# LATTICES OF RELATIVE COLOUR-FAMILIES AND ANTIVARIETIES

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#### Abstract

We consider general properties of lattices of relative colour-families and antivarieties. Several results generalise the corresponding assertions about colour-families of undirected loopless graphs, see [1]. Conditions are indicated under which relative colour-families form a lattice. We prove that such a lattice is distributive. In the class of lattices of antivarieties of relation structures of finite signature, we distinguish the most complicated (universal) objects. Meet decompositions in lattices of colour-families are considered. A criterion is found for existence of irredundant meet decompositions. A connection is found between meet decompositions and bases for anti-identities.

**Keywords:** colour-family, antivariety, lattice of antivarieties, meet decomposition, basis for anti-identities.

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#### 1. Preliminary facts

Throughout the article, by a *structure* we mean a relation structure of a fixed signature  $\sigma = (r_j)_{j \in J}$ . A structure is said to be *finite* if its universe is a finite set. A *homomorphism* from a structure  $\mathcal{A}$  into a structure  $\mathcal{B}$  is a map  $\varphi : \mathcal{A} \to \mathcal{B}$  such that  $(\varphi(a_1), \ldots, \varphi(a_n)) \in r_j^{\mathcal{B}}$  for all  $j \in J$  and

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 $a_1, \ldots, a_n \in A$  with  $(a_1, \ldots, a_n) \in r_j^{\mathcal{A}}$ . If there exists a homomorphism from  $\mathcal{A}$  into  $\mathcal{B}$  then we write  $\mathcal{A} \to \mathcal{B}$ ; otherwise, we write  $\mathcal{A} \to \mathcal{B}$ .

For every class  $\mathbf{K}$ , let  $\mathbf{K}_{f}$  denote the set of isomorphism types of finite structures in  $\mathbf{K}$ . On  $\mathbf{K}_{f}$ , define an equivalence relation  $\equiv$  as follows:  $\mathcal{A} \equiv \mathcal{B}$  if and only if  $\mathcal{A} \to \mathcal{B}$  and  $\mathcal{B} \to \mathcal{A}$ . The relation  $\to$  induces a partial order  $\leq$  on the quotient set  $\mathbf{K}_{f}/_{\equiv}$ . Let  $\operatorname{Core}(\mathbf{K})$  denote the resulting partially ordered set.

In the sequel, it is convenient to consider an isomorphic partially ordered set whose universe is the set of cores of finite structures in **K**. Recall [2, Section 2] that a finite structure is a *core* if all its endomorphisms are automorphisms. A structure  $\mathcal{A}$  is a *core of a structure*  $\mathcal{B}$  if  $\mathcal{A}$  is a minimal retract of  $\mathcal{B}$  (with respect to set inclusion). Simple properties of cores can be found, for example, in [2, Proposition 2.1]. It is easy to see that, in every coset  $\mathcal{G}/_{\equiv}$ , there exists a unique (up to isomorphism) core. We denote this core by Core( $\mathcal{G}$ ). The map defined by the rule  $\mathcal{G}/_{\equiv} \mapsto \text{Core}(\mathcal{G})$  is an isomorphism.

Let **K** be a class of structures. For every  $A \in \mathbf{K}$ , let

$$[\mathbf{K} \to \mathcal{A}] = \{\mathcal{B} \in \mathbf{K} : \mathcal{B} \to \mathcal{A}\}.$$

If there is no ambiguity or **K** is the class of all structures of a given signature then we write  $[\rightarrow \mathcal{A}]$  instead of  $[\mathbf{K} \rightarrow \mathcal{A}]$ . For every set  $\mathbf{A} \subseteq \mathbf{K}$ , let  $[\mathbf{K} \rightarrow \mathbf{A}] = \bigcup_{\mathcal{A} \in \mathbf{A}} [\mathbf{K} \rightarrow \mathcal{A}]$ . If **A** is a finite set of finite structures then  $[\mathbf{K} \rightarrow \mathbf{A}]$ is called a **K**-colour-family. If  $|\mathbf{A}| = 1$  then the **K**-colour-family  $[\mathbf{K} \rightarrow \mathbf{A}]$ is said to be principal. Let  $L_0(\mathbf{K})$  denote the partially ordered set of all **K**-colour-families with respect to set inclusion. (If  $L_0(\mathbf{K})$  has no greatest element then by  $L_0(\mathbf{K})$  we mean the set of all **K**-colour-families with a new greatest element  $\mathbf{1}_{\mathbf{K}}$ ).

We recall the definition of operations with structures from [3]. Let  $\mathcal{A}$  and  $\mathcal{B}$  be structures and let  $(\mathcal{A}_i)_{i \in I}$  be a family of structures.

On the disjoint union of the universes of  $\mathcal{A}$  and  $\mathcal{B}$ , define a structure  $\mathcal{A} + \mathcal{B}$  of signature  $\sigma$  as follows:  $(a_1, \ldots, a_n) \in r^{\mathcal{A}+\mathcal{B}}$  if and only if either  $(a_1, \ldots, a_n) \in r^{\mathcal{A}}$  or  $(a_1, \ldots, a_n) \in r^{\mathcal{B}}$ . The resulting structure is called the sum of  $\mathcal{A}$  and  $\mathcal{B}$ . We have

(1) 
$$\mathcal{A} + \mathcal{B} \to \mathcal{C} \iff \mathcal{A} \to \mathcal{C} \text{ and } \mathcal{B} \to \mathcal{C}$$

for every structure C. A structure  $\mathcal{A}$  is said to be *connected* if it cannot be represented in the form  $\mathcal{A}_1 + \mathcal{A}_2$ , where  $\mathcal{A} \not\rightarrow \mathcal{A}_i$ , i = 1, 2.

On the Cartesian product of the universes of  $\mathcal{A}_i$ ,  $i \in I$ , define a structure  $\prod_{i \in I} \mathcal{A}_i$  of signature  $\sigma$  as follows:  $(a_1, \ldots, a_n) \in r^{\prod_{i \in I} \mathcal{A}_i}$  if and only if  $(a_1(i), \ldots, a_n(i)) \in r^{\mathcal{A}_i}$  for every  $i \in I$ . The resulting structure is called the *product* of the family  $(\mathcal{A}_i)_{i \in I}$ . We have

(2) 
$$\mathcal{C} \to \prod_{i \in I} \mathcal{A}_i \iff \mathcal{C} \to \mathcal{A}_i \text{ for all } i \in I$$

for every structure  $\mathcal{C}$ .

On the set  $A^B$  of all functions from B into A, define a structure  $\mathcal{A}^{\mathcal{B}}$  of signature  $\sigma$  as follows:  $(f_1, \ldots, f_n) \in r^{\mathcal{A}^{\mathcal{B}}}$  if and only if  $(f_1(b_1), \ldots, f_n(b_n)) \in r^{\mathcal{A}}$  for all  $b_1, \ldots, b_n \in B$  with  $(b_1, \ldots, b_n) \in r^{\mathcal{B}}$ . The resulting structure is called the *exponent* of  $\mathcal{A}$  by  $\mathcal{B}$ . We have

$$(3) \qquad \qquad \mathcal{C} \to \mathcal{A}^{\mathcal{B}} \iff \mathcal{B} \times \mathcal{C} \to \mathcal{A}$$

for every structure C.

