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THE TABLE OF CHARACTERS OF SOME QUASIGROUPS

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Abstract

It is known that $(\mathbb{Z}_n, -_n)$ are examples of entropic quasigroups which are not groups. In this paper we describe the table of characters for quasigroups $(\mathbb{Z}_n, -_n)$.

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1. Introduction

The theory of characters of finite quasigroup has been already considered by J.D.H. Smith in [3].

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A *quasigroup* $(Q, \cdot)$ is a set Q equipped with a binary *multiplication* operation denoted by $\cdot$ or juxtaposition of the two arguments, in which specification of any two of x, y, z in the equation $x \cdot y = z$ determines the third uniquely.

A quasigroup $(Q, \cdot)$ is called *entropic* if

$$(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t)$$

for all $x, y, z, t \in Q$.

Let $(Q, \cdot)$ be a finite quasigroup. Now we describe how to obtain the character table of Q (see [3], Chapter 5).

Let $R\colon Q \to Q!$; $x \mapsto R(x)$ and $L\colon Q \to Q!$; $x \mapsto L(x)$, where $R(x)(q) = q \cdot x$ and $L(x)(q) = x \cdot q$. Then the subgroup $G = Mlt(Q, \cdot)$ of $Q!$ generated by the union $R(Q) \cup L(Q)$ is called the *multiplication group* of the quasigroup $(Q, \cdot)$.

The group G acts onto $Q \times Q$ in the following way:

$$g\colon Q \times Q \to Q \times Q; \quad (x, y) \mapsto (g(x), g(y)).$$

The orbits $\{C_1, \ldots, C_s\}$ of G on $Q \times Q$ under this action are called the *conjugacy classess* of Q.

We consider the incidence matrix a_i of the conjugacy class C_i. This is $0-1$-matrix having 1 as its xy-component if $(x, y) \in C_i$ and 0 otherwise.

The space $\mathbb{C}Q$ can be decomposed as a direct sum of subspaces E_j such that

(a) $\forall_{1 \le i \le s}, \exists_{\xi_{ij} \in \mathbb{C}} E_j(a_i - \xi_{ij} I) = \{0\}$;

(b) $\forall_{j \ne k}, \exists_i \xi_{ij} \ne \xi_{ik}$;

(c) $E_1 = \mathbb{C}\Big(\sum_{q \in Q} q\Big)$.

To get (a) and (b), decompose $\mathbb{C}Q$ into a_1-eigenspaces, then decompose each of these into a_2-eigenspaces,and so on. In the case of quasigroup $(\mathbb{Z}_n, -_n)$ it is enough to end this process with a_2-eigenspaces. Let $e_j\colon \mathbb{C}Q \to E_j$ be the projection onto E_j. Define $(s \times s)$-matrix $\Xi = (\xi_{ij})$ by $a_i = \sum_{j=1}^{s} \xi_{ij} e_j$.

Finally the *character table* of the quasigroup Q is the complex $(s \times s)$ matrix Ψ with components

$$\psi_{il} = (f_i)^{\frac{1}{2}} \xi_{li} n_l^{-1},$$

for $i, l = 1, \dots, s$, where $f_i = dim_{\mathbb{C}} E_i$ and $n_l = \frac{|C_l|}{|Q|}$.

For more details see [1, 3, 5].

In this paper we find the character tables of quasigroups $(\mathbb{Z}_n, -_n)$.

If $i, j \in \mathbb{Z}_n$ then

$$i -_n j = \begin{cases} i - j & \text{for } i \geq j \\ n + i - j & \text{for } i < j \end{cases}.$$

Every quasigroup $(\mathbb{Z}_n, -_n)$ has the following conjugacy classes:

$$C_i = \{(k,t) \in \mathbb{Z}_n^2 \colon |k - t| = i - 1 \text{ or } |k - t| = n - i + 1\}$$

for $i = 1, \dots, \left[\frac{n}{2}\right]$.

One can check that $|C_j| = n$ if $j = 1$ or $(j = \frac{n}{2} + 1$ and $2|n)$ and $|C_j| = 2n$ otherwise.

This is a „road map" through the lemmas in this paper:

$$\begin{tikzcd}
 & L2 \arrow[d] & L4 \arrow[r] & L6 \arrow[d] & L5 \arrow[l] & L4 \arrow[d] & & L7 \arrow[d] \\
L1 \arrow[r] & L3 \arrow[rr] & & L9 \arrow[rr] & & T1 \arrow[r] & L12 \arrow[r] & T2 \\
 & & L8 \arrow[ur] & & L4 \arrow[r] & L10 \arrow[r] \arrow[ur] & L11 \arrow[u] &
\end{tikzcd}$$

2. Notations

For $n \in \mathbb{N}$, $0 \le m \le \left[\frac{n}{2}\right]$ and $m \in \mathbb{N}$ let

$$x_{n,m} = \begin{cases} 2\cos\frac{2m\pi}{n} & \text{if} \quad 2|n \\ \\ (-1)^m 2\cos\frac{m\pi}{n} & \text{otherwise.} \end{cases}$$

For $n \in \mathbb{N}$ define the function $g_n \colon \mathbb{Z} \to \{0, 1, \ldots, \left[\frac{n}{2}\right]\}$ in the following way $g_n(x) = dist(x, n\mathbb{Z})$. Let a_i be the incidence matrix of the conjugacy class C_i. This is $0-1$-matrix having 1 as its xy-component if $(x, y) \in C_i$ and 0 otherwise. Let w_n be the characteristic polynomial of a_2.

3. Main theorem

In this section we prove a recursive formula for the characteristic polynomial of the matrix a_2. Before that we give and prove necessary lemmas.

