for some $g_i$ of the form* (9) *and $k_F \in \mathbb{Z}$.*

The Hosszú-Gluskin Theorem was used by K. Głazek and his co-workers to calculation of $n$-ary groups of some special types. For example, in [33] K. Głazek and B. Gleichgewicht calculated all ternary semigroups and groups for which their ternary operation $f$ is a ternary polynomial over an infinite commutative integral domain with identity. In the case when $f$ is a ternary

polynomial over an infinite commutative field $K$ the operation $f$ has the form $f(x,y,z) = x+y+z+d$ or $f(x,y,z) = axyz$, where $a \neq 0$ and $d$ are fixed. In the first case $(K,f)$ is a ternary group, in the second $(K-\{0\},f)$.

**Theorem 16** (K. Głazek, J. Michalski, 1984)**.** *Let $m$ be odd and let $(G;f)$ be an $n$-ary group. Then the operation $f$ has the form*

$$f(x_1^n) = x_1 \cdot x_2^{-1} \cdot x_3 \cdot \ldots \cdot x_{n-1}^{-1} \cdot x_n,$$

*where $(G;\cdot)$ is an abelian group, if and only if $f$ is idempotent and*

$$f\left(x_1^i, y, y, x_{i+3}^n\right) = f\left(x_1^i, z, z, x_{i+3}^n\right)$$

*for all $0 \leqslant i \leqslant n-2$. In this case $(G;\cdot) = ret_a(G;f)$ for some $a \in G$.*

**Theorem 17** (K. Głazek, J. Michalski, 1984)**.** *Let $(G;\cdot)$ be a group and let $t_1, \ldots, t_n$ be fixed integers. Then $G$ with the operation*

$$f(x_1^m) = (x_1)^{t_1} \cdot (x_2)^{t_2} \cdot \ldots \cdot (x_{n-1})^{t_{n-1}} \cdot (x_n)^{t_n},$$

*is an $n$-ary group if and only if*

(1) $x^{t_1} = x = x^{t_n}$,

(2) $t_j = k^j$ *for some integer $k$ and all $j = 2, \ldots, n-1$,*

(3) $(x \cdot y)^k = x^k \cdot y^k$.

In this case we say that $(G;f)$ is derived from the $k$-exponential group. Basing on the results obtained in [20] one can prove

**Proposition 18** (W.A. Dudek, K. Głazek, 2006)**.** *An $n$-ary group $(G;f)$ is derived from the $k$-exponential $(k>0)$ group $(G;\cdot)$ if and only if in $(G;f)$ there exists an idempotent $a$ such that*

$$f_{(k)}\left(\overset{(n-2)}{a}, x, \overset{(n-2)}{a}, x, \ldots, \overset{(n-2)}{a}, x, a\right) = x$$

*for all $x \in G$. In this case $(G;\cdot) = ret_a(G;f)$.*

The following theorem proved in [19] plays the fundamental role in the calculation of non-isomorphic $n$-ary groups.

**Theorem 19** (W.A. Dudek, J. Michalski, 1984). *Two $n$-ary groups $(G_1, f_1)$, $(G_2, f_2)$ are isomorphic if and only if for some $a \in G_1$ and $b \in G_2$ there exists an isomorphism $h : ret_a(G_1, f_1) \to ret_b(G_2, f_2)$ such that*

$$h(a) = b,$$

$$h\Big(f_1(\overline{a}, \ldots, \overline{a})\Big) = f_2\Big(\overline{b}, \ldots, \overline{b}\Big),$$

$$h\left(f_1\left(\overline{a}, x, \overset{(n-2)}{a}\right)\right) = f_2\left(\overline{b}, h(x), \overset{(n-2)}{b}\right).$$

This theorem reduces the problem of the calculation of non-isomorphic $n$-ary groups to the classification of their binary retracts. As an illustration of results obtained by K. Głazek and his co-workers we present the complete list of $n$-ary groups $\langle\varphi, b\rangle$-derived from cyclic groups.

Let $(\mathbb{Z}_k, +)$ by the cyclic group modulo $k$. Consider the following three $n$-ary operations:

$$f_a(x_1^n) \equiv (x_1 + \ldots + x_n + a) \,(\mathrm{mod}\, k),$$

$$g_d(x_1^n) \equiv (x_1 + dx_2 + \ldots + d^{n-2}x_{n-1} + x_n) \,(\mathrm{mod}\, k),$$

$$g_{d,c}(x_1^n) \equiv (x_1 + dx_2 + \ldots + d^{n-2}x_{n-1} + x_n + c) \,(\mathrm{mod}\, k),$$

where $a \in \mathbb{Z}_k$, $c, d \in \mathbb{Z}_k \setminus \{0, 1\}$, $d^{n-1} \equiv 1 \,(\mathrm{mod}\, k)$. Additionally, for the operation $g_{d,c}$ we assume that $dc \equiv c \,(\mathrm{mod}\, k)$ holds. By Theorem 13, $(\mathbb{Z}_k, f_a)$, $(\mathbb{Z}_k, g_d)$ and $(\mathbb{Z}_k, g_{d,c})$ are $n$-ary groups.