## 2. LATTICES OF COLOUR-FAMILIES

In this section, we show that partially ordered sets of **K**-colour-families are usually lattices and study their lattice-theoretical properties.

Lemma 1. If K is a class satisfying the condition

(4) 
$$\mathcal{A} \times \mathcal{B} \in \mathbf{K} \text{ for all finite } \mathcal{A}, \mathcal{B} \in \mathbf{K}$$

then  $L_0(\mathbf{K})$  is a distributive lattice with respect to the set-theoretical operations.

**Proof.** Let  $\mathbf{K}_1 = [\mathbf{K} \to (\mathcal{A}_i)_{i < n}], \mathbf{K}_2 = [\mathbf{K} \to (\mathcal{B}_j)_{j < m}]$ . It is clear that  $\mathbf{K}_1 \cup \mathbf{K}_2$  is a **K**-colour-family, i.e.,  $\mathbf{K}_1 \vee \mathbf{K}_2 = \mathbf{K}_1 \cup \mathbf{K}_2$ .

Let  $\mathcal{A} \in \mathbf{K}_1 \cap \mathbf{K}_2$ . Then  $\mathcal{A} \in \mathbf{K}$  and there exist i < n and j < msuch that  $\mathcal{A} \to \mathcal{A}_i$  and  $\mathcal{A} \to \mathcal{B}_j$ . Hence,  $\mathcal{A} \to \mathcal{A}_i \times \mathcal{B}_j$ , cf. (2). By (4), we have  $\mathcal{A}_i \times \mathcal{B}_j \in \mathbf{K}$ . Conversely, if  $\mathcal{A} \to \mathcal{A}_i \times \mathcal{B}_j$ , where  $\mathcal{A} \in \mathbf{K}$ , i < n, and j < m, then we have  $\mathcal{A} \to \mathcal{A}_i$  and  $\mathcal{A} \to \mathcal{B}_j$ . Hence,  $\mathbf{K}_1 \cap \mathbf{K}_2 = \begin{bmatrix} \mathbf{K} \to (\mathcal{A}_i \times \mathcal{B}_j)_{i < n} \end{bmatrix}$ . In view of (4), we have  $\mathcal{A}_i \times \mathcal{B}_j \in \mathbf{K}$  for all i < nand j < m. **Remark 2.** Another lattice of (principal) colour-families was considered in [4, 5]. In fact, the universe of that lattice is the set of cores, the meet operation coincides with the meet operation in  $L_0(\mathbf{K})$ , while the join operation corresponds to the sum of relation structures. That lattice is distributive too.

Let L be a lattice and let  $a, b \in L$ . By a *relative pseudocomplement* of a with respect to b we mean an element a \* b such that

$$a \wedge x \leqslant b \iff x \leqslant a * b$$

for all  $x \in L$ . If a relative pseudocomplement exists for every pair of elements of L then L is said to be a *relatively pseudocomplemented lattice*.

**Lemma 3.** If  $\mathbf{K}$  is a class satisfying (4) and the condition

(5) 
$$\mathcal{A}^{\mathcal{B}} \in \mathbf{K} \text{ for all finite } \mathcal{A}, \mathcal{B} \in \mathbf{K} \text{ with } \mathcal{B} \nrightarrow \mathcal{A}$$

then  $L_0(\mathbf{K})$  is a relatively pseudocomplemented lattice.

**Proof.** Let  $\mathbf{K}_1 = [\mathbf{K} \to (\mathcal{A}_i)_{i < n}]$  and let  $\mathbf{K}_2 = [\mathbf{K} \to (\mathcal{B}_j)_{j < m}]$ . We introduce the notation  $\mathbf{K}^i = [\mathbf{K} \to (\mathcal{B}_j^{\mathcal{A}_i})_{j < m}]$ . By (5), we have  $\mathcal{B}_j^{\mathcal{A}_i} \in \mathbf{K}$  provided  $\mathcal{A}_i \to \mathcal{B}_j$ . If  $\mathcal{A}_i \to \mathcal{B}_j$  then  $\mathcal{A}_i \times \mathcal{C} \to \mathcal{B}_j$  for every  $\mathcal{C} \in \mathbf{K}$ . By (3), we obtain  $\mathcal{C} \to \mathcal{B}_j^{\mathcal{A}_i}$ , i.e.,  $[\mathbf{K} \to \mathcal{B}_j^{\mathcal{A}_i}] = \mathbf{K}$  is the greatest element of  $L_0(\mathbf{K})$ . Therefore,  $[\mathbf{K} \to \mathcal{B}_j^{\mathcal{A}_i}] \in L_0(\mathbf{K})$  for all i < n and j < m.

By Lemma 1, we have  $[\mathbf{K} \to \mathcal{A}_i] \cap \mathbf{K}^i = \left[\mathbf{K} \to (\mathcal{B}_j^{\mathcal{A}_i} \times \mathcal{A}_i)_{j < m}\right]$ . It is immediate from (3) that  $\mathcal{B}_j^{\mathcal{A}_i} \times \mathcal{A}_i \to \mathcal{B}_j$ . Hence,  $[\mathbf{K} \to \mathcal{A}_i] \cap \mathbf{K}^i \subseteq \mathbf{K}_2$ . Let  $\mathbf{K}_3 = [\mathbf{K} \to (\mathcal{C}_k)_{k < l}]$  be a **K**-colour-family such that  $[\mathbf{K} \to \mathcal{A}_i] \cap \mathbf{K}_3 \subseteq \mathbf{K}_2$ . By Lemma 1, we have  $[\mathbf{K} \to \mathcal{A}_i] \cap \mathbf{K}_3 = [\mathbf{K} \to (\mathcal{A}_i \times \mathcal{C}_k)_{k < l}]$ . Therefore, for every k < l, there exists a j < m such that  $\mathcal{A}_i \times \mathcal{C}_k \to \mathcal{B}_j$ . By definition,  $\mathcal{C}_k \to \mathcal{B}_j^{\mathcal{A}_i}$ . Thus,  $\mathbf{K}_3 \subseteq \mathbf{K}^i$ .

We have proven that  $\mathbf{K}^i$  is a pseudocomplement of  $[\mathbf{K} \to \mathcal{A}_i]$  with respect to  $\mathbf{K}_2$ . For every distributive lattice L and elements  $a, b, c \in L$ , if a \* c and b \* c exist then so does  $(a \lor b) * c$  and  $(a \lor b) * c = (a * c) \land (b * c)$ , cf. [6, Theorem 9.2.3]. Hence,  $\mathbf{K}_1 * \mathbf{K}_2 = \bigcap_{i < n} \mathbf{K}^i$ .

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#### 3. LATTICES OF ANTIVARIETIES

Recall [2] that a  $\mathbf{K}$ -antivariety is a class defined in  $\mathbf{K}$  by some (possibly, empty) set of *anti-identities*, i.e., sentences of the form

$$\forall x_1 \dots \forall x_n (\neg R_1(\overline{x}) \vee \dots \vee \neg R_m(\overline{x})),$$

where each  $R_i(\overline{x})$  is an atomic formula. By [2, Theorem 1.2], for every universal Horn class **K**, a subclass **K'** is a **K**-antivariety if and only if  $\mathbf{K}' = \mathbf{K} \cap \mathbf{H}^{-1}\mathbf{SP}_{\mathbf{u}}^*(\mathbf{K}')$ , where  $\mathbf{H}^{-1}$ , **S**, and  $\mathbf{P}_{\mathbf{u}}^*$  are operators for taking homomorphic pre-images, substructures, and nontrivial ultraproducts. In particular, each **K**-colour-family is a **K**-antivariety. Let  $\mathbf{L}(\mathbf{K})$  denote the partially ordered set of all **K**-antivarieties with respect to set inclusion.