Lemma 1. *For every $n \ge 3$ we have*

$$w_{n+2}(x) = -xw_{n+1}(x) - w_n(x) + (-1)^n(2x - 4).$$

Proof. Let $v_n = (b_{ij})_{1 \le i,j \le n}$ be the matrix such that

$$b_{ij} = \begin{cases} 0 & \text{for} \quad |i - j| \ge 2 \\ 1 & \text{for} \quad |i - j| = 1 \\ -x & \text{for} \quad i = j\,. \end{cases}$$

By Laplace's expansion of the determinant along 1 column we have

$$v_n(x) = -xv_{n-1} - v_{n-2}(x)\,. \tag{1}$$

Using again Laplace's formula to expand the determinant along 1 column and 1 row we have

$$\begin{aligned} w_n(x) &= -xv_{n-1}(x) - (v_{n-2} + (-1)^n) + (-1)^{n+1}(1 + (-1)^n v_{n-2}(x)) \\ &= -xv_{n-1}(x) - 2v_{n-2}(x) + 2 \cdot (-1)^{n+1} . \end{aligned} \tag{2}$$

Now we obtain

$$\begin{aligned} w_{n+2}(x) &\overset{(2)}{=} -xv_{n+1} - 2v_n + 2 \cdot (-1)^{n+1} \overset{(1)}{=} -x(-xv_n(x) - v_{n-1}(x)) \\ &-2v_n(x) + 2 \cdot (-1)^{n+1} = v_n(x)(x^2 - 2) + xv_{n-1}(x) + 2 \cdot (-1)^{n+1} \\ &= \underbrace{x^2 v_n(x) + 2xv_{n-1} - 2x(-1)^n}_{=-xw_{n+1}(x)} -2v_n(x) - xv_{n-1}(x) + 2x(-1)^n \\ &+2(-1)^{n+1} \overset{(2)}{=} -xw_{n+1}(x) + \underbrace{xv_{n-1}(x) + 2v_{n-2}(x) - 2(-1)^{n+1}}_{=-w_n(x)} \\ &\underbrace{-2xv_{n-1}(x) - 2v_n(x) - 2v_{n-2}(x)}_{=0 \; by \; (1)} +4(-1)^{n+1} + 2x(-1)^n \\ &= -xw_{n+1}(x) - w_n(x) + (-1)^n(2x - 4). \end{aligned}$$

■

Let $u_n(x)$ be a polynomial such that $u_{2n+2}(x) = u_{2n+1}(x) - u_{2n}(x)$, $u_{2n+1}(x) = (x+2)u_{2n}(x) - u_{2n-1}(x)$ and $u_1(x) = u_2(x) = 1$.

Lemma 2. *For every $n \in \mathbb{N}$ we have*

(a) $$(x+2)u_{2n}(x)u_{2n+1}(x) = u_{2n+1}^2(x) + (x+2)u_{2n}^2(x) - 1,$$

(b) $$(x+2)u_{2n+2}(x)u_{2n+1}(x) = u_{2n+1}^2(x) + (x+2)u_{2n+2}^2(x) - 1.$$

Proof. For $n = 1$ it is clear. Assume that lemma is true for n. We prove this lemma for $n + 1$.

$u_{2n+3}^2(x) + (x+2)u_{2n+2}^2(x) - 1 = ((x+2)u_{2n+2}(x) - u_{2n+1}(x))u_{2n+3}(x) +$

$(x+2)u_{2n+2}^2(x) - 1 = (x+2)u_{2n+2}(x)u_{2n+3}(x) - u_{2n+1}(x)((x+2)u_{2n+2}(x)$

$-u_{2n+1}(x)) + (x+2)u_{2n+2}^2(x) - 1 \overset{by (b)}{=} (x+2)u_{2n+2}(x)u_{2n+3}(x)$

$-(u_{2n+1}^2(x) + (x+2)u_{2n+2}^2(x) - 1) + u_{2n+1}^2(x) + (x+2)u_{2n+2}^2(x) - 1$

$= (x+2)u_{2n+2}(x)u_{2n+3}(x),$

hence (a) is true for $n + 1$.

$$\begin{aligned}
u_{2n+3}^2(x) &+ (x+2)u_{2n+4}^2(x) - 1 = u_{2n+3}^2(x) + (x+2)u_{2n+4}(x)(u_{2n+3}(x) \\
&-u_{2n+2}(x)) - 1 = u_{2n+3}^2(x) + (x+2)u_{2n+4}(x)u_{2n+3}(x) \\
&-(x+2)u_{2n+4}(x)u_{2n+2}(x) - 1 \\
&= u_{2n+3}^2(x) + (x+2)u_{2n+4}(x)u_{2n+3}(x) \\
&-(x+2)(u_{2n+3}(x) - u_{2n+2}(x))u_{2n+2}(x) - 1 \\
&= u_{2n+3}^2(x) + (x+2)u_{2n+2}^2(x) - 1 + (x+2)u_{2n+4}(x)u_{2n+3}(x) \\
&-(x+2)u_{2n+3}(x)u_{2n+2}(x) \overset{by\ (a)\ for\ n+1}{=} (x+2)u_{2n+2}(x)u_{2n+3}(x) \\
&+(x+2)u_{2n+4}(x)u_{2n+3}(x) - (x+2)u_{2n+3}(x)u_{2n+2}(x) \\
&= (x+2)u_{2n+4}(x)u_{2n+3}(x)
\end{aligned}$$

so we obtain (b) for $n + 1$. ∎

Now we pass to the lemma expressing polynomial w_n by u_n.