**Theorem 20** (K. Głazek, J. Michalski, I. Sierocki, 1984)**.** *A $k$-element $n$-ary group $(G; f)$ is $\langle\varphi, b\rangle$-derived from the cyclic group of order $k$ if and only if it is isomorphic to exactly one $n$-ary group of the form $(\mathbb{Z}_k, f_a)$, $(\mathbb{Z}_k, g_d)$ or $(\mathbb{Z}_k, g_{d,c})$, where $d|gcd(k, n-1)$ and $c|k$.*

An infinite cyclic group can be identified with the group $(\mathbb{Z}, +)$. This group has only two automorphisms: $\varphi(x) = x$ and $\varphi(x) = -x$. So, according to

Theorem 13, $n$-ary groups defined on $\mathbb{Z}$ have the form $(\mathbb{Z}, f_a)$ or $(\mathbb{Z}, g_{-1})$, where

$$g_{-1}(x_1^n) = x_1 - x_2 + x_3 - x_4 + \ldots + x_n,$$

and $n$ is odd. Since $n$-ary groups $(\mathbb{Z}, f_a)$ and $(\mathbb{Z}, f_b)$ are isomorphic for $a \equiv b \,(\mathrm{mod}\,(n-1))$, we have $n-1$ non-isomorphic $n$-ary groups of this form. Isomorphisms have the form $\varphi_k(x) = x - k$.

So, we have proved

**Theorem 21** (W.A. Dudek, K. Głazek, 2006)**.** *An $n$-ary group $\langle \varphi, b\rangle$-derived from the infinite cyclic group $(\mathbb{Z}, +)$ is isomorphic to $n$-ary group $(\mathbb{Z}, f_a)$, where $0 \leqslant a \leqslant (n-2)$, or with $(\mathbb{Z}, g_{-1})$, where $n$ is odd.*

A very similar result, but in incorrect form, was firstly formulated in [38].

Denote by $Inn\,(G;\cdot)$ the group of all inner automorphisms of $(G;\cdot)$, by $Out\,(G;\cdot)$ the factor group of $Aut\,(G;\cdot)$ by $Inn\,(G;\cdot)$, and by $Out_n\,(G;\cdot)$ the set of all cosets $\overline{\gamma} \in Out\,(G;\cdot)$ containing $\gamma$ such that $\gamma^{n-1} \in Inn\,(G;\cdot)$. Then, the number of pairwise non-isomorphic $n$-ary groups $\langle \varphi, b\rangle$-derived from a centerless group $(G;\cdot)$, i.e., a group for which $|Cent\,(G;\cdot)\,| = 1$, is smallest or equal to $s =\mid Out_n\,(G;\cdot)\mid$. It is equal to $s$ if and only if $Out\,(G;\cdot)$ is abelian (for details see [38]).

For every $n$ and $k \neq 2, 6$, there exists exactly one $n$-ary group which is $\langle \varphi, b\rangle$-derived from $S_k$ (for $k = 2$ and $k = 6$ we have one or two such $n$-ary groups relatively to evenness of $n$).

Any finite group is uniquely determined by its multiplication table which in fact is a Latin square. In the case of $n$-ary groups the role of multiplication tables play $n$-dimensional cubes. So, the problem of enumeration of all finite $n$-ary groups can be reduced to the problem of enumeration of the corresponding cubes. But it is rather difficult problem. The better approach was suggested by K. Głazek and J. Michalski. They proposed a method based on the Hosszú-Gluskin Theorem and our Theorem 19, i.e., the calculation of all non-isomorphic $n$-ary groups by the classification of their retracts and automorphisms of these retracts. Using this method they obtain in [35, 36, 37] a full classification of all non-isomorphic $n$-ary groups on sets with at most 7 elements. The complete list of such $n$-groups (with some comments) one can find in [16].

## 6. COVERING GROUP

A binary group $(G^*, \circ)$ is said to be a *covering group* for the $n$-ary group $(G; f)$ if there exists an embedding $\tau : G \to G^*$ such that $\tau(G)$ is a generating set of $G^*$ and $\tau(f(x_1^n)) = \tau(x_1) \circ \tau(x_2) \circ \ldots \circ \tau(x_n)$ for every $x_1, \ldots, x_n \in G$. $(G^*, \circ)$ is a *universal covering group* (or a *free covering group*) if for any covering group $(G^\bullet, \diamond)$ there exists a homomorphism from $G^*$ onto $G^\bullet$ such that the following diagram is commutative:

$$\begin{array}{ccc} & G & \\ \swarrow & \circlearrowleft & \searrow \\ G^* & - - \to & G^\bullet \\ & \text{onto} & \end{array}$$

E.L. Post proved in [49] that for every $n$-ary group $(G; f)$ there exist a covering group $(G^*, \circ)$ and its normal subgroup $G_0$ such that $G^*/G_0$ is a cyclic group of order $n-1$ and $f(x_1^n) = x_1 \circ x_2 \circ \ldots \circ x_n$ for all $x_1, \ldots, x_n \in G$, where $G$ is identified with the generator of the group $G^*/G_0$. So, the theory of $n$-ary groups is closely related to the theory of *cyclic extensions of groups*, but these theories are not equivalent.