**Lemma 4.** For every universal Horn class  $\mathbf{K}$ , the partially ordered set  $L(\mathbf{K})$  is a distributive lattice with respect to the set-theoretical operations.

**Proof.** It is clear that  $\mathbf{K}_1 \wedge \mathbf{K}_2 = \mathbf{K}_1 \cap \mathbf{K}_2$  and  $\mathbf{K}_1 \cup \mathbf{K}_2 \subseteq \mathbf{K}_1 \vee \mathbf{K}_2$ for all  $\mathbf{K}_1, \mathbf{K}_2 \in L(\mathbf{K})$ . Since  $\mathbf{K}_1$  and  $\mathbf{K}_2$  are elementary classes, the class  $\mathbf{K}_1 \cup \mathbf{K}_2$  is elementary too. In particular,  $\mathbf{P}^*_{\mathbf{u}}(\mathbf{K}_1 \cup \mathbf{K}_2) \subseteq \mathbf{K}_1 \cup \mathbf{K}_2$ . By [2, Theorem 1.2], we have  $\mathbf{K}_1 \vee \mathbf{K}_2 = \mathbf{K} \cap \mathbf{H}^{-1}\mathbf{S}\mathbf{P}^*_{\mathbf{u}}(\mathbf{K}_1 \cup \mathbf{K}_2)$ . Hence, for every  $\mathcal{A} \in \mathbf{K}_1 \vee \mathbf{K}_2$ , there exists a  $\mathcal{B} \in \mathbf{K}_1 \cup \mathbf{K}_2$  such that  $\mathcal{A} \to \mathcal{B}$ . Since  $\mathbf{K}_1$  and  $\mathbf{K}_2$  are closed under  $\mathbf{H}^{-1}\mathbf{S}$  in  $\mathbf{K}$ , we obtain  $\mathcal{A} \in \mathbf{K}_1 \cup \mathbf{K}_2$  and, consequently,  $\mathbf{K}_1 \vee \mathbf{K}_2 = \mathbf{K}_1 \cup \mathbf{K}_2$ .

The proof of the following lemma is similar to that of [1, Lemma 3.4]

**Lemma 5.** For every universal Horn class **K** and **K**-antivariety **X**, we have  $\mathbf{X} = \mathbf{H}^{-1}\mathbf{SP}^*_{\mathbf{u}}(\operatorname{Core}(\mathbf{X})) \cap \mathbf{K}$ .

If  $\sigma$  is a finite signature then the partially ordered set  $Core(\mathbf{K})$  and the lattices  $L_0(\mathbf{K})$  and  $L(\mathbf{K})$  are related as follows.

**Lemma 6.** For every universal Horn class  $\mathbf{K}$  of finite signature, the lattice  $L(\mathbf{K})$  is isomorphic to the ideal lattice  $I(L_0(\mathbf{K}))$  of the lattice  $L_0(\mathbf{K})$  and to the lattice  $I_o(\text{Core}(\mathbf{K}))$  of order ideals of the partially ordered set  $\text{Core}(\mathbf{K})$ .

This lemma generalises [1, Theorem 3.6] to the case of arbitrary relation structures of finite signature. The proof follows the lines of the proof in [1].

**Proof.** Put  $\varphi(\mathbf{K}') = {\mathbf{A} \in \mathcal{L}_0(\mathbf{K}) : \mathbf{A} \subseteq \mathbf{K}'}$  for every  $\mathbf{K}' \in \mathcal{L}(\mathbf{K})$  and put  $\psi(J) = \bigvee {\mathbf{A} \in \mathcal{L}_0(\mathbf{K}) : \mathbf{A} \in J}$  for every ideal J of  $\mathcal{L}_0(\mathbf{K})$ . It is easy to see that  $\varphi$  is a map from  $\mathcal{L}(\mathbf{K})$  into  $\mathcal{I}(\mathcal{L}_0(\mathbf{K}))$  and  $\psi$  is a map from  $\mathcal{I}(\mathcal{L}_0(\mathbf{K}))$  into  $\mathcal{L}(\mathbf{K})$ ; moreover, both maps are monotone. We prove that  $\varphi = \psi^{-1}$ , which implies that  $\varphi$  and  $\psi$  are isomorphisms.

It is clear that  $\varphi \psi(J) \supseteq J$  and  $\psi \varphi(\mathbf{K}') \subseteq \mathbf{K}'$  for all  $J \in I(L_0(\mathbf{K}))$  and  $\mathbf{K}' \in L(\mathbf{K})$ .

Let  $\mathcal{A} \in \mathbf{K}'$  be a finite structure. Then  $[\mathbf{K} \to \mathcal{A}] \subseteq \mathbf{K}'$  because  $\mathbf{K}'$  is a **K**-antivariety. We have  $\mathcal{A} \in [\mathbf{K} \to \mathcal{A}] \subseteq \bigcup \{\mathbf{A} \in L_0(\mathbf{K}) : \mathbf{A} \subseteq \mathbf{K}'\}$ . Since the **K**-antivariety  $\psi \varphi(\mathbf{K}')$  is generated by its finite structures, we obtain  $\psi \varphi(\mathbf{K}') = \mathbf{K}'$ .

Let  $\mathbf{A} \in \varphi \psi(J)$ , i.e.,  $\mathbf{A} \in \mathcal{L}_0(\mathbf{K})$  and  $\mathbf{A} \subseteq \bigvee_{\mathbf{B} \in J} \mathbf{B}$ . Then  $\mathbf{A} = [\mathbf{K} \to (\mathcal{A}_i)_{i < n}]$ , where each finite structure  $\mathcal{A}_i$  belongs to  $\bigvee_{\mathbf{B} \in J} \mathbf{B}$ . By [2, Theorem 1.2], for every i < n, there exist a family  $(\mathcal{B}_{ij})_{j \in J_i} \subseteq \bigcup_{\mathbf{B} \in J} \mathbf{B}$  and an ultrafilter  $U_i$  over  $J_i$  such that  $\mathcal{A}_i \to \prod_{j \in J_i} \mathcal{B}_{ij}$ . Since  $\mathcal{A}_i$  is a finite structure of finite signature, from [1, Lemma 3.2] it follows that there exists a  $j(i) \in J_i$  such that  $\mathcal{A}_i \to \mathcal{B}_{j(i)}$ . Hence,  $\mathbf{A} \subseteq [\mathbf{K} \to (\mathcal{B}_{j(i)})_{i < n}]$ . Since  $\mathcal{B}_{j(i)} \in \bigcup_{\mathbf{B} \in J} \mathbf{B}$ , we obtain  $[\mathbf{K} \to \mathcal{B}_{j(i)}] \in J$ . Since J is an ideal, we conclude that  $\bigvee_{i < n} [\mathbf{K} \to \mathcal{B}_{j(i)}] \in J$ . Thus,  $\mathbf{A} \in J$  and, consequently,  $\varphi \psi(J) = J$ .