Lemma 3. *For every* $n \geq 1$

$$(\alpha) \qquad w_{2n+1}(x) = (2-x)u_{2n+1}^2(x),$$

$$(\beta) \qquad w_{2n}(x) = (x^2-4)u_{2n}^2(x).$$

Proof. For $n = 2$ it is obvious. Assume that lemma is true for n. We prove lemma for $n+1$. Using Lemma 1 and Lemma 2 we have

$$\begin{aligned} w_{2n+2}(x) &\overset{L1}{=} -xw_{2n+1}(x) - w_{2n}(x) + 2x - 4 = -x(2-x)u_{2n+1}^2(x) \\ &- (x^2-4)u_{2n}^2(x) + 2x - 4 \\ &\overset{L2a}{=} (x^2-2x)u_{2n+1}^2(x) - (x^2-4)u_{2n}^2(x) + 2x - 4 \\ &+ (2x-4)(u_{2n+1}^2(x) - (x+2)u_{2n}(x)u_{2n+1}(x) - 1 + u_{2n}^2(x)(x+2)) \\ &= (x^2-4)u_{2n+1}^2(x) + (x^2-4)u_{2n}^2(x) - 2(x^2-4)u_{2n}(x)u_{2n+1}(x) \\ &= (x^2-4)(u_{2n+1}^2(x)u_{2n}^2(x) - 2u_{2n}(x)u_{2n+1}(x)) \\ &= (x^2-4)(u_{2n+1}(x) - u_{2n}(x))^2 = (x^2-4)u_{2n+2}^2(x) \end{aligned}$$

so we obtain (β) for $n+1$.

By Lemma 1 and 2 and (β) for $n+1$ we have

$$\begin{aligned} (2-x)u_{2n+3}^2(x) &= (2-x)((x+2)u_{2n+2}(x) - u_{2n+1}(x))^2 \\ &\overset{L2b}{=} (2-x)((x+2)u_{2n+2}(x) - u_{2n+1}(x))^2 \\ &+ (2x-4)((x+2)u_{2n+2}^2(x) + u_{2n+1}^2(x) \\ &- 1 - (x+2)u_{2n+2}(x)u_{2n+1}(x)) = \end{aligned}$$

$$= (x-2)(-(x+2)^2 u_{2n+2}^2(x)$$

$$+ 2(x+2)u_{2n+1}(x)u_{2n+2}(x) - u_{2n+1}^2(x)$$

$$+ 2(x+2)u_{2n+2}^2(x) + 2u_{2n+1}^2(x) - 2$$

$$- 2(x+2)u_{2n+2}(x)u_{2n+1}(x)$$

$$= (x-2)(u_{2n+2}^2(x)(-x^2-2x) + u_{2n+1}^2 - 2)$$

$$= -x(x^2-4)u_{2n+2}^2(x) - (2-x)u_{2n+1}^2(x) - 2x + 4$$

$$\stackrel{(\beta)}{=} -xw_{2n+2}(x) - w_{2n+1}(x) - 2x + 4 \stackrel{L1}{=} w_{2n+3}(x)$$

hence (α) is true for $n+1$. ■

Lemma 4. *Let $n \in N$ and $0 \leq j, k \leq \left[\frac{n}{2}\right]$. Then*

$$x_{n,j} \cdot x_{n,k} = x_{n,|k-j|} + x_{n,g_n(k+j)}.$$

Proof. Consider the following cases:

1. n is odd and $j + k \leq \left[\frac{n}{2}\right]$. Then

$$x_{n,j} \cdot x_{n,k} = 2\cos\left(\frac{2j\pi}{n}\right) 2\cos\left(\frac{2k\pi}{n}\right)$$

$$= 2\left(\cos\left(\frac{2(j-k)\pi}{n}\right) + \cos\left(\frac{2(j+k)\pi}{n}\right)\right)$$

$$= x_{n,|k-j|} + x_{n,g_n(k+j)}.$$

2. n is odd and $j+k > \left[\frac{n}{2}\right]$. Then $g_n(j+k) = n-(j+k)$ and

$$x_{n,j} \cdot x_{n,k} = 2\cos\left(\frac{2j\pi}{n}\right) 2\cos\left(\frac{2k\pi}{n}\right)$$

$$= 2\left(\cos\left(\frac{2(j-k)\pi}{n}\right) + \cos\left(\frac{2(j+k)\pi}{n}\right)\right)$$

$$= 2\left(\cos\left(\frac{2(j-k)\pi}{n}\right) + \cos\left(2\pi - \frac{2(j+k)\pi}{n}\right)\right)$$

$$= 2\cos\left(\frac{2(j-k)\pi}{n}\right) + \cos\left(\frac{2(n-(j+k))\pi}{n}\right) = x_{n,|k-j|} + x_{n,g_n(k+j)}.$$

3. n is even and $j+k \leq \left[\frac{n}{2}\right]$. Then

$$x_{n,j} \cdot x_{n,k} = (-1)^j 2\cos\left(\frac{j\pi}{n}\right) (-1)^k 2\cos\left(\frac{k\pi}{n}\right)$$

$$= (-1)^{j+k} 2\left(\cos\left(\frac{(j-k)\pi}{n}\right) + \cos\left(\frac{(j+k)\pi}{n}\right) = x_{n,|k-j|} + x_{n,g_n(j+k)}\ .$$

4. n is even and $j+k > \left[\frac{n}{2}\right]$. Then

$$x_{n,j} \cdot x_{n,k} = (-1)^j 2\cos\left(\frac{j\pi}{n}\right) (-1)^k 2\cos\left(\frac{k\pi}{n}\right)$$

$$= (-1)^{j+k} 2\left(\cos\left(\tfrac{(j-k)\pi}{n}\right) + \cos\left(\tfrac{(j+k)\pi}{n}\right)\right)$$

$$= (-1)^{k-j} 2\cos\left(\tfrac{(j-k)\pi}{n}\right) + (-1)^{j+k} 2(-1)\cos\left(\pi - \tfrac{(j+k)\pi}{n}\right)$$

$$= (-1)^{k-j} 2\cos\left(\frac{(j-k)\pi}{n}\right) + (-1)^{n-(j+k)} 2\cos\left(\pi - \frac{(j+k)\pi}{n}\right)$$

$$= x_{n,|k-j|} + x_{n,g_n(j+k)}.$$

■

Lemma 5. *Let $n \in \mathbb{N}$, $y \in \mathbb{Z}$ and $j \in \{0, 1, \ldots, \left[\frac{n}{2}\right]\}$. Then*

$$\{g_n(j + g_n(y)), |g_n(y) - j|\} = \{g_n(y - j), g_n(y + j)\}.$$

Proof. There exists $k \in \mathbb{Z}$ such that $kn \leq y \leq kn + n$. Let us consider the following cases:

1. If $y - kn \leq \left[\frac{n}{2}\right]$ then $g_n(y) = y - kn$ and

$$g_n(y + j) = dist(y + j, n\mathbb{Z}) = dist(y - kn + j, n\mathbb{Z})$$

$$= dist(g_n(y) + j, n\mathbb{Z}) = g_n(g_n(y) + j)$$

and

$$g_n(y - j) = dist(y - j, n\mathbb{Z}) = dist(y - kn - j, n\mathbb{Z})$$

$$= dist(g_n(y) - j, n\mathbb{Z}) = |g_n(y) - j|.$$

2. If $kn + n - y \leq \left[\frac{n}{2}\right]$ then $g_n(y) = kn + n - x$ and

$$g_n(y - j) = dist(y - j, n\mathbb{Z}) = dist(j - y, n\mathbb{Z})$$

$$= dist(kn + n - y + j, n\mathbb{Z}) = dist(g_n(y) + j, n\mathbb{Z}) = g_n(g_n(y) + j)$$

and

$$g_n(y + j) = dist(y + j, n\mathbb{Z}) = dist(-y - j, n\mathbb{Z}) =$$

$$dist(kn + n - y - j, n\mathbb{Z}) = dist(g_n(y) - j, n\mathbb{Z}) = |g_n(y) - j|.$$

■

Now we find eigenvectors for the matrix a_2.

Let $n \in \mathbb{N}$ and $0 \leq j \leq \left[\frac{n}{2}\right]$. Let

$$v_{n,j} = [x_{n,g_n(0)}, x_{n,g_n(j)}, x_{n,g_n(2j)}, \ldots, x_{n,g_n(kj)}, \ldots, x_{n,g_n((n-1)j)}] \in \mathbb{C}^n.$$

Lemma 6. *Let* $0 \leq j \leq \left[\frac{n}{2}\right]$. *Then vector* $v_{n,j}$ *is an eigenvector of the matrix* a_2 *corresponding to an eigenvalue* $x_{n,j}$.

Proof. We must show that

$$(*) \qquad x_{n,j} \cdot x_{n,g_n(kj)} = x_{n,g_n((k-1)j)} + x_{n,g_n((k+1)j)}$$

for $k = 1, 2, \ldots, n-1$ and

$$(**) \qquad x_{n,j} \cdot x_{n,g_n(0)} = x_{n,g_n(j)} + x_{n,g_n((n-1)j)}$$

and

$$(***) \qquad x_{n,j} \cdot x_{n,g_n((n-1)j)} = x_{n,g_n(0)} + x_{n,g_n((n-2)j)}.$$

By Lemma 4 we have

$$x_{n,j} \cdot x_{n,g_n(kj)} = x_{n,|j-g_n(kj)|} + x_{n,g_n(j+g_n(kj))}.$$

Hence

$$x_{n,j} \cdot x_{n,g_n(kj)} = x_{n,g_n((k-1)j)} + x_{n,g_n((k+1)j)},$$

by Lemma 5 for $y = kj$ and this ends the proof of $(*)$.

Obviously $g_n(0) = 0$ and $g_n(j) = j$. Moreover

$$g_n((n-1)j) = dist(nj - j, n\mathbb{Z}) = dist(-j, n\mathbb{Z}) = dist(j, n\mathbb{Z}) = g_n(j).$$

Therefore

$$x_{n,j} \cdot x_{n,g_n(0)} = x_{n,j} \cdot x_{n,0} = x_{n,j} + x_{n,g_n(j)} = x_{n,g_n(j)} + x_{n,g_n((n-1)j)}$$

and $(**)$ was proved.

We have

$$x_{n,j} \cdot x_{n,g_n((n-1)j)} = x_{n,j} \cdot x_{n,g_n(j)} = x_{n,j} \cdot x_{n,j} = x_{n,0} + x_{n,g_n(2j)}$$

$$= x_{n,g_n(0)} + x_{n,g(-2j)} = x_{n,g_n(0)} + x_{n,g_n((n-2)j)}$$

and $(***)$ was shown. ■

Notice that if the vector $[y_1, y_2, \ldots, y_n]$ is an eigenvector for the matrix a_2 then the vector $[y_n, y_1, y_2, \ldots, y_{n-1}]$ is also an eigenvector for the matrix a_2.

Let $n \in \mathbb{N}$ and $0 \leq j \leq \left[\frac{n}{2}\right]$. Let

$u_{n,j} =$

$[x_{n,g_n((n-1)j))}, x_{n,g_n(0)}, x_{n,g_n(j)}, x_{n,g_n(2j)}, \ldots, x_{n,g_n(kj)}, \ldots, x_{n,g_n((n-2)j))}] \in \mathbb{C}^n.$

Let $E_{n,j+1} = lin(v_{n,j}, u_{n,j})$ and $e_{n,j+1}$ be a matrix of the projection $\mathbb{C}^n$ onto $E_{n,j+1}$.

Lemma 7.

$$dimE_{n,j} = \begin{cases} 1 & for\ j = 1\ or\ (j = \frac{n}{2} + 1\ and\ 2|n) \\ \\ 2 & otherwise. \end{cases}$$

Proof. If $j = 1$ then $E_{n,1} = lin(v_{n,0}, u_{n,0}) = lin([x_{n,0}, \ldots, x_{n,0}], [x_{n,0}, \ldots, x_{n,0}])$, so $dimE_{n,1} = 1$.

If $2|n$ and $j = \frac{n}{2} + 1$ then $v_{n,j-1} = [2, -2, 2, \ldots, (-1)^{n+1}2]$ (since $x_{n,\frac{n}{2}} = -2$, $x_{n,0} = 2$ and $g_n(\frac{nk}{2}) = 0$ for k odd and $g_n(\frac{nk}{2}) = \frac{n}{2}$ for k even) and $u_{n,j-1} = (-1)^{n+1}v_{n,j-1}$ hence $dimE_{n,j} = 1$.