Indeed, the above Post's theorem shows that for any $n$-ary group $(G; f)$ we have the sequence

$$O \longrightarrow (G_0, \circ) \longrightarrow (G^*, \circ) \xrightarrow{\zeta} (\mathbb{Z}_n, +_n) \longrightarrow O,$$

where $(G^*, \circ)$ is the free covering group of $(G; f)$ and $G = \zeta^{-1}(1)$. We have also

$$\begin{array}{ccccc} & & (G_1^*, \circ) & & \\ & \nearrow & \uparrow & \searrow & \\ (G_0, \circ) & \circlearrowleft & \circlearrowleft\circlearrowleft & & (\mathbb{Z}_n, +_n) \\ & \searrow & \downarrow & \nearrow & \\ & & (G_2^*, \circ) & & \end{array}$$

where we use

- $\circlearrowleft$ for the equivalence of extensions,
- $\circlearrowleft$ for the isomorphism of suitable $n$-ary groups.

Of course, two $n$-ary groups determined in the above-mentioned sense by two equivalent cyclic extensions are isomorphic. However, as observed K. Głazek, two non-equivalent cyclic extensions can determine two isomorphic $n$-ary groups. Below we give an example of two non-equivalent cyclic extensions induced by two isomorphic 4-ary groups.

**Example 1** (W.A. Dudek, K. Głazek, 2006)**.** Consider two cyclic extensions of the cyclic group $\mathbb{Z}_3$ by $\mathbb{Z}_3$:

$$0 \longrightarrow \mathbb{Z}_3 \xrightarrow{\alpha} \mathbb{Z}_9 \xrightarrow{\beta_1} \mathbb{Z}_3 \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z}_3 \xrightarrow{\alpha} \mathbb{Z}_9 \xrightarrow{\beta_2} \mathbb{Z}_3 \longrightarrow 0,$$

where the homomorphisms $\alpha$, $\beta_1$ and $\beta_2$ are given by:

$$\alpha(x) = 3x \qquad \text{for} \quad x \in \mathbb{Z}_3,$$

$$\beta_1(x) = x(\mathrm{mod}\, 3) \qquad \text{for} \quad x \in \mathbb{Z}_9,$$

$$\beta_2(x) = 2x(\mathrm{mod}\, 3) \quad \text{for} \quad x \in \mathbb{Z}_9.$$

It is easy to verify that the sets $\beta_1^{-1}(1) = \{1, 4, 7\}$ and $\beta_1^{-1}(1) = \{2, 5, 8\}$ with the operation $g(x, y, z, v) = (x+y+z+v)(\mathrm{mod}\, 9)$ are 4-ary groups. These 4-ary groups are isomorphic. The isomorphism $\varphi : (\beta_1^{-1}(1), g) \longrightarrow (\beta_2^{-1}(1), g)$ has the form $\varphi(x) = 2x(\mathrm{mod}\, 9)$. Nevertheless, the above-mentioned extensions are not equivalent because there is no automorphism $\lambda$ of $\mathbb{Z}_9$ such that $\lambda \circ \alpha = \alpha$ and $\beta_2 \circ \lambda = \beta_1$.

The Post's construction of a covering group for given $n$-ary group $(G; f)$ is based on the following equivalence relation defined on the set of all finite sequences of elements of $G$:

$$x_1^k \sim y_1^m \Longleftrightarrow f_{(\cdot)}\Big(z_1^s, x_1^k, u_1^t\Big) = f_{(\cdot)}\Big(z_1^s, y_1^m, u_1^t\Big)$$

for some $z_1^s, u_1^t \in G$. Since $k \equiv m(\mathrm{mod}(n-1))$, each sequence is equivalent with some sequence of the length $i = 1, 2, \ldots, n-1$. Defining on the set $G^*$ of such obtained equivalence classes the operation $[x_1^i] * [y_1^j] = [x_1^i y_1^j]$ we obtain a covering group of $(G; f)$ (for details see [49]).

Basing on this method K. Głazek and B. Gleichgewicht proposed in [31] (see also [27]) other more simple construction of a covering group for ternary group. Namely, for a ternary group $(G; f)$ they consider the set $G^2 \cup G$ with the operation

$$x \circ y = \langle x, y\rangle$$

$$x \circ \langle y, z\rangle = f(x, y, z)$$

$$\langle x, y\rangle \circ z = f(x, y, z)$$

$$\langle x, y\rangle \circ \langle z, u\rangle = \langle f(x, y, z), u\rangle$$

and the relation $\sim$ defined as follows:

$$x \sim y \Longleftrightarrow x = y$$

$$\langle x, y\rangle \sim \langle x', y'\rangle \Longleftrightarrow (\exists a, b \in G)\ (f(x', a, b) = x \wedge f(a, b, y) = y).$$

Then $\sim$ is a congruence on the algebra $(G^2 \cup G, \circ)$ and $(G^2 \cup G/\sim, \tilde{\circ})$, where $\tilde{\circ}$ is the operation induced by $\circ$, is a universal covering group for $(G; f)$. Obviously this construction can be extended to an arbitrary $n$-ary group, but, as it was observed by W.A. Dudek at the Głazek's seminar in 1977, for $n > 3$ this construction coincides with the Post's construction. Nevertheless this construction gives the impulse to the nice construction presented by J. Michalski (cf. [45]).

