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PSEUDOCOMPLEMENTS IN SUM-ORDERED PARTIAL SEMIRINGS

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Abstract

We study a particular way of introducing pseudocomplementation in ordered semigroups with zero, and characterise the class of those pseudocomplemented semigroups, termed g-semigroups here, that admit a Glivenko type theorem (the pseudocomplements form a Boolean algebra). Some further results are obtained for g-semirings – those sum-ordered partially additive semirings whose multiplicative part is a g-semigroup. In particular, we introduce the notion of a partial Stone semiring and show that several well-known elementary characteristics of Stone algebras have analogues for such semirings.

Keywords: Glivenko theorem, partial monoid, partial semiring, pseudocomplementation, semigroup, Stone semiring, sum-ordering.

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1. Introduction

In [3], the present author introduced the notion of a semiring-like logic (SR-logic). It was defined to be a bounded sum-ordered partially additive semiring (see the next section for definitions). An example of such a structure is any multiplicative semilattice, or an (upper) semilattice ordered semigroup, with zero. Another example is the interval $[0,1]$ of reals with the restricted (hence, partial) addition and the ordinary multiplication.

If it happens that the semiring is unital and the unit is its greatest element, one arrives at the so called integral semiring like logic (ISR-logic).

The zero and unit elements of an ISR-logic R may be interpreted as the truth values F and T respectively. Furthermore, addition and multiplication in R play, respectively, the roles of disjunction and conjunction. The main purpose of [3] was to demonstrate that the algebraic concept of an existential quantifier, which is well-developed for lattice-like structures, has a significance also in SR-logics. In the present paper, we look for an appropriate algebraic analogue of negation in semiring-like logics.

Benson and Manes have shown in [12] that the standard definition of a complementation in lattices can be adapted also for the interval $[0,1]$ of a sum-ordered semiring with unit. Henceforth, their generalised notion of a complementation fits well into ISR-logics. However, the complementation obtained in this manner is normally only partial; moreover, it is total only if the ISR-logic is a Boolean algebra (see Proposition 4 below and the subsequent observation).

Another widely studied kind of complementation in lattices is pseudocomplementation. One possible way of treating pseudocomplements in semigroups is to define the pseudocomplement of an element a of an ordered semigroup with zero to be the residual of 0 by a. In the paper [2] by Blyth, residuals of zero were termed pseudo-residuals, and the case when the left and the right pseudoresiduals coincide was examined thoroughly. In particular, those semigroups that admit a Glivenko type theorem were characterised; it was also shown there that the most general pseudocomplemented distributive upper-semilattice ordered band which satisfies a weak form of the Stone identity is one in which the Boolean algebra of pseudoresiduals is a subsemilattice (in fact, a subalgebra).

A more abstract approach (pseudocomplements via ideals) was initiated in [13]; see also Section 2 in [11]. A more recent paper on pseudocomplements in semigroups (and even in rings) is [10].

Being inspired by [2], we, however, deal there with another kind of a pseudocomplementation, which, generally, is weaker as far as positive semigroups are considered. Nevertheless, it is fruitful enough; in particular, most results obtained in [2] still have their analogues for pseudocomplemented semigroups. Actually, two main theorems of [2] are improved, and their proofs are considerably simplified.

The paper is structured as follows. In Section 2, the necessary information on sum-ordered partially additive semirings is collected. Pseudocomplemented semigroups are the subject of Section 3, and Glivenko style theorem for a subclass of them is proved in Section 4. Section 5 contains a few general results on pseudocomplements in sum-ordered partially additive semirings. The final section is devoted to Stone semirings, a generalisation of Stone algebras which covers also distributive Stone semigroups in the sense of [2].

2. PRELIMINARIES: PARTIALLY ADDITIVE SEMIRINGS

A *partial Abelian monoid* [4] is a partial algebra $(A, +, 0)$, where A is a non-empty set, $+$ is a partial binary operation on A and 0 is an element of A, subject to the following axioms (we write $a \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} b$ to mean that $a + b$ is defined in A):

if $x \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} y$, then $y \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} x$ and $x + y = y + x$,

if $y \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} z$ and $x \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} (y+z)$, then $x \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} y$, $(x+y) \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} z$ and $x+(y+z)=(x+y)+z$,

$x \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} 0$ and $x + 0 = x$.