For proving the fact that  $L(\mathbf{K})$  and  $I_o(Core(\mathbf{K}))$  are isomorphic put

$$\varphi_0(\mathbf{K}') = \{ \operatorname{Core}(\mathcal{A}) : \mathcal{A} \in \mathbf{K}' \}, \quad \psi_0(J) = \mathbf{H}^{-1} \mathbf{SP}^*_{\mathbf{u}}(J) \cap \mathbf{K}.$$

By Lemma 5 and [1, Lemma 3.2], we have  $\varphi_0^{-1} = \psi_0$ . It is clear that  $\varphi_0$  and  $\psi_0$  are monotone. Therefore, the lattices  $L(\mathbf{K})$  and  $I_o(Core(\mathbf{K}))$  are isomorphic.

**Corollary 7.** For every universal Horn class  $\mathbf{K}$  of finite signature, the lattice  $L(\mathbf{K})$  is relatively pseudocomplemented. For all  $\mathbf{K}_1, \mathbf{K}_2 \in L(\mathbf{K})$ , the following equality holds:  $\mathbf{K}_1 * \mathbf{K}_2 = \mathbf{H}^{-1} \mathbf{SP}^*_{\mathbf{u}} \{ \mathcal{A} \in \mathbf{K}_f : \mathcal{A} \times \mathcal{B} \in \mathbf{K}_2 \text{ for all } \mathcal{B} \in (\mathbf{K}_1)_f \} \cap \mathbf{K}.$ 

**Proof.** By [7, Corollary II.1.4], we have  $I * J = \{a \in L : a \land i \in J \text{ for all } i \in I\}$  for every distributive lattice L and ideals I and J of L. This equality, together with Lemma 6, yields the required assertion.

#### 4. Complexity of lattices of antivarieties

In this section, we introduce the notion of a universal (the most complicated) lattice among the lattices of antivarieties of relation structures of finite signature and give examples of universal lattices.

Let **K** be a class of structures. By the category **K** we mean the category whose objects are structures in **K** and morphisms are homomorphisms. A one-to-one functor  $\Phi$  from a category  $\mathbf{K}_1$  into a category  $\mathbf{K}_2$  is called a *full embedding* if, for every morphism  $\alpha : \Phi(\mathcal{A}) \to \Phi(\mathcal{B})$  in  $\mathbf{K}_2$ , there exists a morphism  $\beta : \mathcal{A} \to \mathcal{B}$  in  $\mathbf{K}_1$  such that  $\Phi(\beta) = \alpha$ . For more information about categories, the reader is referred to [8]. By **G** we denote the class (and the category) of undirected loopless graphs.

**Lemma 8.** Let **K** be a class of structures and let there exist a full embedding  $\Phi : \mathbf{G} \to \mathbf{K}$  such that, for every finite graph  $\mathfrak{G}$ , the structure  $\Phi(\mathfrak{G})$  if finite. Then there exists an embedding  $\varphi : \operatorname{Core}(\mathbf{G}) \to \operatorname{Core}(\mathbf{K})$ .

**Proof.** Put  $\varphi(\mathfrak{G}) = \operatorname{Core}(\Phi(\mathfrak{G}))$  for every  $\mathfrak{G} \in \operatorname{Core}(\mathbf{G})$ . It is clear that  $\varphi$  is a map from  $\operatorname{Core}(\mathbf{G})$  into  $\operatorname{Core}(\mathbf{K})$ . We show that  $\varphi$  is an embedding.

Let  $\mathcal{G} \leq \mathcal{H}$ , where  $\mathcal{G}, \mathcal{H} \in \text{Core}(\mathbf{G})$ , and let  $\psi$  be the corresponding homomorphism. Then the composition

$$\varphi(\mathfrak{G}) = \operatorname{Core}(\Phi(\mathfrak{G})) \xrightarrow{e} \Phi(\mathfrak{G}) \xrightarrow{\Phi(\psi)} \Phi(\mathfrak{H}) \xrightarrow{r} \operatorname{Core}(\Phi(\mathfrak{H})) = \varphi(\mathfrak{H})$$

is a homomorphism from  $\varphi(\mathfrak{G})$  into  $\varphi(\mathfrak{H})$ . Hence,  $\varphi(\mathfrak{G}) \leq \varphi(\mathfrak{H})$ .

Let  $\varphi(\mathfrak{G}) \leq \varphi(\mathfrak{H})$  for some  $\mathfrak{G}, \mathfrak{H} \in \text{Core}(\mathbf{G})$  and let  $\psi$  be the corresponding homomorphism. Then the composition

$$\Phi(\mathfrak{G}) \xrightarrow{r} \operatorname{Core}(\Phi(\mathfrak{G})) = \varphi(\mathfrak{G}) \xrightarrow{\psi} \varphi(\mathfrak{H}) = \operatorname{Core}(\Phi(\mathfrak{H})) \xrightarrow{e} \Phi(\mathfrak{H})$$

is a homomorphism from  $\Phi(\mathfrak{G})$  into  $\Phi(\mathfrak{H})$ . Denote this homomorphism by  $\alpha$ . Since  $\Phi$  is a full embedding, we have  $\alpha = \Phi(\beta)$  for some homomorphism  $\beta : \mathfrak{G} \to \mathfrak{H}$ . Hence,  $\mathfrak{G} \leq \mathfrak{H}$ .

It remains to show that  $\varphi$  is a one-to-one map. If  $\varphi(\mathfrak{G}) = \varphi(\mathfrak{H})$  then  $\varphi(\mathfrak{G}) \leq \varphi(\mathfrak{H})$  and  $\varphi(\mathfrak{H}) \leq \varphi(\mathfrak{G})$ . By the above,  $\mathfrak{G} \leq \mathfrak{H}$  and  $\mathfrak{H} \leq \mathfrak{G}$ . Hence,  $\mathfrak{G} = \mathfrak{H}$ .

A category **K** satisfying the conditions of Lemma 8 is said to be *finite-to-finite universal*. As is known [9] (see also [10, Theorem 2.10]), the partially ordered set Core(**G**) is  $\omega$ -universal, i.e., each countable partially ordered set

is embeddable into  $\text{Core}(\mathbf{G})$ . By Lemma 8, for every finite-to-finite universal category  $\mathbf{K}$ , the partially ordered set  $\text{Core}(\mathbf{K})$  is  $\omega$ -universal.

Recall that a lattice L is called a *factor* of a lattice K if L is a homomorphic image of a suitable sublattice of K. We say that  $L(\mathbf{K})$  is a *universal* lattice if, for every universal Horn class  $\mathbf{K}'$  of relation structures of finite signature, the lattice  $L(\mathbf{K}')$  is a factor of the lattice  $L(\mathbf{K})$ .

**Theorem 9.** Let  $\mathbf{K}$  be a universal Horn class of relation structures of finite signature. If  $\mathbf{K}$  is a finite-to-finite universal category then  $L(\mathbf{K})$  is a universal lattice.

**Proof.** By Lemma 8, for every universal Horn class  $\mathbf{K}'$  of relation structures of finite signature, there exists an embedding  $\varphi : \operatorname{Core}(\mathbf{K}') \to \operatorname{Core}(\mathbf{K})$ .