**Theorem 22** (J. Michalski, 1981)**.** *Let $c$ ba an arbitrary fixed element of an $n$-ary group $(G; f)$. Then the set $G \times \mathbb{Z}_{n-1}$ with the operation*

$$\langle x, s\rangle * \langle y, t\rangle = \left\langle f\left(x, \overset{(s)}{c}, y, \overset{(t)}{c}, \overline{c}, \overset{(n-1-s\diamond t)}{c}\right), s \diamond t\right\rangle,$$

*where $s\diamond t = (s+t+1)(\mathrm{mod}(n-1))$, is a universal covering group for $(G; f)$.*

It is clear that $G \times \{n-2\}$ is a normal subgroup of $G \times \mathbb{Z}_{n-1}$. The set of all classes (in the Post's construction) induced by the sequences of the length $i+1$ can be identified with $G \times \{i\}$. So, the set $G$ can be identified with $G \times \{0\}$, i.e., $\tau(G)$ is a coset of $G \times \mathbb{Z}_{n-1}$ modulo $G \times \{n-1\}$. Since for $\tau(x) = \langle x, 0\rangle$ we have $\tau(f(x_1^n)) = \tau(x_1) * \tau(x_2) * \ldots * \tau(x_n)$, an $n$-ary group $(G; f)$ can be identified with an $n$-ary group derived from $G \times \{0\}$.

In [19] (see also [20]) it is proved that for every $c \in G$ the groups $ret_c(G; f)$ and $G \times \{n-1\}$ are isomorphic.

## 7. Representations

Let $(G; g, \tilde{})$ be a general algebra with one $n$-ary operation $g$ and one unary operation $\tilde{}: G \to G$. If $\psi, \varphi_1, \ldots, \varphi_n : G \to H$ are homomorphism of $(G; g, \tilde{})$ into a semiabelian $n$-ary group $(H; f, ^-)$, then, as it was observed by K. Głazek and B. Gleichgewicht in [32], the mapping

$$\widehat{f}(\varphi_1, \ldots, \varphi_n) : G \to H$$

defined by

$$\widehat{f}(\varphi_1,\dots,\varphi_n)(x) = f(\varphi_1(x),\dots,\varphi_n(x))$$

and the mapping $\widehat{\psi} : G \to H$ defined by

$$\widehat{\psi}(x) = \overline{\psi(x)}$$

are also homomorphisms of $(G; g, ^{\sim})$ into $(H; f, ^{-})$. Moreover, the algebra $(F; \widehat{f}, ^{\widehat{}})$ of all homomorphisms of $(G; g, ^{\sim})$ into $(H; f, ^{-})$ belongs to the same variety as the algebra $(H; f, ^{-})$, so it is semiabelian too. The set of all endomorphisms of a semiabelian $n$-ary group forms an $(n,2)$-nearring $(E; \widehat{f}, \circ)$ with unity, where $\circ$ is the superposition of endomorphisms.

Recall that an abstract algebra $(E; f, \cdot)$ is an $(n,2)$-nearring if $(E; f)$ is an $n$-ary group, $(E; \cdot)$ is a binary semigroup and the following two identities

$$y \cdot f(x_1^n) = f(y \cdot x_1, y \cdot x_2, \dots, y \cdot x_n),$$

$$f(x_1^n) \cdot y = f(x_1 \cdot y, x_2 \cdot y, \dots, x_n \cdot y)$$

hold. In the case when $(E; f)$ is a commutative $n$-ary group an $(n,2)$-nearring is called an $(n,2)$-*ring*. Every $(n,2)$-ring $(E; f, \cdot)$ with a cancellable element (with respect to the multiplication $\cdot$ ) is isomorphic to an $(n,2)$-ring of some endomorphisms of the $n$-ary group $(E; f)$ (cf. [32]).

Further study of homomorphisms of $n$-ary groups was continued by A.M. Gal'mak (cf. for example [22, 24, 25]). Most of his results are based on the Głazek's observation (cf. [29] or [30]) that every weak homomorphism between commutative $n$-ary groups is an ordinary homomorphism and the following Post's construction of polyadic substitutions.

As it is well known an ordinary substitution, finite or infinite, is a one-to-one map from the set $A$ onto $A$. Let $A_1, A_2, \dots, A_{n-1}$ be a finite sequence of sets of the same cardinality. The sequence $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{n-1})$ of maps

$$\sigma_1 : A_1 \to A_2, \quad \sigma_2 : A_2 \to A_3, \ \dots, \ \sigma_{n-1} : A_{n-1} \to A_1$$

is called a *$n$-ary substitution*. The superposition of two $n$-ary substitutions is the sequence $\tau = (\tau_1, \tau_2, \dots, \tau_{n-1})$ of maps

$$\tau_1 : A_1 \to A_3, \quad \tau_2 : A_2 \to A_4, \ \dots, \ \tau_{n-2} : A_{n-2} \to A_1, \quad \tau_{n-1} : A_{n-1} \to A_2.$$

The set of $n$-ary substitutions is closed with respect to the superposition of $n$ such substitutions. In fact it is an $n$-ary group with respect to this

operation. Moreover, each $n$-ary group is isomorphic to the $n$-ary group of substitutions of some set $A$ (cf. [49]). Each $n$-ary group is isomorphic to the $n$-ary group of some translations too [22].