A partial Abelian monoid is said to be *zero-sum free* [8], if $y = 0$ whenever $x + y = 0$. We call it *strongly zero-sum free*, if $x + y = x$ whenever $x + y + z = x$.

Proposition 1. *A partial Abelian monoid $(A, +, 0)$ is strongly zero-sum free if and only if the relation $\leq$ on it defined by*

$x \leq y$ if and only if $y = x + u$ for some u

is a partial order. If it is the case, then A is ordered:

(1) *if $x \leq y$ and $y \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} z$, then $x \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} z$ and $x + z \leq y + z$.*

and positive: $0 \leq x$ for all x.

Following [12], we call this relation $\leq$ on a strongly zero-sum free partial Abelian monoid A the *sum-ordering* of A (other terms used in the literature are natural or difference-ordering).

A semigroup $(S,\cdot)$ is said to be *ordered*, if it is equipped with a partial ordering with respect to which the multiplication $\cdot$ is both left and right isotone: if $x \leq y$, then $xz \leq yz$ and $zx \leq zy$. An ordered semigroup with zero (i.e., with an element 0 such that $x0 = 0 = 0x$ for all $x \in S$) is

- *positive*, if $0 \leq x$ for all x,
- *bounded*, if it is positive and has the largest element,
- *quasi-integral*, or negatively ordered, if $xy \leq x, y$ for all x, y,
- *integral*, if it is a monoid and the multiplicative unit is the largest element in S.

We will be interested mainly in positive semigroups. Clearly, a quasi-integral semigroup with zero is positive, and an integral semigroup is quasi-integral, hence, bounded. A bounded meet semilattice is an example of an integral semigroup.

A *sum-ordered partially additive semiring* (sopasr, for short) is a partial algebra $(R,+,\cdot,0)$, where $(R,+,0)$ is a sum-ordered partial Abelian monoid, $(R,\cdot,0)$ is a semigroup with zero and $\cdot$ is left and right distributive over $+$:

$$\text{if} \quad x \downarrow y, \quad \text{then} \quad xz \downarrow yz, \quad zx \downarrow zy \quad \text{and}$$

$$(x+y)z = xz + yz, \;\; z(x+y) = zx + zy.$$

The underlying semigroup of $(R,\cdot)$ of a sopasr is ordered. See [12] and [3] for examples of partial additive monoids and sopasrs.

A sopasr is said to be

- *unital*, if it has a two-sided multiplicative unit,
- *saturated* if $x \downarrow y$ whenever $x \perp y$ (we write $x \perp y$ to mean that $xy = 0 = yx$),
- *bounded*, *quasi-integral* or *integral* if such is its underlying multiplicative semigroup.

It follows from [12, Lemma 3.6] that

Proposition 2. *An integral sopasr is saturated.*

In a unital sopasr R, an element b is said to be a *complement* of a, if $a \perp b$ and $a + b$ exists and equals to 1. As shown in Section 3 of [12], no element has more than one complement. The set C of elements having a complement is called the *center* of R; clearly, $C \subset [0, 1]$. According to [12, Theorem 3.7], this notion of a center coincides with that of the so called Birkhoff center of the interval $[0, 1]$ considered as a poset (see [1, 14] for the latter notion).

We call an element a of A *central*, if it belongs to C. Let a^- stand for the complement of a central element a.

Proposition 3 ([12], Lemma 3.5). *Suppose that a is a central element of R. Then*

(a) *a commutes with all elements of the interval* $[0, 1]$,

(b) *if $x \leq a$, then $x \perp a^-$.*

It is easily seen that, moreover,

$$x \perp a^- \text{ if and only if } ax = x = xa. \tag{2}$$

Indeed, if $ax = x = xa$, then $a^-x = a^-ax = 0$, and likewise $xa^- = 0$. Conversely, if $a^-x = 0 = xa^-$, then $ax = ax + a^-x = (a + a^-)x = x$, and likewise $xa = x$.

We consider a Boolean algebra as an algebra $(B, \circ, ', e)$ of type (2,1,0) such that $(B, \circ, e)$ is a semilattice (i.e., a commutative band) with the neutral element e and, for all $x, y \in B$,

$$x \circ y = x \text{ if and only if } x \circ y' = e;$$

the operation $'$ then becomes the Boolean complementation. This description of Boolean algebras comes back to [5]; see also Exercise 2 for §§ 10–11 in [1, p. 46].