By Lemma 6, we may consider the lattice of order ideals of the partially ordered set of cores instead of the lattice of antivarieties. For every  $I \in I_o(\text{Core}(\mathbf{K}'))$ , put

$$\psi(I) = \{ \mathcal{H} \in \operatorname{Core}(\mathbf{K}) : \mathcal{H} \le \varphi(\mathcal{G}) \text{ for some } \mathcal{G} \in I \}.$$

It is easy to verify that, for every order ideal I of  $Core(\mathbf{K}')$ , the set  $\psi(I)$  is an order ideal of  $Core(\mathbf{K})$ .

We prove that  $\psi$  is one-to-one. Let  $\psi(I) = \psi(J)$ . For every  $\mathcal{H} \in I$ , we have  $\varphi(\mathcal{H}) \in \psi(I) = \psi(J)$ , i.e., there exists an element  $\mathcal{G} \in J$  such that  $\varphi(\mathcal{H}) \leq \varphi(\mathcal{G})$ . Since  $\varphi$  is an embedding, we have  $\mathcal{H} \leq \mathcal{G}$ , i.e.,  $\mathcal{H} \in J$ . We have proven that  $I \subseteq J$ . The proof of the converse inclusion is similar.

We prove that  $\psi$  is a join homomorphism. Consequently, the join semilattice of  $I_o(Core(\mathbf{K}'))$  is embeddable into the join semilattice of  $I_o(Core(\mathbf{K}))$ . The inclusion  $\psi(I) \lor \psi(J) \subseteq \psi(I \lor J)$  is obvious. Conversely, let  $\mathcal{H} \in \psi(I \lor J)$ . Then there exists a  $\mathcal{G} \in I \lor J$  such that  $\mathcal{H} \leq \varphi(\mathcal{G})$ . Since  $I \lor J = I \cup J$ , we obtain  $\mathcal{H} \in \psi(I) \cup \psi(J) = \psi(I) \lor \psi(J)$ .

Let *L* be the sublattice of  $I_o(\mathbf{K})$  generated by the set  $\{\psi(I) : I \in I_o(\operatorname{Core}(\mathbf{K}'))\}$ . Then, for every  $X \in L$ , there exist a lattice term  $t(v_0, \ldots, v_{n-1})$  and order ideals  $J_0, \ldots, J_{n-1}$  of  $\operatorname{Core}(\mathbf{K}')$  such that  $X = t(\psi(J_0), \ldots, \psi(J_{n-1}))$ .

We prove that, for every lattice term  $t(v_0, \ldots, v_{n-1})$  and order ideals  $J_0, \ldots, J_{n-1}$  of Core(**K**'), the equality

(6) 
$$t(\psi(J_0),\ldots,\psi(J_{n-1})) \cap \varphi(\operatorname{Core}(\mathbf{K}')) = \varphi(t(J_0,\ldots,J_{n-1}))$$

holds, where  $\varphi(M) = \{\varphi(m) : m \in M\}$  for every set M.

We use induction on the length of the term. Let  $t(v_0, \ldots, v_{n-1}) = v_i$ . Then the right-hand side of (6) is  $\varphi(J_i)$  and the left-hand side of (6) is  $\psi(J_i) \cap \varphi(\operatorname{Core}(\mathbf{K}'))$ . It is clear that  $\varphi(J_i) \subseteq \psi(J_i) \cap \varphi(\operatorname{Core}(\mathbf{K}'))$ . Conversely, let  $\mathcal{H} \in \psi(J_i) \cap \varphi(\operatorname{Core}(\mathbf{K}'))$ . Then  $\mathcal{H} = \varphi(\mathcal{G})$  for some  $\mathcal{G} \in \operatorname{Core}(\mathbf{K}')$ . Let J be the least ideal of  $\operatorname{Core}(\mathbf{K}')$  containing  $J_i \cup \{\mathcal{G}\}$ . We have  $\psi(J) = \psi(J_i) \cup \{\mathcal{A} \in \operatorname{Core}(\mathbf{K}') : \mathcal{A} \leq \varphi(\mathcal{G})\}$ . Since  $\varphi(\mathcal{G}) = \mathcal{H} \in \psi(J_i)$ , we obtain  $\psi(J) = \psi(J_i)$ . Since  $\psi$  is a one-to-one map, we have  $J = J_i$ , i.e.,  $\mathcal{G} \in J_i$ . Therefore,  $\mathcal{H} \in \varphi(J_i) \subseteq \psi(J_i)$ .

Assume that  $t = t_1 \wedge t_2$  or  $t = t_1 \vee t_2$  for some terms  $t_1$  and  $t_2$ . We introduce the notation

$$Y_i = t_i(\psi(J_0), \dots, \psi(J_{n-1})), \quad X_i = t_i(J_0, \dots, J_{n-1}),$$

where i = 1, 2. By induction,  $Y_i \cap \text{Core}(\mathbf{K}') = \varphi(X_i), i = 1, 2$ .

If  $t = t_1 \wedge t_2$  then  $t(\psi(J_0), \ldots, \psi(J_{n-1})) = Y_1 \cap Y_2, t(J_0, \ldots, J_{n-1}) = X_1 \cap X_2$ . By induction,  $Y_1 \cap Y_2 \cap \varphi(\operatorname{Core}(\mathbf{K}')) = \varphi(X_1) \cap \varphi(X_2) \supseteq \varphi(X_1 \cap X_2)$ . For every  $\mathcal{A} \in \varphi(X_1) \cap \varphi(X_2)$ , there exist  $\mathcal{A}_i \in X_i, i = 1, 2$ , such that  $\mathcal{A} = \varphi(\mathcal{A}_1) = \varphi(\mathcal{A}_2)$ . Since  $\varphi$  is a one-to-one map, we obtain  $\mathcal{A}_1 = \mathcal{A}_2 \in X_1 \cap X_2$ .

It  $t = t_1 \vee t_2$  then  $t(\psi(J_0), \ldots, \psi(J_{n-1})) = Y_1 \cup Y_2, t(J_0, \ldots, J_{n-1}) = X_1 \cup X_2$ . By induction,  $(Y_1 \cup Y_2) \cap \varphi(\operatorname{Core}(\mathbf{K}')) = (Y_1 \cap \varphi(\operatorname{Core}(\mathbf{K}'))) \cup (Y_2 \cap \varphi(\operatorname{Core}(\mathbf{K}'))) = \varphi(X_1) \cup \varphi(X_2) \subseteq \varphi(X_1 \vee X_2)$ . The converse inclusion is an easy consequence of the equality  $X_1 \vee X_2 = X_1 \cup X_2$ .

Since the operations of the lattice of order ideals are the set-theoretical operations, the union and the intersection, from (6) we obtain

(7) 
$$\varphi^{-1}(t(\psi(J_0),\ldots,\psi(J_{n-1}))) \cap \varphi(\operatorname{Core}(\mathbf{K}'))) = t(J_0,\ldots,J_{n-1}).$$

Let  $X = t(\psi(J_0), \ldots, \psi(J_{n-1})) \in L$ . Put  $\alpha(X) = \varphi^{-1}(X \cap \varphi(\operatorname{Core}(\mathbf{K}')))$ . It is immediate from (7) that  $\alpha$  is a map from L onto  $I_o(\operatorname{Core}(\mathbf{K}'))$ . By (6),  $\alpha$  is a homomorphism.

We present an example showing that the converse to Theorem 9 is not true. Namely, we indicate a quasivariety  $\mathbf{K}$  of loopless digraphs such that  $\mathbf{K}$  is not a finite-to-finite universal category but  $L(\mathbf{K})$  is a universal lattice.