Let now $(A; f)$ be an $n$-ary group. The Cartesian product $A^{n-1}$ endowed with the $n$-ary operation $g$ defined as the skew product in the matrix $[a_{ij}]_{n\times(n-1)}$, i.e.,

$$g((a_{11}, a_{12}, \dots, a_{1\,n-1}), (a_{21}, a_{22}, \dots, a_{2\,n-1}), \dots, (a_{n1}, a_{n2}, \dots, a_{n\,n-1}))$$

$$= (f(a_{11}, a_{22}, a_{33}, \dots, a_{n-1\,n-1}, a_{n1}), f(a_{12}, a_{23}, a_{34}, \dots, a_{n-2\,n-1}, a_{n-1\,1}, a_{n2}),$$

$$\dots, f(a_{1\,n-1}, a_{21}, a_{32}, \dots, a_{n-1\,n-2} a_{n\,n-1})),$$

is an $n$-ary group (cf. [49]) which is called *diagonal*. The diagonal $n$-ary group of invertible linear transformations of a complex vector space is used in [40] to the description of one-dimensional representations of cyclic $n$-groups. The invariant subspaces of a representation $\rho$, sequences of $\rho$-invariant subspaces, the covering representation $\widehat{\rho}$ and the relations between $\rho$ and $\widehat{\rho}$ are discussed in [60].

Matrix representations of ternary groups are described in [4] (see also [3]). In these representations each matrix is determined by two elements. Below we present the general concept of such representations.

Let $V$ be a complex vector space and $\operatorname{End} V$ denotes a set of $\mathbb{C}-$linear endomorphisms of $V$.

**Definition 23.** A *left bi-element representation* of an $n$-ary group $(G; f)$ in a vector space $V$ is a map $\Pi^L : G^{n-1} \to \operatorname{End} V$ such that

$$\Pi^L\left(a_1^{n-1}\right) \circ \Pi^L\left(b_1^{n-1}\right) = \Pi^L\left(f\left(a_1^{n-1}, b_1\right), b_2^{n-1}\right), \tag{11}$$

$$\Pi^L\left(\overset{(n-2)}{a}, \overline{a}\right) = \mathrm{id}_V \tag{12}$$

for all $a, a_1, \dots, a_{n-1}, b_1, \dots, b_{n-1} \in G$.

Note that the axioms considered in the above definition are the natural ones satisfied by left multiplications $x \mapsto f(a_1^{n-1}, x)$.

Using (12) and the associativity of the operation $f$ it is not difficult to see that

$$\Pi^L\left(f\left(a_1^{n-1}\right), a_n^{2n-2}\right) = \Pi^L\left(a_1^i, f\left(a_{i+1}^{n+i-1}\right), a_{n+i}^{2n-2}\right)$$

for all $a_1, \ldots, a_{2n-2} \in G$ and $i = 1, 2, \ldots, n-1$. Moreover, from (12), by (5), we have also

$$\Pi^L\left(\overset{(j-1)}{a}, \overline{a}, \overset{(n-j-1)}{a}\right) = \mathrm{id}_V$$

for all $a \in G$ and $j = 1, 2, \ldots, n-1$.

**Lemma 24.** *A left bi-element representation of an $n$-ary group is uniquely determined by two elements.*

***Proof.*** According to the definition of an $n$-ary group for every $a, a_1, \ldots, a_{n-1}, b \in G$ there exists $c \in G$ such that

$$f\left(a_1^{n-1}, a\right) = f\left(\overset{(n-2)}{b}, c, a\right).$$

So,

$$\begin{aligned}\Pi^L\left(a_1^{n-1}\right) &= \Pi^L\left(a_1^{n-1}\right) \circ \Pi^L\left(\overset{(n-2)}{a}, \overline{a}\right) = \Pi^L\left(f\left(a_1^{n-1}, a\right), \overset{(n-3)}{a}, \overline{a}\right)\\ &= \Pi^L\left(f\left(\overset{(n-2)}{b}, c, a\right), \overset{(n-3)}{a}, \overline{a}\right) = \Pi^L\left(\overset{(n-2)}{b}, c\right) \circ \Pi^L\left(\overset{(n-2)}{a}, \overline{a}\right)\\ &= \Pi^L\left(\overset{(n-2)}{b}, c\right),\end{aligned}$$

which completes the proof. ∎

**Corollary 25.** $\Pi^L\left(a_1^{n-1}\right) = \Pi^L\left(b_1^{n-1}\right) \Longleftrightarrow f\left(a_1^{n-1}, a\right) = f\left(b_1^{n-1}, a\right) \forall a \in G.$

**Proposition 26.** *Let $(G; f)$ be an $n$-ary group derived from a binary group $(G; \odot)$. There is one-to-one correspondence between representations of $(G; \odot)$ and left bi-element representations of $(G; f)$.*