Proposition 4 ([12], Theorem 9). *Suppose that R is a unital sopasr. Then the algebra $(C, \cdot, ^-, 0)$ is a Boolean algebra whose Boolean ordering coincides with $\leq$:*

$$x \leq y \text{ if and only if } xy = x.$$

The join operation $\triangledown$ in C is characterised by

$$x \triangledown y = xy + x^-y + xy^- = x + x^-y. \tag{3}$$

One may conclude from this proposition that isolating those integral sopasrs in which complementation is total is not of much interest: in this case, the center coincides with the whole semiring, which is then nothing else than a Boolean algebra.

Theorem 11 of [12] gives more information on interrelations between operations $+$ and $\triangledown$ in the center.

Proposition 5. *The following conditions are equivalent:*

(a) *C is closed under existing sums,*

(b) *$+$ is idempotent in C: if $x \in C$ and $x \mathrel{\underset{\circ}{\downarrow}} x$, then $x + x = x$,*

(c) *if $x, y \in C$ and $x \mathrel{\underset{\circ}{\downarrow}} y$, then $x + y = x \triangledown y$.*

3. Pseudocomplemented semigroups

Let $(S, \cdot, 0, \leq)$ be a an ordered semigroup with zero. It is

- 0-*commutative*, if $xy = 0$ implies that $yx = 0$,
- *zero-root free*, if $x = 0$ whenever $x^2 = 0$

Elements a and b of S are said to be *orthogonal* if $ab = 0 = ba$; we write $a \perp b$ if this is the case. The subsequent lemma justifies the choice of the term and notation.

Proposition 6. *The relation $\perp$ has the following properties:*

(a) *if $x \perp y$, then $y \perp x$,*

(b) *if S is positive, $x \leq y$ and $y \perp z$, then $x \perp z$,*

(c) *$0 \perp x$,*

(d) *if S is a monoid, then $x = 0$ whenever $x \perp y$ for all y,*

(e) *S is zero-root free if and only if $x = 0$ whenever $x \perp x$.*

An element a^* of S is called a *pseudocomplement* of a if, for all $x \in S$,

$$x \leq a^* \text{ if and only if } x \perp a. \tag{4}$$

Therefore, in a positive semigroup the pseudocomplement of a is the largest element orthogonal to a (see Proposition 6(b)). The semigroup S is said to be *pseudocomplemented* (or a *p-semigroup*, for short) if every element in it has the pseudocomplement. Let B stand for the set of all pseudocomplements.

There is a related type of semigroups already studied in the literature. The zero element of S is said to be *equiresidual* if, for all $x \in S$, both $\max\{y\colon\ xy \leq 0\}$ and $\max\{y\colon\ yx \leq 0\}$ exist and are equal (see [2, p. 442]). If it is the case, the semigroup itself could be called 0*-equiresiduated*.

Proposition 7. *If S is positive, then the following conditions on S are equivalent:*

(a) *S is* 0*-equiresiduated,*

(b) *S is pseudocomplemented and* 0*-commutative.*

If these conditions are fulfilled, then the residuals $\max\{y\colon\ xy \leq 0\}$ and $\max\{y\colon\ yx \leq 0\}$ coincide with x^*.

Lemma 8. *Suppose that S is pseudocomplemented. Then*

(a) $x^* \perp x$, [(4)]

(b) $x \leq x^{**}$, [(a),P6(a)]

(c) if $x \leq y$, then $y^* \leq x^*$, [(b),(4),P6(a),(4)]

(d) $x^{***} = x^*$, [(b),(b),(c)]

(e) if $x \leq y^*$, then $y \leq x^*$, [(c),(b)]

(f) $0 \leq x^*$, [(4),P6(c)]

(g) $x \leq 0^*$, [(f),(e)]

(h) $0^{**} \leq x^*$, [(g),(c)]

(i) if $x \le 0^{**}$, then $x \perp y$ for all y, [(h),(4)]

(j) $x \le 0^{**}$ if and only if $x^* = 0^*$, [(e),(g),(b)]

(k) S is zero-root free if and only if $x \le x^*$ implies that $x = 0$, [(4)]

If S has the neutral element 1*, then, moreover,*

(l) $1^* = 0$, [(a)]

(m) $0^{**} = 0$, [(l),(d)]

(n) $x \le 1^{**}$. [(l),(g)]

It is now easily seen from (b), (c), (d) that the mapping $\gamma\colon\ x \mapsto x^{**}$ is a closure operator and that B is its set of closed elements. Moreover, the kernel equivalence ρ of the mapping $x \mapsto x^*$ coincides with that of γ. The items (a), (b) and (c) of the subsequent lemma are slight generalisations of Lemma 7 and a portion of Theorem 2 of [2].