**Example 10.** Let  $\sigma$  consist of one binary relation symbol r. Denote by **K** the quasivariety of structures of the signature  $\sigma$  defined by the quasi-identities

$$\begin{aligned} &\forall x \forall y \big( r(x, x) \to x \approx y \big), \\ &\forall x \forall y \forall z \big( r(x, y) \& r(x, z) \to y \approx z \big) \\ &\forall x \forall y \forall z \big( r(y, x) \& r(z, x) \to y \approx z \big) \end{aligned}$$

Let  $\mathcal{C}_n$ ,  $n \ge 2$ , denote the *cycle* of length n, i.e., the structure whose universe is  $C_n = \{0, 1, \ldots, n-1\}$  and  $(i, j) \in r^{\mathcal{C}_n}$  if and only if  $i+1 \equiv j \pmod{n}$ . It is easy to see that, for every  $n \ge 2$ , we have  $\mathcal{C}_n \in \mathbf{K}$ .

Let  $\mathbb{P}$  denote the set of prime numbers. Denote by  $a_p, p \in \mathbb{P}$ , the **K**-colour-family  $[\mathbf{K} \to \mathbb{C}_p]$ . Let L be the sublattice of  $L_0(\mathbf{K})$  generated by the elements  $(a_p)_{p \in \mathbb{P}}$ .

We show that the distributive lattice L is freely generated by the set  $(a_p)_{p\in\mathbb{P}}$ . Since  $|\mathbb{P}| = \omega$ , this means that the free distributive lattice  $F_{\mathbf{D}}(\omega)$  of countable rank is embedded into  $L_0(\mathbf{K})$ . We use [7, Theorem II.2.3]. It suffices to verify that, for all finite nonempty subsets  $I, J \subseteq \mathbb{P}$ , from  $\bigwedge_{i \in I} a_i \leq \bigvee_{i \in J} a_j$  it follows that  $I \cap J \neq \emptyset$ .

Let *I* and *J* be finite and nonempty. Assume that  $\bigwedge_{i \in I} a_i \leq \bigvee_{j \in J} a_j$ . By Lemma 1, we have  $\bigwedge_{i \in I} a_i = [\mathbf{K} \to \prod_{i \in I} \mathcal{C}_i]$  and  $\bigvee_{j \in J} a_j = [\mathbf{K} \to (\mathcal{C}_j)_{j \in J}]$ . Let  $k = \prod_{i \in I} i$ . It is easy to see that  $\prod_{i \in I} \mathcal{C}_i \simeq \mathcal{C}_k$  (cf., for example, [11]). Since  $\bigwedge_{i \in I} a_i \leq \bigvee_{j \in J} a_j$ , there exists a prime  $j \in J$  with  $\mathcal{C}_k \in [\mathbf{K} \to \mathcal{C}_j]$ . We have  $\mathcal{C}_k \to \mathcal{C}_j$  if and only if j divides k (cf., for example, [12]). Since j is prime and k is a product of distinct primes, we conclude that  $j \in I$ . Thus,  $I \cap J \neq \emptyset$ .

We show that the ideal lattice  $I(F_{\mathbf{D}}(\omega))$  of the free distributive lattice of countable rank is embeddable into  $L(\mathbf{K})$ . Let L and K be distributive lattices and let  $\varphi : L \to K$  be an embedding. Define a map  $\psi : I(L) \to I(K)$ by the following rule:  $\psi(I)$  is the ideal of K generated by  $\varphi(I)$ . Using the definition of an ideal generated by a set, we easily find that  $\psi$  is an embedding. In particular,  $I(F_{\mathbf{D}}(\omega))$  is embeddable into  $I(L_0(\mathbf{K}))$ . The latter lattice is isomorphic to  $L(\mathbf{K})$  in view of Lemma 6.

We show that the lattice  $L(\mathbf{G})$  of antivarieties of undirected loopless graphs is a homomorphic image of the lattice  $I(F_{\mathbf{D}}(\omega))$ . Since  $L_0(\mathbf{G})$  is a countable distributive lattice, there exists a homomorphism from  $F_{\mathbf{D}}(\omega)$  onto  $L_0(\mathbf{G})$ . As above, this homomorphism induces a homomorphism between the corresponding ideal lattices. It remains to use Lemma 6. Therefore, L(G) is a factor of L(K). We conclude that L(K) is a universal lattice. The class of rigid objects in the category K consists of trivial structures and finite directed chains only (cf. [8, Exercise IV.1.6]). Therefore, the category K is not universal and, consequently, is not finite-to-finite universal.

# 5. IRREDUNDANT MEET DECOMPOSITIONS IN LATTICES OF COLOUR-FAMILIES

Recall that **G** denotes the universal Horn class and the category of undirected loopless graphs. The study of the lattice  $L_0(\mathbf{G})$  was initiated in [1]. It was proven that this lattice possesses neither completely join irreducible nor completely meet irreducible nonzero elements. A simple description for join irreducible colour-families was found. The question on meet irreducible elements turned to be closely connected with a well-known problem in the graph theory, Hedetniemi's conjecture [13].

Here, we consider meet decompositions of **K**-colour-families with the help of Lemma 3, which says that the lattice of **K**-colour-families is relatively pseudocomplemented. A similar approach was first used in [4].

We present necessary definitions. By a *meet decomposition* of an element  $x \in L$ , where L is an arbitrary lattice, we mean a representation

(8) 
$$x = \bigwedge_{i \in I} m_i,$$

where  $(m_i)_{i \in I}$  is a family of meet irreducible elements, i.e., for each  $i \in I$ , we have  $m_i \neq 1$  and from  $m_i = a \wedge b$  it follows that either  $m_i = a$  or  $m_i = b$ . A meet decomposition (8) is *irredundant* if  $x < \bigwedge_{i \in J} m_i$  for every proper subset  $J \subsetneq I$ . For distributive relatively pseudocomplemented lattices, the following criterion for meet irreducibility of elements is known [14].

**Proposition 11.** Let L be a distributive relatively pseudocomplemented lattice and let  $m \in L$ . The element m is meet irreducible if and only if x \* m = m for every  $x \in L$  with  $x \nleq m$ .

Throughout this section, we assume that L is an arbitrary distributive relatively pseudocomplemented lattice. Let  $\lor$ ,  $\land$ , and  $\ast$  denote the operations of L. For every  $x \in L$ , let  $\text{Reg}(x) = \{y \ast x : y \in L\}$ , i.e., let Reg(x) denote the set of regular elements of the principal filter [x) of L. For all  $u, v \in \text{Reg}(x)$ , put

 $u + v = ((u \lor v) * x) * x, \quad u \cdot v = u \land v, \quad 0 = x, \quad 1 = 1_L, \quad u' = u * x.$ 

The set  $\operatorname{Reg}(x)$  with the operations +,  $\cdot$ , and ' and constants 1 and 0 is a Boolean algebra; moreover, the map r from L to  $\operatorname{Reg}(x)$  defined by the rule r(y) = (y \* x) \* x is a homomorphism between Heyting algebras [6, Theorem 8.4.3].

We mention the following relationship between meet irreducible elements of L and dual atoms of Reg(x) [4, Theorem 6].