Otherwise

$$\det \begin{bmatrix} x_{n,g_n(0)} & x_{n,g_n(j-1)} \\ \\ x_{n,g_n((n-1)(j-1))} & x_{n,g_n(0)} \end{bmatrix} = x_{n,0}^2 - x_{n,j-1}^2 = 4 - x_{n,j-1}^2 \neq 0$$

hence $v_{n,j-1}$ and $u_{n,j-1}$ are linear independent vectors. ∎

Observe that $dimE_{n,1} + \ldots + dimE_{n,\left[\frac{n}{2}\right]+1} = n$ and $\mathbb{C}^n = E_{n,1} \oplus \ldots \oplus E_{n,\left[\frac{n}{2}\right]+1}$.

Lemma 8. *If* $n = 2r + 1$ *and* $r > 3$ *then*

$$u_n(x) = x^r + x^{r-1} + (1-r)x^{r-2} + \ldots$$

If $n = 2r$ *and* $r > 2$ *then*

$$u_n(x) = x^{r-1} + 0 \cdot x^{r-2} + (2-r)x^{r-3} + \ldots$$

Proof. $u_5(x) = x^2 + x - 1$ and $u_6(x) = x^2 - 1$. Therefore lemma is true for $n = 5$ and $n = 6$.

If lemma is true for $n = 2r$ and $n = 2r - 1$ then

$$u_{2r+1}(x) = (x+2)u_{2r} - u_{2r-1}(x)$$

$$= (x+2)(x^{r-1}+(2-r)x^{r-3}+\dots) - (x^{r-1}+x^{r-2}+(1-(r-1))x^{r-3}+\dots)$$

$$= x^r + (2-1)x^{r-1} + ((2-r)-1)x^{r-2} + \dots$$

and

$$u_{2r+2}(x) = u_{2r+1}(x) - u_{2r}(x)$$

$$= x^r + x^{r-1} + (1-r)x^{r-2} + \dots - (x^{r-1} + (2-r)x^{r-3} + \dots)$$

$$= x^r+(1-1)x^{r-1}+(1-r-0)x^{r-2} + \dots = x^r + 0 \cdot x^{r-1}+(1-r)x^{r-2}+ \dots$$

Hence lemma is true for $n = 2r + 1$ and $n = 2r + 2$. ■

Lemma 9.

$$x_{n,0}^2 + \dots + x_{n,[\frac{n}{2}]}^2 = \begin{cases} n+2 \ for \ n \ even \\ n+4 \ for \ n \ odd \end{cases}.$$

Proof. Consider the following cases:

1. If n is even and $n = 2k + 1$. By Lemma 6 we know that $x_{n,1}, \dots, x_{n,k}$ are eigenvalues of the matrix a_2. Hence they are roots of w_n. Obviously $x_{n_i} \neq 2$ for $i = 1, \dots, k$, so by Lemma 3 they are roots of u_n. Therefore we have

$$(*) \qquad u_n(x) = (x - x_{n,1})(x - x_{n,2}) \dots (x - x_{n,k}).$$

Using Lemma 8 we obtain $x_{n,1} + \dots + x_{n,k} = -1$ and $\sum_{1 \le i < j \le k} x_{n,i}x_{n,j} = 1 - k$.

Hence

$$x_{n,1}^2 + \dots + x_{n,k}^2 = (x_{n,1} + \dots + x_{n,k})^2 - 2 \sum_{1 \le i < j \le k} x_{n,i}x_{n,j}$$

$$= 1 - 2(1-k) = 2k - 1 = n - 2$$

and $x_{n,0}^2 + x_{n,1}^2 + \dots + x_{n,k}^2 = 4 + n - 2 = n + 2$.

2. Assume that n is odd and $n = 2k$. Then

$$(**) \qquad u_n(x) = (x - x_{n,1})(x - x_{n,2}) \dots (x - x_{n,k-1})$$

because $x_{n,1}, \dots, x_{n,k-1}$ are eigenvalues of the matrix a_2 by Lemma 6, hence they are roots of w_n and by Lemma 3 they are also roots of u_n. So by Lemma 8 we have $(**)$.

By Lemma 8 it turns out that $x_{n,1} + \dots + x_{n,k-1} = 0$ and $\sum_{1 \le i < j \le k-1} x_{n,i}x_{n,j} = 2 - k$.

Hence

$$x_{n,1}^2 + \dots + x_{n,k-1}^2 = (x_{n,1} + \dots + x_{n,k-1})^2 - 2 \sum_{1 \le i < j \le k-1} x_{n,i}x_{n,j}$$

$$= 0 - 2(2 - k) = 2k - 4 = n - 4$$

and $x_{n,0}^2 + x_{n,1}^2 + \dots + x_{n,k-1}^2 + x_{n,k} = 4 + (n - 4) + 4 = n + 4$.

■

Lemma 10. *Let $n, k \in \mathbb{N}$ and $gcd(n, k) = 1$. Let $A = \{0, 1, \dots, \left[\frac{n}{2}\right]\}$ and $f\colon A \to A$ be a function such that $f(x) = g_n(kx)$. Then f is a bijection.*

Proof. It is sufficient to show that f is $1 - 1$. Suppose $i, j \in A$, $i < j$ and $f(i) = f(j)$. Let $x = dist(ik, n\mathbb{Z}) = dist(jk, n\mathbb{Z})$. There exist $p, q \in \mathbb{Z}$ such that $|ik - pn| = |jk - qn|$.

If $ik - pn = jk - qn$ then $(i - j)k = (p - q)n$ hence $n|j - i$ (since $gcd(n, k) = 1$) but $j - i \in A$ and we have a contradiction.