Lemma 9. *Let S be a* 0*-commutative p-semigroup. Then*

(a) *for all $x, y \in S$, $xy^* \le y^*$ and $y^*x \le y^*$,*

(b) *ρ is a congruence of S,*

(c) *if B is closed under $\cdot$, then γ is an endomorphism of $(S, \cdot)$,*

Proof.

(a) By Lemma 8(a), $yy^* = 0 = y^*y$. Hence, $yxy^* = yy^*x = 0 = xy^*y$, and then $xy^* \le y^*$ by (4). Likewise $y^*x \le y^*$.

(b) Suppose that $x\,\rho\,x_1$ and $y\,\rho\,y_1$. By Lemma 8(a) and the assumption, $0 = xy(xy)^* = y(xy)^*x$, wherefrom $y(xy)^* \le x^* = x_1^*$ and, furthermore, $x_1y(xy)^* = 0 = y(xy)^*x_1 = (xy)^*x_1y$. Therefore, $(xy)^* \le (x_1y)^*$. Likewise, $(x_1y)^* \le (xy)^*$; thus, $xy\,\rho\,x_1y$ and, similarly, $x_1y\,\rho\,x_1y_1$. Hence, $xy\,\rho\,x_1y_1$.

(c) In virtue of (b) and Lemma 8(d),

$$(x^{**}y^{**})^* = (xy)^*, \tag{5}$$

wherefrom, by the supposition of (c) and Lemma 8(d), $(xy)^{**} = x^{**}y^{**}$. ■

At last, we shall say that a p-semigroup S is

- *-idempotent, if $(x^*)^2 = x^*$ for all $x \in S$,
- *-quasi-integral, if $x^*y^* \leq x^*, y^*$ for all $x, y \in S$,
- *-saturated, if $x^* \perp y^*$ implies $x^* \mathrel{⫰} y^*$ for all $x, y \in S$.

4. Glivenko theorem

According to Theorem 1 of [2], the subset B of closed elements of a 0-equiresiduated semigroup S forms a Boolean algebra if and only if the quotient semigroup S/ρ is idempotent, i.e., if $(x^2)\,\rho\, x$ for all $x \in S$. A semigroup satisfying this latter condition was termed a Glivenko semigroup in [2].

It is worthwhile to note that a 0-commutative p-semigroup (see Proposition 7) always is a Glivenko semigroup if it is zero-root free. Indeed, let $x, y \in S$. It is immediate that if $y \perp x$, then $y \perp x^2$. Conversely, assume that $x^2 \perp y$, then $x^2 \perp y^2$ and, in virtue of 0-commutativity, $(xy)^2 = 0 = (yx)^2$. If S is also zero-root free, we conclude that $x \perp y$. Therefore, $y \leq (x^2)^*$ if and only if $y \leq x^*$ (see (4)), wherefrom $(x^2)^* = x^*$.

On the other hand, every unital Glivenko semigroup S is zero-root free: if $x^2 = 0$, then $1 \perp x^2$ and, further, $1 \leq (x^2)^* = x^*$. Hence, $1 \perp x$, wherefrom $x = 0$.

So, we have arrived at

Theorem 10. *A 0-commutative p-monoid is a Glivenko monoid if and only if it is zero-root free.*

We weaken the notion of a Glivenko semigroup and call a *g-semigroup* any $*$-quasi-integral p-semigroup S such that $(x^2)\,\rho\, x$ in B, i.e., for all $x \in S$,

$$\big((x^*)^2\big)^* = x^{**}. \tag{6}$$

In the case when the p-semigroup is 0-commutative, the first of the two conditions is superfluous (see Lemma 9(a)), while the second one alone guarantees that the semigroup is Glivenko: for every $x \in S$, $x^{**}\,\rho\, x$ (see Lemma 8(d)), and, in virtue of Lemma 9(b), $(x^2)^* = ((x^{**})^2)^* = x^{***} = x^*$.

The two theorems below show that the concept of a g-semigroup is still adequate. By the way, the proof of mentioned Theorem 1 in [2] was two pages long. Due to the simple presentation of a Boolean algebra (see Section 1), our proofs also will be considerably simpler.