**Proposition 12.** Let  $x, y \in L$  and let x < y. The element y is a dual atom of the Boolean algebra  $\operatorname{Reg}(x)$  if and only if y \* x > x and y is meet irreducible in L.

Recall [15] that a Boolean algebra A is *atomic* if, for every nonzero element  $a \in A$ , there exists an atom b such that  $b \leq a$ . An element a is said to be *atomless* if  $a \neq 0$  and there is no atom b such that  $b \leq a$ .

**Theorem 13.** Let L be a distributive pseudocomplemented lattice and let  $a \in L$ . The element a admits an irredundant meet decomposition in L if and only if the Boolean algebra Reg(a) is atomic.

**Proof.** Let  $a = \bigwedge_{i \in I} m_i$  be an irredundant meet decomposition. We prove that  $m_i * a > a$  for every  $i \in I$ . In view of Proposition 12, this means that each  $m_i$ ,  $i \in I$ , is a dual atom of  $\operatorname{Reg}(a)$ . Since  $a = \bigwedge_{i \in I} m_i$ , we have  $m_i * a = m_i * (\bigwedge_{i \in I} m_i) = \bigwedge_{m_j \notin m_i} m_j = \bigwedge_{j \neq i} m_j$  (cf. [16, IV.7.2 (8)]). Since the meet decomposition is irredundant, we have  $m_i * a = \bigwedge_{j \neq i} m_j > a$ .

Assume that there exists an atomless element  $b \in \text{Reg}(a)$ . Since the complement of a dual atom is an atom, we conclude that  $m_i * a \notin b$  for all  $i \in I$ . Hence,  $b \wedge (m_i * a) = a$  for all  $i \in I$ . By the definition of a relative pseudocomplement, we have  $b \leq (m_i * a) * a = m_i$  for all  $i \in I$ . Therefore,  $b \leq \bigwedge_{i \in I} m_i = a$ . This proves that Reg(a) possesses no atomless element, i.e., Reg(a) is an atomic Boolean algebra.

Conversely, assume that  $\operatorname{Reg}(a)$  is an atomic Boolean algebra. Let  $(a_i)_{i\in I}$  be the set of atoms. If  $|\operatorname{Reg}(a)| = 2$  then the element a is meet irreducible in view of Proposition 11. In the sequel, we assume that  $|\operatorname{Reg}(a)| > 2$ . We denote  $m_i = a_i * a, i \in I$ . For every  $i \in I$ , the element  $m_i$  is a dual atom of  $\operatorname{Reg}(a)$ ; moreover, each dual atom is of the form  $m_i, i \in I$ .

By Proposition 12, we have  $m_i * a > a$  and  $m_i$  is meet irreducible for every  $i \in I$ . It remains to show that  $a = \bigwedge_{i \in I} m_i$  (in the lattice L). It is clear that a is a lower bound for  $(m_i)_{i \in I}$ . If a is not the greatest lower bound then there exists a lower bound  $b_0 \in L$  for  $(m_i)_{i \in I}$  such that  $b_0 \notin a$ . Consider the element  $(b_0 * a) * a \in \text{Reg}(a)$ . Since  $b_0 \leqslant m_i$ , we conclude that  $(b_0 * a) * a \leqslant (m_i * a) * a = m_i, i \in I$ . Hence,  $b = (b_0 * a) * a \in \text{Reg}(a)$  is a lower bound for  $(m_i)_{i \in I}$ . Since  $b_0 \notin a$  and  $b_0 \leqslant b$ , we find that  $b \neq a$ , i.e., b > a. Since Reg(a) is an atomic Boolean algebra, there exists an atom  $a_j$  such that  $a_j \leqslant b$ . Passing to the complements, we find that  $b' \leqslant m_j$ . Since  $b \leqslant m_i$  for all  $i \in I$ , we obtain  $b + b' = 1 \leqslant m_j + m_j = m_j < 1$ , a contradiction.

Similar questions for undirected loopless graphs were considered in [17], where the notion of the *level of nonmultiplicativity* of a graph was introduced. In our terminology, the level of nonmultiplicativity of a graph  $\mathcal{G}$  is the number of dual atoms of the Boolean algebra  $\operatorname{Reg}([\mathbf{G} \to \mathcal{G}])$ . In [17], the following conjecture is stated: *The level of nonmultiplicativity of each finite* graph is finite. In connection with Theorem 13, we formulate the following

**Problem 14.** Let  $\mathbf{K}$  be a universal Horn class of relation structures of finite signature. Is this true that, for every  $\mathbf{K}$ -colour-family  $\mathbf{A}$ , the following conditions are equivalent:

- (1) there exists an irredundant meet decomposition  $\mathbf{A} = \bigwedge_{i \in I} \mathbf{M}_i$ ,
- (2) there exists a finite meet decomposition  $\mathbf{A} = \bigwedge_{i < n} \mathbf{M}_i, n < \omega$ ,
- (3) the Boolean algebra  $\operatorname{Reg}(\mathbf{A})$  is finite?

In the next section, we find a connection between this problem and existence of independent bases for anti-identities.

### 6. Anti-identities of finite structures

Recall that a set  $\Sigma$  of anti-identities is a *basis for anti-identities* of a class **K** if **K** is the class of structures in which all anti-identities of  $\Sigma$  are valid, i.e.,  $\mathbf{K} = \text{Mod}(\Sigma)$ . By a basis for anti-identities of a structure  $\mathcal{A}$  we mean a basis for anti-identities of the antivariety generated by  $\mathcal{A}$ . A basis  $\Sigma$  is said to be *independent* if, for every  $\varphi \in \Sigma$ , the proper inclusion  $\text{Mod}(\Sigma) \subsetneq \text{Mod}(\Sigma \setminus \{\varphi\})$  holds.

A structure  $\mathcal{A}$  is said to be *weakly atomic compact* if every locally consistent in  $\mathcal{A}$  set of atomic formulas is consistent in  $\mathcal{A}$ .

We reduce the question on existence of an independent basis for antiidentities of a finite relation structure of finite signature to Problem 14.

Let  $\mathcal{A}$  be a finite relation structure of finite signature and let  $\Sigma = (\varphi_i)_{i \in I}$ be an independent basis for anti-identities of  $\mathcal{A}$ . With each anti-identity  $\varphi_i$ ,  $i \in I$ , we associate a finitely presented structure  $\mathcal{B}_i$  as follows:

Let  $\varphi_i := \forall \overline{x}(\neg \psi_1(\overline{x}) \lor \cdots \lor \neg \psi_n(\overline{x}))$ ; then  $\mathcal{B}_i$  is the structure defined by generators  $\overline{x}$  and relations  $\psi_1(\overline{x}), \ldots, \psi_n(\overline{x})$ .

It is easy to see that the antivariety defined by  $\Sigma$  coincides with the class

$$\bigcap_{i \in I} [\mathcal{B}_i \not\rightarrow] = \{ \mathcal{B} : \mathcal{B}_i \not\rightarrow \mathcal{B} \text{ for all } i \in I \}.$$

Since  $\Sigma$  is an independent basis, we have  $\mathcal{B}_i \to \mathcal{B}_j$  if and only if i = j.