***Proof.*** Because $(G; f)$ is derived from $(G; \odot)$, then

$$x \odot y = f\left(x, \overset{(n-2)}{e}, y\right) \quad \text{and} \quad \overline{e} = e,$$

where $e$ is the identity of $(G;\odot)$. If $\pi$ is a representation of $(G;\odot)$, then (as it is not difficult to see)

$$\Pi^L\left(x_1^{n-1}\right) = \pi(x_1)\circ\pi(x_2)\circ\ldots\circ\pi(x_{n-1}) \tag{13}$$

is a left bi-element representation of $(G;f)$. Conversely, if $\Pi^L$ is a left bi-element representation of $(G;f)$, then

$$\pi(x) = \Pi^L\left(\overset{(n-2)}{e},x\right)$$

is a representation of $(G;\odot)$ and (13) is satisfied. ■

**Proposition 27.** *Any left bi-element representations of an $n$-ary group $(G;f)$ induces a representation of its retract.*

***Proof.*** Let $(G;\odot) = ret_a(G;f)$ for some fixed $a \in G$. According to Lemma 24, for all $a_1,\ldots,a_{n-1}\in G$ there exists $c\in G$ such that

$$\Pi^L\left(a_1^{n-1}\right) = \Pi^L\left(\overset{(n-2)}{a},c\right) = \pi(c).$$

Then $\pi(\overline{a}) = \mathrm{id}_V$ and

$$\pi(x)\circ\pi(y) = \Pi^L\left(\overset{(n-2)}{a},x\right)\circ\Pi^L\left(\overset{(n-2)}{a},y\right) = \Pi^L\left(f\left(\overset{(n-2)}{a},x,a\right),\overset{(n-3)}{a},y\right)$$

$$= \Pi^L\left(\overset{(n-2)}{a},f\left(x,\overset{(n-2)}{a},y\right)\right) = \Pi^L\left(\overset{(n-2)}{a},x\odot y\right) = \pi(x\odot y),$$

which proves that $\pi$ is a representation of $(G;\odot)$. ■

For ternary groups also the converse statement is true: every representation of $ret_a(G;f)$ induces a left bi-element representation of $(G;f)$ (cf. [4]). All such bi-element representations are invertible.

**Definition 28.** A *right bi-element representation* of an $n$-ary group $(G;f)$ in $V$ is a map $\Pi^R : G^{n-1}\to \mathrm{End}\ V$ such that

$$\Pi^R\Big(a_1^{n-1}\Big)\circ\Pi^R\Big(b_1^{n-1}\Big)=\Pi^R\Big(a_1^{n-2},f\Big(a_{n-1},b_1^{n-1}\Big)\Big),$$

$$\Pi^R\left(\overset{(n-2)}{a},\overline{a}\right)=\mathrm{id}_V$$

for all $a,a_1,\ldots,a_{n-1},b_1,\ldots,b_{n-1}\in G$.

It is clear that left and right bi-element representations are dual.

**Example 2.** Let $(G;f)$ be an $n$-ary group and $\mathbb{C}G$ denote a vector space spanned by $G$. It means that any element $u$ of $\mathbb{C}G$ can be uniquely presented in the form $u=\sum_{i=1}^m k_iy_i$, with $k_i\in\mathbb{C}$, $y_i\in G$, $m\in N$ (we do not assume that $G$ has finite rank). Moreover, $\mathbb{C}G$ is an $n$-ary (group) algebra (for details see [61] or [62]). Then left and right bi-element regular representations can be immediately defined by means of this structure

$$\Pi^L_{reg}\Big(a_1^{n-1}\Big)u=\sum_{i=1}^m k_if\Big(a_1^{n-1},y_i\Big), \tag{14}$$

$$\Pi^R_{reg}\Big(a_1^{n-1}\Big)u=\sum_{i=1}^m k_if\Big(y_i,a_1^{n-1}\Big). \tag{15}$$

Left (right) regular bi-element representations of a ternary group are unitary [4].

**Example 3.** Consider an $n$-ary group $(G;f)$, where $n$ is odd, $G=\mathbb{Z}_3=\{0,1,2\}$ and $f(x_1^n)=\sum_{i=1}^n(-1)^{i+1}x_i(\mathrm{mod}\,3)$. It is clear that

$$\Pi^L\Big(x_1^{n-1}\Big)=\Pi^R(x_{n-1},x_{n-2},\ldots,x_2,x_1).$$

From the proof of Lemma 24 it follows that it is sufficient to consider the representations of the form $\Pi^L\big(\overset{(n-2)}{a},b\big)$. By Corollary 25 in our case

$$\Pi^L\left(\overset{(n-2)}{a},b\right)=\Pi^L\left(\overset{(n-2)}{c},d\right)\Longleftrightarrow(a-b)=(c-d)(\mathrm{mod}\,3).$$

Straightforward calculations give the left regular representation in the manifest matrix form, where $\Pi^L(a,b)$ means $\Pi^L\big(\overset{(n-2)}{a},b\big)$.

$$\Pi^L_{reg}(0,0) = \Pi^L_{reg}(2,2) = \Pi^L_{reg}(1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [1] \oplus [1] \oplus [1],$$

$$\Pi^L_{reg}(2,0) = \Pi^L_{reg}(1,2) = \Pi^L_{reg}(0,1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= [1] \oplus \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = [1] \oplus \left[-\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right] \oplus \left[-\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right],$$

$$\Pi^L_{reg}(2,1) = \Pi^L_{reg}(1,0) = \Pi^L_{reg}(0,2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= [1] \oplus \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = [1] \oplus \left[-\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right] \oplus \left[-\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right].$$

Observe that for an $n$-ary (group) algebra $\mathbb{C}G$ from Example 2 we can consider additionally middle representations of the form

$$\Pi^M_{reg}\left(a_1^{n-1}\right)u = \sum_{i=1}^{m} k_i f\left(a_1^{j-1}, y_i, a_j^{n-1}\right),$$

where $j = 2, 3, \ldots, n-1$ is fixed.