Theorem 11. *Let S be a p-semigroup. It is a g-semigroup if and only if the meet $x \wedge y$ of any two elements $x, y \in B$ exists in S and equals to $(xy)^{**}$. If it is the case, then $\wedge$ coincides with $\cdot$ on B if and only if the subset B is closed under multiplication.*

Proof. Suppose that S is a g-semigroup and that $x, y \in B$. As S is *-quasi-integral, $xy \leq x$, wherefrom $(xy)^{**} \leq x^{**} = x$ by Proposition 8(c,d). Likewise $(xy)^{**} \leq y$. On the other hand, if $u \leq x, y$ for some $u \in S$, then $u^{**} \leq x, y$ and $(u^{**})^2 \leq xy$. It follows that $(xy)^{**} \geq ((u^{**})^2)^{**} = u^{**} \geq u$. So, $(xy)^{**}$ is the greatest lower bound of x and y.

The converse is trivial: if $x \wedge y = (xy)^{**}$ for all $x, y \in B$, then $(x^2)^* = (x^2)^{***} = (x \wedge x)^* = x^*$, and $xy \leq (xy)^{**} = x \wedge y \leq x, y$.

To prove the other statement, it suffices to note that, in every p-semigroup, $xy \in B$ if and only if $xy = (xy)^{**}$. ∎

Corollary 12. *Suppose that S is a g-semigroup. Then B is a subsemigroup of S if and only if S is *-idempotent. If it is the case, then $0 \in B$.*

Proof. If B is a subsemigroup of S then, for every $x \in B$, $x^2 = x \wedge x = x$. Conversely, suppose that the multiplication is idempotent in B and that $x, y \in B$. Just as in the previous proof, $(xy)^{**} \leq x, y$. Then $xy \geq ((xy)^{**})^2 = (xy)^{**}$, wherefrom by Lemma 8(b) $xy = (xy)^{**} \in B$.

Furthermore, $0^{**} \perp 0^{**}$ (see Lemma 8(i)). Now, if B is a subsemigroup of S, then $0 = 0^{**}0^{**} = 0^{**} \wedge 0^{**} = 0^{**}$. ∎

For Glivenko semigroups, this equivalence was proved in [2, Theorem 2]. In the next corollary, by a subsemilattice of S we mean a subsemigroup of S which happens to be a commutative band. Such a subsemilattice is said to be *natural*, if its natural ordering coincides with $\leq$.

Corollary 13. *A p-semigroup S is a *-idempotent g-semigroup if and only if B is a natural subsemilattice of S. If it is the case, then, for all $x \in S$,*

$$x^*0^* = x^* = 0^*x^*. \tag{7}$$

Proof. If S is a *-idempotent g-semigroup, then B is its subsemigruop by Corollary 12, and is even a natural semilattice by Theorem 11.

Conversely, if B a semilattice with respect to both $\cdot$ and $\leq$, then S is *-quasi-integral and *-idempotent; hence it satisfies also (6). As to (7), the element 0^* is the greatest element in B by Lemma 8(g). ∎

The next Glivenko type theorem explains our interest in *g*-semigroups.

Theorem 14. *Suppose that S is a g-semigroup. Then the algebra* $(B, \wedge, ^*, 0^{**})$ *is a Boolean algebra whose Boolean ordering coincides with* $\leq$*.*

Proof. We already know that $(B, \wedge)$ is a semilattice w.r.t. $\leq$ and that 0^{**} is its least element. It remains to prove that, for all $x, y \in B$,

$$x \leq y \text{ if and only if } x \wedge y^* = 0^{**}.$$

Suppose that $x, y \in B$. If $x \leq y$, then $xy^* \leq yy^* = 0$ and $0^* \leq (xy^*)^* \leq 0^*$ by Lemma 8(c,g). Henceforth, $(xy^*)^{**} = 0^{**}$. Conversely, if $(xy^*)^{**} = 0^{**}$, then $xy^* \leq 0^{**}$. By Lemma 8(i), then $x^2y^* = 0$. As $\wedge$ is commutative, likewise $y^*x^2 = 0$, Henceforth, $y^* \perp x^2$, i.e., $y^* \leq (x^2)^* = x^*$ and, by Lemma 8(c), $x^{**} \leq y^{**}$. As $x, y \in B$, it follows that $x \leq y$. ∎