If $ik - pn = -jk + qn$ then $(i + j)k = (p + q)n$ hence $n|i + j$ but $i, j \in A$ so $0 < i + j \le \left[\frac{n}{2}\right] + \left[\frac{n}{2}\right] - 1 < n$ and we obtain a contradiction. ■

Lemma 11. *Let $n, k, p \in \mathbb{N}$ and $0 \le k \le \left[\frac{n}{2}\right]$. Then $|v_{pn,pk}|^2 = p|v_{n.k}|^2$.*

Proof. Let us note that $g_{pn}(px) = dist(px, pn\mathbb{Z}) = p \cdot dist(x, n\mathbb{Z}) = pg_n(x)$ for any $x \in \mathbb{Z}$, $v_{np,kp} = [(x_{pn,g_{pn}((i-1)pk))})_{i=1,\dots,pn}] = [(x_{pn,pg_n((i-1)k)})_{i=1,\dots,pn}]$ and $g_n((n + i - 1)k) = g_n((i - 1)k)$.

Consider the following cases:

1. If $2|n$ then $x_{pn,pj} = 2\cos(\frac{2pj\pi}{pn}) = 2\cos(\frac{2j\pi}{n}) = x_{n,j}$. Hence $v_{np,nk} = [(x_{n,g_n((i-1)k)})_{i=1,\dots,pn}]$ and $v_{pn,pk} = \underbrace{[v_{n,k}, v_{n,k}, \dots, v_{n,k}]}_{p-times}$ and $|v_{n,k}|^2 = p|v_{n,k}|^2$.
2. If $2\not|n$ and $2\not|p$ then $x_{pn,pj} = (-1)^{pj}2\cos(\frac{pj\pi}{pn}) = (-1)^j 2\cos(\frac{j\pi}{n}) = x_{n,j}$, $v_{pn,pk} = \underbrace{[v_{n,k}, v_{n,k}, \dots, v_{n,k}]}_{p-times}$ and $|v_{n,k}|^2 = p|v_{n,k}|^2$.
3. If $2\not|n$ and $2|p$ then $x_{pn,pj} = 2\cos(\frac{2pj\pi}{pn}) = 2\cos(\frac{2j\pi}{n}) = 2(2\cos^2(\frac{j\pi}{n}) - 1) = 4\cos^2(\frac{j\pi}{n}) - 2 = x_{n,j}^2 - 2$ and by Lemma 4 we have $x_{pn,pj} = x_{n,0} + x_{n,g_n(2j)} - 2 = x_{n,g_n(2j)}$. By Lemma 10 $v_{pn,pk} = \underbrace{[\widetilde{v_{n,k}}, \widetilde{v_{n,k}}, \dots, \widetilde{v_{n,k}}]}_{p-times}$ (since $gcd(2,n) = 1$), where coordinates of $\widetilde{v_{n,k}}$ arise as a result of the permutation of coordinates of $v_{n,k}$. Hence $|v_{pn,pk}|^2 = p|v_{n.k}|^2$.

■

Theorem 1. *Let $n \in \mathbb{N}$ and $0 \le j \le \left[\frac{n}{2}\right]$. Then*

$$|v_{n,j}|^2 = \begin{cases} 4n & for\ j = 0\ or\ (j = \frac{n}{2}\ and\ 2|n) \\ 2n & otherwise. \end{cases}$$

Proof. Assume that $gcd(n,j) = 1$.

Let $n = 2r+1$. According to the fact that $g_n((i-1)k) = g_n((n-i+1)k)$ and by Lemma 10 we have

$$|v_{n,j}|^2 = |[x_{n,0}, x_{n,1}, \dots, x_{n,r}, x_{n,r}, \dots, x_{n,1}]|^2 = 2(x_{n,0}^2 + \dots + x_{n,r}^2) - x_{n,0}^2$$
$$= 2(n+2) - 4 = 2n$$

using Lemma 9.

Let $n = 2r$. Then

$$|v_{n,j}|^2 = |[x_{n,0}, x_{n,1}, \dots, x_{n,r-1}, x_{n,r}, x_{n,r-1}, \dots, x_{n,1}]|^2,$$

by Lemma 10. Hence

$$|v_{n,j}|^2 = 2(x_{n,0}^2 + \dots + x_{n,r}^2) - x_{n,0}^2 - x_{n,r}^2 = 2(n+4) - 4 - 4 = 2n,$$

by Lemma 9.

Assume that $gcd(n, j) \neq 1$. Let $p = gcd(n, j)$, $n = pn'$, $j = pj'$, where $gcd(n', j') = 1$. By Lemma 11 we have $|v_{n,j}|^2 = p|v_{n',j'}|^2$.

One needs to consider the following cases:

1. If $j = 0$ then $v_{n,j} = [\underbrace{2, 2, \ldots 2}_{n-times}]$ and $|v_{n,j}|^2 = 4n$.
2. If $2 \not| n$ and $j \neq 0$ then $2 \not| n'$ and $|v_{n,j}|^2 = p|v_{n',j'}|^2 = p2n' = 2n$.
3. If $2|n$, $2 \not| n'$ and $j \neq 0$ then $2|p$ and $j \neq \frac{n}{2}$ (because if $j = \frac{n}{2}$ then $\frac{p}{2}n' = j = pj' = \frac{p}{2}2j'$ and $n' = 2j'$ but $2 \not| n'$). Hence $|v_{n,j}|^2 = p|v_{n',j'}|^2 = p2n' = 2n$.
4. If $2|n$, $2|n'$ and $j' = \frac{n'}{2}$ then $j = pj' = p\frac{n'}{2} = \frac{n}{2}$ and $|v_{n,j}|^2 = p|v_{n',j'}|^2 = p4n' = 4n$.
5. If $2|n$, $2|n'$, $j \neq 0$ and $j' \neq \frac{n'}{2}$ then $j' \neq 0$, $j \neq \frac{n}{2}$ and $|v_{n,j}|^2 = p|v_{n',j'}|^2 = p2n' = 2n$. ■

Let $b = (b_{ij})_{1 \leq i,j \leq n} \in M_n(\mathbb{C})$ be a matrix. Then let $\bar{b} = [b_{11}, \ldots, b_{1n}]$. Obviously $^-$ is a linear operation.