Since in this case we obtain complicated formulas we restrict our attention to middle bi-element representations of ternary groups described in [3] and [4].

**Definition 29.** A *middle bi-element representation* of a ternary group $(G; f)$ in $V$ is a map $\Pi^M : G \times G \to \operatorname{End} V$ such that

$$\Pi^M(a_3,b_3)\circ\Pi^M(a_2,b_2)\circ\Pi^M(a_1,b_1)=\Pi^M\left(f(a_3,a_2,a_1),f(b_1,b_2,b_3)\right),$$

$$\Pi^M(a,b)\circ\Pi^M\left(\overline{a},\overline{b}\right)=\Pi^M\left(\overline{a},\overline{b}\right)\circ\Pi^M(a,b)=\mathrm{id}_V$$

for all $a,a_1,a_2,a_3,b,b_1,b_2,b_3\in G$.

The composition of two middle bi-element representations is not a middle representation, but in some cases described in [4] it is a left bi-element representation. Obviously, $\Pi^M(a,b)=\Pi^M(c,d)$ if and only if $f(a,y,b)=f(c,y,d)$ for every $y\in G$.

**Example 4.** Let $G=\mathbb{Z}_3$ and $f(x,y,z)=(x-y+z)(\mathrm{mod}\,3)$. Then $(G;f)$ is a ternary group in which

$$\Pi^M(a,b)=\Pi^M(c,d)\Longleftrightarrow(a+b)=(c+d)(\mathrm{mod}\,3).$$

So,

$$\Pi^M_{reg}(0,0)=\Pi^M_{reg}(1,2)=\Pi^M_{reg}(2,1)=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}=[1]\oplus\begin{bmatrix}-1&0\\0&1\end{bmatrix},$$

$$\Pi^M_{reg}(0,1)=\Pi^M_{reg}(1,0)=\Pi^M_{reg}(2,2)=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}=[1]\oplus\begin{bmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{bmatrix},$$

$$\Pi^M_{reg}(0,2)=\Pi^M_{reg}(2,0)=\Pi^M_{reg}(1,1)=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}=[1]\oplus\begin{bmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{bmatrix}.$$

This representation $\Pi^M_{reg}$ is equivalent to the orthogonal direct sum of two irreducible representations, i.e., one-dimensional trivial and two-dimensional.

In this example for all $x$ we have $\Pi^M(x,\overline{x}) = \Pi^M(x,x) \neq \mathrm{id}_V$, but $\Pi^M(x,y) \circ \Pi^M(x,y) = \mathrm{id}_V$ for all $x, y$.

Putting in the above example $\gamma^M_{k+l} = \Pi^M_{reg}(k,l)$ for $k, l \in \mathbb{Z}_3$, we obtain the identity

$$\gamma^M_i \circ \gamma^M_j \circ \gamma^M_k = \gamma^M_{f(i,j,k)}, \quad i,j,k \in \mathbb{Z}_3,$$

which in some sense can be treated as a *ternary analog of Clifford algebra* [3]. Any matrix representation of this identity gives rise to the middle representation $\Pi^M(k,l) = \gamma_{k+l}$.

Different connections between middle bi-element representations of a ternary group $(G; f)$ and representations of its retract and covering group are described in [4]. For example, any middle bi-element representation of a ternary group $(G; f)$ derived from a group $(G; \cdot)$ has the form $\Pi^M(a,b) = \pi(a) \circ \rho(b^{-1})$, where $\pi(x) = \Pi^M(x,e)$ and $\rho(x) = \Pi^M(e,\overline{x})$ are pairwise commuting representations of $(G; \cdot)$.

## 8. $\mathcal{Q}$-INDEPENDENT SETS IN $HG$-ALGEBRAS

Let $\mathfrak{A} = (A; \mathbb{F})$ be an algebra $\emptyset \neq X \subseteq A$. The set $X$ is said to be $\mathcal{M}$-*independent* if

$$(\forall n \in \mathbb{N},\ n \leqslant card(X))\ \Big(\forall f,g \in \mathbb{T}^{(n)}(\mathfrak{A})\Big)\ (\forall \underbrace{a_1,\ldots,a_n}_{\neq} \in X)$$
$$\Big[f(a_1^n) = g(a_1^n) \Longrightarrow f = g\Big].$$

This condition is equivalent to each of the following ones:

(a) $(\forall n \in \mathbb{N}, n \leqslant card(X))(\forall f,g \in \mathbb{T}^{(n)}(\mathfrak{A}))(\forall p : X \to A)(\forall a_1,\ldots,a_n \in X)$

$$\Big[f(a_1^n) = g(a_1^n) \Longrightarrow f(p(a_1),\ldots,p(a_n)) = g(p(a_1),\ldots,p(a_n))\Big],$$

(b) $(\forall p \in A^X)\ (\exists \bar{p} \in Hom(\langle X\rangle_{\mathfrak{A}}, \mathfrak{A}))\ \bar{p}|_X = p$, where $\langle X\rangle_{\mathfrak{A}}$ is a subalgebra of $\mathfrak{A}$ generated by $X$.