Given a g-semigroup S, let $\curlyvee$ stand for the join operation in B. It follows from Theorem 11 that, for all $x, y \in S$,

$$x^* \wedge y^* = (x^*y^*)^{**}, \quad x^* \curlyvee y^* = (x^{**}y^{**})^*. \tag{8}$$

If the subset B is closed under $\cdot$, then even

$$x^* \wedge y^* = x^*y^*, \quad (x^* \curlyvee y^*)^* = x^{**}y^{**}. \tag{9}$$

If S is 0-commutative, then by (8) and (5),

$$x^* \curlyvee y^* = (xy)^*. \tag{10}$$

The following connection between $\curlyvee$ and $\vee$, the partial join operation in S, generalises Lemma 16 in [2], where the operation $\vee$ was assumed to be total, and is proved in a similar fashion.

Lemma 15. *In a g-semigroup, if $x^* \vee y^*$ exists for some $x, y \in S$, then*

$$(x^* \vee y^*)^* = (x^* \curlyvee y^*)^*.$$

Proof. Assume that $x^* \vee y^* = z$. As $x^*, y^* \leq x^* \curlyvee y^*$, it follows that $z \leq x^* \curlyvee y^*$ and, furthermore, $(x^* \curlyvee y^*)^* \leq z^*$. On the other hand, $x^*, y^* \leq z$, wherefrom $z^* \leq x^{**}, y^{**}$. Therefore, $z^* \leq x^{**} \wedge y^{**} = (x^* \curlyvee y^*)^*$ by (8). ∎

5. Pseudocomplementation in sum-ordered semirings

A *(partial) pseudocomplemented semiring* (or a *(partial) p-semiring*, for short) is a sopasr R whose multiplicative part is a p-semigroup w.r.t. the sum ordering $\leq$ of R. It follows from Lemma 8(g) that then R is bounded, i.e, is an SR-like logic (see Section 1). An example of a p-semiring is given at the end of the next section.

Lemma 16. *Let R be a p-semiring. For all $x, y, z \in R$,*

(a) *if $x \between y$, then $(x + y) \perp z$ if and only if $x \perp z$ and $y \perp z$,*

(b) *if $x \between y$, then the meet of x^* and y^* in R exists, and $(x+y)^* = x^* \wedge y^*$,*

(c) *$x \between x^*$ if and only if x is compatible with all elements orthogonal to x,*

(d) *if $x \between x^*$, then $(x + x^*)^* = 0^{**}$.*

Moreover, addition is stable w.r.t. ρ: if $x^* = x_1^*$, $y^* = y_1^*$, and if $x \between y$ and $x_1 \between y_1$, then $(x + y)^* = (x_1 + y_1)^*$.

Proof.

(a) As R is zero-sum free,

$$(x+y) \perp z \Leftrightarrow xz + xy = 0 = zx + zy \Leftrightarrow xz = xy = 0 = zx = zy \Leftrightarrow x, y \perp z.$$

(b) Follows by (4) from (a).

(c) Follows by (4) from Lemma 8(a) and the first part of (1).

(d) Follows from (b) and Lemma 8(h).

The last statement of the lemma is an immediate consequence of (b). ∎

A p-semiring whose multiplicative part is a g-semigroup, is said to be a *(partial) g-semiring.* In a g-semiring, if $x^* \downarrow_\circ y^*$, then

$$(x^* + y^*)^* = x^{**} \wedge y^{**}, \quad (x^* + y^*)^{**} = x^* \curlyvee y^* \tag{11}$$

(see Lemma 16(b) and (8)). Consequently, by Lemma 15,

$$(x^* + y^*)^* = (x^* \vee y^*)^*, \tag{12}$$

provided $x^* \vee y^*$ also exists.

Corollary 17. *In a g-semiring R, the following conditions are equivalent:*

(a) *B is closed under existing sums,*

(b) *for every $x, y \in R$, if $x^* \downarrow_\circ y^*$, then $x^* + y^* = x^* \curlyvee y^*$,*

If R is also 0-commutative, then any of these conditions is equivalent to

(c) *$x^* + y^* = (xy)^*$ whenever $x^* \downarrow_\circ y^*$.*

Proof.

(a) → (b). In virtue of (11).

(b) → (a). Trivial.