For $1 \leq i \leq \left[\frac{n}{2}\right]$ let e_i be a matrix of the projection of $\mathbb{C}^n$ onto $E_{n,i}$. We know (see [3]) that $lin(a_1, \ldots, a_{[\frac{n}{2}]+1}) = lin(e_1, \ldots, e_{[\frac{n}{2}]+1})$.

Let $n \in \mathbb{N}$, $1 \leq i \leq \left[\frac{n}{2}\right]$ and $a_i = \sum_{j=1}^{[\frac{n}{2}]+1} \xi_{ij} e_j$.

Lemma 12. *Let $n \in \mathbb{N}$ and $1 \leq i, j \leq \left[\frac{n}{2}\right] + 1$. Then*

$$\xi_{i,j} = \begin{cases} 1 \ for \ i = 1 \ or \ \left(i = \dfrac{n}{2} + 1, (2|n) \ and \ j = 1\right) \\ 2 \ for \ j = 1 \ and \ i \neq 1 \ and \ \left(if \ 2|n \ then \ i \neq \dfrac{n}{2} + 1\right) \\ \dfrac{1}{2} x_{n,g_n((i-1)(j-1))} \ for \ 2|n \ and \ i = \dfrac{n}{2} + 1 \ and \ j \neq 1 \\ x_{n,g_n((i-1)(j-1))} \ otherwise. \end{cases}$$

Proof. It is obvious that

$$\bar{a}_i = \sum_{j=1}^{[\frac{n}{2}]+1} \xi_{ij}\bar{e}_j \quad \text{and} \quad \bar{e}_j = \frac{[1,0,\dots,0]\circ v_{n,j-1}}{|v_{n,j-1}|^2}v_{n,j-1} = \frac{2v_{n,j-1}}{|v_{n,j-1}|^2},$$

where $\circ$ means the scalar product of vectors. Using Theorem 1 we have

$$\bar{e}_j = \begin{cases} \dfrac{1}{2n}v_{n,j-1} & \text{for } j=1 \text{ or } \left(j=\dfrac{n}{2}+1 \text{ and } 2|n\right) \\ \dfrac{1}{n}v_{n,j-1} & \text{otherwise.} \end{cases}$$

Hence $\bar{e}_1,\dots,\bar{e}_{[\frac{n}{2}]+1}$ are pairwise orthogonal. Therefore $\xi_{i,j} = \dfrac{\bar{a}_i\circ\bar{e}_j}{|\bar{e}_j|^2}$.

Consider the following cases:

1. If $i=1$ and $j=1$ or $(j=\frac{n}{2}+1$ and $2|n)$ then

$$\xi_{1,j} = \frac{\bar{a}_1\circ\bar{e}_j}{|\bar{e}_j|^2} = \frac{\frac{1}{2n}2}{\frac{1}{4n^2}|v_{n,j-1}|} = \frac{\frac{1}{n}}{\frac{1}{4n^2}4n} = 1.$$

2. If $i=1$ and $j\neq 1$ and $(j\neq\frac{n}{2}+1$ if $2|n)$ then

$$\xi_{1,j} = \frac{\bar{a}_1\circ\bar{e}_j}{|\bar{e}_j|^2} = \frac{\frac{1}{n}2}{\frac{1}{n^2}|v_{n,j-1}|} = \frac{\frac{2}{n}}{\frac{1}{n^2}2n} = 1.$$

3. If $i=\frac{n}{2}+1$, $2|n$ and $j=1$ then

$$\xi_{i,1} = \frac{\bar{a}_i\circ\bar{e}_1}{|\bar{e}_1|^2} = \frac{\frac{1}{n}}{\frac{1}{4n^2}4n} = 1.$$

4. If $j=1$ and $i\neq 1$ and $(i\neq\frac{n}{2}+1$ if $2|n)$ then

$$\xi_{i,1} = \frac{\bar{a}_i\circ\bar{e}_1}{|\bar{e}_1|^2} = \frac{\frac{2}{n}}{\frac{1}{4n^2}4n} = 2.$$

5. If $2|n$ and $i = \frac{n}{2}+1$, $j \neq 1$ and $j \neq \frac{n}{2}+1$ then

$$\xi_{i,j} = \frac{\bar{a}_i \circ \bar{e}_j}{|\bar{e}_j|^2} = \frac{\frac{1}{n} x_{n,g_n(\frac{n}{2}(j-1))}}{\frac{1}{n^2}|v_{n,j-1}|^2} = \frac{1}{2} x_{n,g_n(\frac{n}{2}(j-1))}.$$

6. If $2|n$ and $i = \frac{n}{2}+1$ and $j = \frac{n}{2}+1$ then

$$\xi_{i,j} = \frac{\bar{a}_i \circ \bar{e}_j}{|\bar{e}_j|^2} = \frac{\frac{1}{2n} x_{n,g_n(\frac{n}{2}\frac{n}{2})}}{\frac{1}{4n^2}|v_{n,j-1}|^2} = \frac{1}{2} x_{n,g_n(\frac{n}{2}\frac{n}{2})}.$$

7. If $2|n$ and $j = \frac{n}{2}+1$, $i \neq 1$ and $i \neq \frac{n}{2}+1$ then $\bar{a}_i = [b_1, \ldots, b_n]$, where

$$b_j = \begin{cases} 1 & \text{for } j = i \text{ or } j = n-i+2 \\ 0 & \text{for } j \neq i \text{ and } j \neq n-i+2. \end{cases}$$

Moreover $v_{n,\frac{n}{2}} = [2, -2, \ldots 2(-1)^{n+1}]$. Hence

$$\xi_{i,j} = \frac{\frac{1}{2n}(2(-1)^{i+1} + 2(-1)^{n-i+1})}{\frac{1}{4n^2}4n} = 2(-1)^{i+1} = x_{n,g_n((i-1)\frac{n}{2})}.$$