The notion of $\mathcal{M}$-independence is stronger than that of independence with respect to the closure operator of such a kind $X \mapsto \langle X\rangle_{\mathfrak{A}}$ (for $X \subseteq A$).

Let $\emptyset \neq X \subseteq A$ and

$$\mathcal{Q}_X \subseteq A^X = \mathcal{M}_X = \{p \mid p : X \to A\},$$

$$\mathcal{Q}(A) = \mathcal{Q} = \bigcup\{\mathcal{Q}_X \mid X \subseteq A\},$$

$$\mathcal{M}(A) = \mathcal{M} = \bigcup\{A^X \mid X \subseteq A\}.$$

For an algebra $\mathfrak{A} = (A; \mathbb{F})$, a mapping $p : X \to A$ belongs to $\mathcal{H}_X(\mathfrak{A})$ if and only if there exists a homomorphism $\bar{p} : \langle X\rangle_{\mathfrak{A}} \to A$ such that $\bar{p}|_X = p$.

The set $X$ is said to be $\mathcal{Q}$-*independent* if $\mathcal{Q}_X \subseteq \mathcal{H}_X(\mathfrak{A})$ or, equivalently,

$$(\forall p \in \mathcal{Q}_X)\ (\forall \text{ finite } n \leqslant card(X))\ \Big(\forall f, g \in \mathbb{T}^{(n)}(\mathfrak{A})\Big)\ (\forall a_1, \ldots, a_n \in X)$$

$$\Big[f(a_1^n) = g(a_1^n) \Longrightarrow f(p(a_1), \ldots, p(a_n)) = g(p(a_1), \ldots, p(a_n))\Big].$$

In the case when $\mathcal{Q} = \bigcup\{p|_X \mid p \in A^A,\ X \subseteq A\}$ and

$$\Big(\forall f, g \in \mathbb{T}^{(1)}(\mathfrak{A})\Big)\ (\forall a \in A)\ \big[f(a) = g(a) \Longrightarrow f(p(a)) = g(p(a))\big],$$

the set $X$ is said to be $\mathcal{G}$-*independent*.

For commutative groups, the notion of $\mathcal{G}$-independence gives us the well-known *linear independence*.

For $HG$-algebras of type $\mathfrak{H} = (G; +, \varphi, b)$, where $(G; +)$ is a commutative group, the equality

$$F_1(x_1, \ldots, x_m) = F_2(x_1, \ldots, x_m)$$

(for two term operations of the form (10) in $\mathfrak{H}$) is equivalent to the equality

$$H(x_1, \ldots, x_m) = 0,$$

where $H \in \mathbb{T}^{(m)}(\mathfrak{H})$, i.e., $H(x_1, \ldots, x_m) = \sum_{i=1}^{m} g_i(x_i) + k_{_H} b$, and 0 denotes the zero of the group $(G; +)$.

Consider a subset $X$ of $G$. Let for $a_1, \ldots, a_m \in X$ the equality

$$H(a_1, \ldots, a_m) = 0$$

hold. Taking into account the mapping $p : X \to \langle X\rangle_{\mathfrak{A}}$ defined by $p(a_i) = 0$ and $p(x) = x$ for $x \in X \setminus \{a_1, \ldots, a_m\}$, we get $k_{_H} b = 0$.

Therefore

$$\sum_{i=1}^{m} g_i(a_i) = 0.$$

Consider the mapping $q_j : X \to \langle X \rangle_{\mathfrak{A}}$ defined for fixed $j \in \{1, \ldots, m\}$ as follows:

$$q_j(x) = \begin{cases} a_j & \text{if} \quad x = a_j, \\ 0 & \text{if} \quad x \neq a_j. \end{cases}$$

We obtain $g_j(a_j) = 0$ for all $j = 1, 2, \ldots, m$. (In the considered case all $q_j$ belong to $\mathcal{M}$ and $\mathcal{G}$ .)

In particular, we can easily observe, by similar considerations, that the following result holds:

**Theorem 30.** (W.A. Dudek, K. Głazek, 2006)**.** *Let $X \subseteq G$ be a subset of the HG-algebra $\mathfrak{H} = (G; +, \varphi, b)$. Then $X$ is $\mathcal{G}$-independent if and only if for any $m \leqslant card(X)$ for all $a_1, \ldots, a_m \in X$ and every term operation $H(x_1, \ldots, x_m) = \sum_{i=1}^{m} g_i(x_i) + k_{_H} b$ the equality*

$$\sum_{i=1}^{m} g_i(a_i) + k_{_H} b = 0 \tag{16}$$

*is equivalent with*

$$(\forall i \in \{1, \ldots, m\})\ (g_i(a) = 0\ \&\ k_{_H} b = 0)\,.$$

*Moreover, $X$ is $\mathcal{M}$-independent in this HG-algebra if and only if for all pairwise different elements $a_1, \ldots, a_m$ from $X$ equality* (16) *implies $g_i(x) = 0$ for all $i = 1, 2, \ldots, m$ and $k_H b = 0$.*

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