(c) ↔ (b). By (10). ∎

Theorem 18. *Suppose that R is a unital g-semiring. Then the center of R, considered as a Boolean algebra, is a subalgebra of B.*

Proof. Let a be a central element of R. As R is quasi-integral, $x \perp a^-$ implies that $x \le a$ in virtue of (2). Together with the converse implication from Proposition 3(b), this means that a is the pseudocomplement of a^- and belongs to B. Likewise, $a^- = (a^{--})^* = a^*$.

Now suppose that $x, y \in C$. Then $x, y \in B$, $xy \in C$ and $xy = (xy)^{--} = (xy)^{-*} = (xy)^{**} = x \wedge y$ by Theorem 11. By Lemma 8(m), $0 = 0^{**}$. Therefore, C is a subalgebra of B, indeed. ∎

6. Partial Stone semirings

We define a *(partial) Stone semiring* to be a *-idempotent g-semiring (i.e., one in which multiplication is idempotent in B) such that the sum $x^* + x^{**}$ exists and equals to 0^* for every x.

In the next theorem, by a subring of a g-semiring we mean a subset containing 0 and closed under products and existing sums.

Theorem 19. *The following conditions for a g-semiring R are equivalent:*

(a) *R is a Stone semiring,*

(b) *multiplication is idempotent in B and, for all $x, y \in S$, the sum of x^*y^*, $x^{**}y^*$ and x^*y^{**} exists; moreover,*

$$x^* \curlyvee y^* = x^*y^* + x^{**}y^* + x^*y^{**} = x^* + x^{**}y^* = y^* + x^*y^{**}, \tag{13}$$

(c) *B is a saturated subsemiring of R.*

Proof.

(a) → (b). Let $x^* \sqcup y^*$ stands for $x^*y^* + x^{**}y^* + x^*y^{**}$ for some $x, y \in R$. This sum exists, for

$$0^* = 0^*0^* = (x^* + x^{**})(y^* + y^{**}) = x^* \sqcup y^* + x^{**}y^{**}$$

(cf. the distributive law for partially additive semirings in Section 2). Moreover,

$$x^* \sqcup y^* = x^*(y^* + y^{**}) + x^{**}y^* = x^*0^* + x^{**}y^* = x^* + x^{**}y^*$$

(see (7)). Likewise, $x^* \sqcup y^* = y^* + x^*y^{**}$. Hence, $x^* \sqcup y^*$ is an upper bound for x^* and y^*. It is even the least upper bound: if $x^*, y^* \leq z^*$ for some $z \in R$, then $x^*z^* = x^*$, $y^*z^* = y^*$ by Corollary 13, and

$$(x^* + x^{**}y^*)z^* = x^*z^* + x^{**}y^*z^* = x^* + x^{**}y^*.$$

(b) $\to$ (a). By substituting x^* for y in (13), we obtain that $x^* + x^{**} = x^* \curlyvee x^{**} = 0^*$.

(a) $\to$ (c). Assume that R is a Stone semiring. By Corollary 12, then B is closed under products and contains 0. Furthermore, if $x^* \perp y^*$ for some $x, y \in S$, then $y^* \leq x^{**}$ by (4), and then $x^* \downarrow_{\circ} y^*$ by (1), as $x^* \downarrow_{\circ} x^{**}$ by the assumption. So R is *-saturated. Let us now see why B is closed also under sums existing in R.

By Lemma 16(c), $(x^* \curlyvee y^*) \downarrow_{\circ} x^{**}y^{**}$ (see (9)), wherefrom $x^* + y^* \downarrow_{\circ} x^{**}y^{**}$ by (11), Lemma 8(b) and (1). Moreover, $x^* + y^* + x^{**}y^{**} = 0^*$. Indeed, in virtue of (7),

$$\begin{aligned} 0^* &= x^* + x^{**} = x^* + x^{**}0^* \\ &= x^* + x^{**}(y^* + y^{**}) = x^* + x^{**}y^* + x^{**}y^{**} \leq x^* + y^* + x^{**}y^{**}. \end{aligned}$$

Therefore,

$$\begin{aligned} (x^* \curlyvee y^*) &= (x^* \curlyvee y^*)0^* \\ &= (x^* \curlyvee y^*)(x^* + y^* + x^{**}y^{**}) \\ &= (x^* \curlyvee y^*)(x^* + y^*) + (x^* \curlyvee y^*)x^{**}y^{**}. \end{aligned}$$

In virtue of (8), $(x^* \curlyvee y^*)x^{**}y^{**} = 0$. Furthermore,

$$\begin{aligned} (x^* \curlyvee y^*)(x^* + y^*) &= (x^* \curlyvee y^*)x^* + (x^* \curlyvee y^*)y^* \\ &= (x^* \curlyvee y^*) \wedge x^* + (x^* \curlyvee y^*) \wedge y^* \\ &= x^* + y^*\,. \end{aligned}$$

Therefore, $x^* + y^* = x^* \curlyvee y^* \in B$, and (a) is fulfilled.