8. If $i \neq 1$, $j \neq 1$, ($i \neq \frac{n}{2}+1$ and $j \neq \frac{n}{2}+1$ if $2|n$) then

$$\xi_{i,j} = \frac{\frac{1}{n}\left(x_{n,g_n((i-1)(j-1))} + x_{n,g_n((n-i+1)(j-1))}\right)}{\frac{1}{n^2}2n}$$
$$= \frac{\frac{2}{n} x_{n,g_n((i-1)(j-1))}}{\frac{1}{n^2}2n} = x_{n,g_n((i-1)(j-1))}.$$

■

Let $f_i = dim_{\mathbb{C}} E_{n,i}$, $n_j = \frac{|C_j|}{n}$ and $\varphi_{i,j} = \sqrt{f_i}\xi_{j,i} n_j^{-1}$ for $i, j \in \{1, \ldots, \left[\frac{n}{2}\right]\}$. Then $(\varphi_{i,j})_{1 \leq i,j \leq \left[\frac{n}{2}\right]}$ is the character table of the quasigroup $(\mathbb{Z}_n, -_n)$.

The next Theorem gives the description of the character table of the quasigroup $(\mathbb{Z}_n, -_n)$.

Theorem 2. *Let $n \in \mathbb{N}$ and $1 \le i, j \le \left[\frac{n}{2}\right] + 1$. Then*

$$\varphi_{i,j} = \begin{cases} 1 \; for \; i = 1 \; or \; (i = \frac{n}{2} + 1, (2|n) \; and \; j = 1) \\ \sqrt{2} \; for \; j = 1 \; and \; i \neq 1 \; and \; \left(if \;\; 2|n \; then \; i \neq \dfrac{n}{2} + 1\right) \\ (-1)^{j-1} \; for \; 2|n \; and \; i = \dfrac{n}{2} + 1 \; and \; j \neq 1 \\ \dfrac{\sqrt{2}}{2} x_{n,g_n((i-1)(j-1))} \; otherwise. \end{cases}$$

Hence for n even we obtain

	$j = 1$	$j \neq 1$
$i = 1$	$\varphi_{i,j} = 1$	$\varphi_{i,j} = 1$
$i \neq 1$	$\varphi_{i,j} = \sqrt{2}$	$\varphi_{i,j} = \frac{\sqrt{2}}{2} x_{n,g_n((i-1)(j-1))}$

and for n odd we have

	$j = 1$	$j \neq 1, j \neq \frac{n}{2} + 1$	$j = \frac{n}{2} + 1$
$i = 1$	$\varphi_{i,j} = 1$	$\varphi_{i,j} = 1$	$\varphi_{i,j} = 1$
$i \neq 1, i \neq \frac{n}{2} + 1$	$\varphi_{i,j} = \sqrt{2}$	$\varphi_{i,j} = \frac{\sqrt{2}}{2} x_{n,g_n((i-1)(j-1))}$	$\varphi_{i,j} = \frac{\sqrt{2}}{2} x_{n,g_n((i-1)(j-1))}$
$i = \frac{n}{2} + 1$	$\varphi_{i,j} = 1$	$\varphi_{i,j} = (-1)^{j-1}$	$\varphi_{i,j} = (-1)^{\frac{n}{2}}$

Proof. We must consider the following cases (we use Lemma 7 to calculate f_i):

1. If $i = 1$ and ($j = 1$ or ($j = \frac{n}{2} + 1$ and $2|n$)) then

$$\varphi_{i,j} = \sqrt{1} \xi_{j,i} \frac{n}{n} = 1.$$

2. If $i=1$, $j\neq 1$ and (if $2|n$ then $j\neq \frac{n}{2}+1$) then

$$\varphi_{i,j}=\sqrt{1}\xi_{j,i}\frac{n}{2n}=1.$$

3. If $2|n$, $i=\frac{n}{2}+1$ and $j=1$ then

$$\varphi_{i,j}=\sqrt{1}\xi_{j,i}\frac{n}{n}=1.$$

4. If $2|n$, $i=\frac{n}{2}+1$ and $j=\frac{n}{2}+1$ then

$$\varphi_{i,j}=\sqrt{1}\xi_{j,i}\frac{n}{n}=\frac{1}{2}x_{n,g_n(\frac{n}{2}\frac{n}{2})}=(-1)^{\frac{n}{2}}=(-1)^{j-1}.$$

5. If $2|n$, $i=\frac{n}{2}+1$ and $j\neq 1$ and $j\neq \frac{n}{2}+1$ then

$$\varphi_{i,j}=\sqrt{1}\xi_{j,i}\frac{n}{2n}=\frac{1}{2}x_{n,g_n((j-1)\frac{n}{2})}=(-1)^{j-1}.$$

6. If $i\neq 1$ and $j=1$ and (if $2|n$ then $i\neq \frac{n}{2}+1$)then

$$\varphi_{i,j}=\sqrt{2}\xi_{j,i}\frac{n}{n}=\sqrt{2}.$$

7. If $i\neq 1$ and $i\neq \frac{n}{2}+1$ and $2|n$ and $j=\frac{n}{2}+1$ then

$$\varphi_{i,j}=\sqrt{2}\xi_{j,i}\frac{n}{n}=\sqrt{2}\frac{1}{2}x_{n,g_n(\frac{n}{2}(i-1))}=\sqrt{2}(-1)^{\frac{n}{2}}=\sqrt{2}\frac{1}{2}x_{n,g_n((i-1)(j-1))}.$$

8. If $i\neq 1$ and (if $2|n$ then $i\neq \frac{n}{2}+1$) and $j\neq 1$ and (if $2|n$ then $j\neq \frac{n}{2}+1$) then

$$\varphi_{i,j}=\sqrt{2}\xi_{j,i}\frac{n}{2n}=\frac{\sqrt{2}}{2}x_{n,g_n((i-1)(j-1))}.$$

■

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