(c) $\to$ (a). By Corollaries 12 and 17(b). ∎

Remark. The identity (13) is a generalisation of (3), and its proof is similar to the proof of the latter one in [12]. The prototype of the implication (a) $\to$ (c) is Theorem 5 in [2]. The implication was proved there under the stronger assumptions that the partial monoid $(R, +)$ is actually an upper semilattice and the semigroup $(R, \cdot)$ is a 0-commutative band (cf. Proposition 7).

Our proof of the crucial identity $x^* + y^* = x^* \curlyvee y^*$ is adapted from [2], but it does not require the preparatory calculations needed there.

An integral p-semiring is a partial Stone semiring if it is *-idempotent and if, for every $x \in S$, $x^* + x^{**} = 1$. As a corollary, we now obtain several characteristics of integral Stone semirings, which are well-known for Stone algebras (see, e.g., Lemma 6.3 in Section 2 of [7]).

Theorem 20. *The following conditions on an integral p-semiring R are equivalent:*

(a) *R is a Stone semiring,*

(b) *R is a g-semiring in which the Boolean algebra of closed elements coincides with that of central elements,*

(c) *B is a subsemiring of R,*

(d) *in B, $\wedge$ coincides with $\cdot$, and $\curlyvee$ is an extension of $+$.*

If R is also 0*-commutative, then any of them is equivalent to*

(e) *R is *-idempotent and $x^* + y^* = (xy)^*$ whenever $x^* \mathrel{\underset{\circ}{\shortmid}} y^*$.*

Proof.

(a) $\to$ (b). In virtue of Theorem 18, C is a subalgebra of B, and $B \subset C$ if and only if R satisfies the Stone identity $x^* + x^{**} = 1$.

(b) $\to$ (a). Assume (b). Then $x^* \wedge y^* = x^*y^*$ for all $x, y \in R$, so that multiplication is idempotent on B. Furthermore, R is saturated (see Proposition 2), so $x^* \mathrel{\underset{\circ}{\shortmid}} x^{**}$. By Proposition 5(c), $x^* + x^{**} = x^* \triangledown x^{**} = 1$.

(a) $\leftrightarrow$ (c). By Proposition 2 and Theorem 19.

(c) $\leftrightarrow$ (d). See Theorem 11 and Corollary 17(a,b) (or either Proposition 5 and (b)).

(d) $\leftrightarrow$ (e). See Corollary 17(b,c), Theorem 11 and Corollary 12. ∎

We end with an example of a Stone semiring that is not integral.

Example. It follows from Example 2.3 of [3] that if addition in a bounded sopasr is idempotent, then it is necessarily total. Therefore, let $L := (L, \vee, 0)$ be a join semilattice with zero, and let $u \neq 0$ be the largest element in L.

Of course, L is strongly zero-sum free, and the corresponding sum-ordering coincides with the semilattice ordering. The operation $\cdot$ on L defined by

$$x \cdot y := \begin{cases} 0 & \text{if } x = 0, \\ y & \text{otherwise} \end{cases}$$

turns L into a semiring. Indeed, both $(x \vee y)z$ and $xz \vee yz$ equal to 0, if $x = y = 0$, and to z otherwise. Likewise, both $z(x \vee y)$ and $zx \vee zy$ equal to 0, if $z = 0$, and to $x \vee y$ otherwise. Furthermore, 0 is also the multiplicative zero in L. Note that the semiring L is quasi-integral and non-unital.

Clearly, $xy = 0$ if and only if $x = 0$ or $y = 0$; hence, the semigroup $(L, \cdot, 0)$ is 0-commutative. In particular, it is zero-root free and, hence, even a Glivenko semigroup (see the second paragraph in Section 4).

Now, $0^* = u$, and $x^* = 0$ if $x \neq 0$. Thus 0 and u are the only closed elements. As multiplication is idempotent in L, this semiring is evidently a Stone semiring.

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