Lemma 15. The following two conditions are equivalent:

- (1)  $[\rightarrow \mathcal{A}] = \bigcap_{i \in I} [\mathcal{B}_i \not\rightarrow],$
- (2) there exists a family of finite structures  $(\mathcal{A}_i)_{i\in I}$  such that  $[\mathcal{B}_i \not\rightarrow] = [\rightarrow \mathcal{A}_i]$  for all  $i \in I$  and  $[\rightarrow \mathcal{A}] = [\rightarrow \prod_{i\in I} \mathcal{A}_i]$ .

**Proof.** It is clear that (2) implies (1). Indeed, if such a family exists then, for every structure C, we have

$$\mathcal{C} \in [\to \mathcal{A}] = \left[ \to \prod_{i \in I} \mathcal{A}_i \right] \iff \mathcal{C} \to \mathcal{A}_i \text{ for all } i \in I \iff \\ \iff \mathcal{C} \in [\mathcal{B}_i \not\rightarrow] \text{ for all } i \in I \iff \\ \iff \mathcal{C} \in \bigcap_{i \in I} [\mathcal{B}_i \not\rightarrow].$$

We prove that (1) implies (2).

Notice that each structure  $\mathcal{B}_i, i \in I$ , is connected. Assume the contrary, i.e., let there exist an element  $i \in I$  such that  $\mathcal{B}_i$  is not connected. Then  $\mathcal{B}_i = \mathcal{B}_i^1 + \mathcal{B}_i^2$  for some structures  $\mathcal{B}_i^k$  with  $\mathcal{B}_i \not\rightarrow \mathcal{B}_i^k, k = 1, 2$ . Since  $\mathcal{B}_j \not\rightarrow \mathcal{B}_i$  provided  $j \neq i$ , we have  $\mathcal{B}_i^k \in [\mathcal{B}_j \not\rightarrow]$  for all  $j \neq i$  and k = 1, 2. Therefore,  $\mathcal{B}_i^k \in \bigcap_{i \in I} [\mathcal{B}_i \not\rightarrow] = [\rightarrow \mathcal{A}], k = 1, 2$ . Thus,  $\mathcal{B}_i \in [\rightarrow \mathcal{A}] \subseteq [\mathcal{B}_i \not\rightarrow]$ , which is a contradiction.

For an arbitrary  $i \in I$ , consider the interval  $[\mathcal{A}, \mathcal{A} + \mathcal{B}_i]$  of the partially ordered set of cores. If there exists a core  $\mathcal{C}$  such that  $\mathcal{A} \to \mathcal{C} \to \mathcal{A} + \mathcal{B}_i$  and  $\mathcal{B}_i \to \mathcal{C} \to \mathcal{A}$  then, by (1), we obtain  $\mathcal{C} \notin \bigcap_{i \in I} [\mathcal{B}_i \to]$ . Hence, there exists a  $j \in I$  such that  $\mathcal{B}_j \to \mathcal{C}$ . It is easy to see that  $i \neq j$ . Since  $\mathcal{B}_j \to \mathcal{C} \to \mathcal{A} + \mathcal{B}_i$ and  $\mathcal{B}_j$  is connected, we conclude that  $\mathcal{B}_j \to \mathcal{B}_i$ , where  $i \neq j$ . Since  $\Sigma$  is an independent basis, we arrive at a contradiction. Thus,  $\mathcal{A} + \mathcal{B}_i$  covers  $\mathcal{A}$  in the partially ordered set of cores (in symbols:  $\mathcal{A} \prec \mathcal{A} + \mathcal{B}_i$ ). Since this is a distributive lattice (cf. Remark 2), we conclude that  $\mathcal{A} \times \mathcal{B}_i \prec \mathcal{B}_i$ . We denote  $\mathcal{C}_i = \mathcal{A} \times \mathcal{B}_i, i \in I$ . By (2) and (3), we obtain  $\mathcal{C}_i \to \mathcal{A} \to \mathcal{C}_i^{\mathcal{B}_i}$  for all  $i \in I$ . By [5, Lemma 2.5], for every  $i \in I$ , the equality  $\left[ \to \mathcal{C}_i^{\mathcal{B}_i} \right] = \left[ \mathcal{B}_i \to \right]$  holds. Put  $\mathcal{A}_i = \mathcal{C}_i^{\mathcal{B}_i}$ . We prove that  $\bigcap_{i \in I} \left[ \mathcal{B}_i \to \right] = \left[ \to \prod_{i \in I} \mathcal{A}_i \right]$ . We deduce

$$\mathcal{D} \in \bigcap_{i \in I} [\mathcal{B}_i \not\rightarrow] \iff \mathcal{D} \in [\mathcal{B}_i \not\rightarrow] \text{ for all } i \in I \iff \\ \iff \mathcal{D} \in [\rightarrow \mathcal{A}_i] \text{ for all } i \in I \iff \\ \iff \mathcal{D} \in \left[ \rightarrow \prod_{i \in I} \mathcal{A}_i \right].$$

Since  $\bigcap_{i \in I} [\mathcal{B}_i \not\rightarrow] = [\rightarrow \mathcal{A}]$ , we obtain  $[\rightarrow \mathcal{A}] = [\rightarrow \prod_{i \in I} \mathcal{A}_i]$ . Moreover, if the structure  $\mathcal{A}$  has no trivial substructure then the structure  $\prod_{i \in I} \mathcal{A}_i$  has no trivial substructure either.

In the sequel, we assume that (equivalent) conditions (1) and (2) of Lemma 15 are satisfied. Without loss of generality, we may assume that  $\mathcal{A}$  is a core. Since  $\mathcal{A}$  is finite, the class  $[\rightarrow \mathcal{A}]$  is elementary. By [2, Proposition 2.2], the structure  $\prod_{i \in I} \mathcal{A}_i$  is weakly atomic compact and  $[\rightarrow \prod_{i \in I} \mathcal{A}_i]$  is the antivariety generated by  $\prod_{i \in I} \mathcal{A}_i$ . In view of [2, Corollary 2.6], there exists a unique (up to isomorphism) core  $\mathcal{A}^*$  of  $\prod_{i \in I} \mathcal{A}_i$ ; moreover, the antivarieties generated by  $\mathcal{A}^*$  and  $\prod_{i \in I} \mathcal{A}_i$  coincide. Therefore, the antivarieties generated by  $\mathcal{A}$  and  $\mathcal{A}^*$  coincide. By [2, Corollary 2.5], the structures  $\mathcal{A}$  and  $\mathcal{A}^*$  are isomorphic.

Since  $\mathcal{A}^*$  is a finite structure of finite signature, there exists a finite subset  $F \subseteq I$  such that  $\mathcal{A}^*$  is embeddable into  $\prod_{i \in F} \mathcal{A}_i$ .

We suggest the following

**Conjecture 16.** The equality  $[\to \mathcal{A}] = [\to \prod_{i \in F'} \mathcal{A}_i]$  holds for some finite subset  $F' \subseteq I$  with  $F \subseteq F'$ .

If this conjecture is true then  $\Sigma$  is a finite basis, which means that every finite relation structure of finite signature having no finite basis for its anti-identities possesses no independent basis for its anti-identities.

We return to Problem 14. Let  $\mathbf{K}_i$  be the principal colour-family generated by  $\mathcal{A}_i, i \in I$ . Then

$$[\rightarrow \mathcal{A}] = \bigwedge_{i \in I} \mathbf{K}_i$$

is an irredundant meet decomposition of  $[\rightarrow \mathcal{A}]$  (in the lattice of colourfamilies). Therefore, if the answer to the question in Problem 14 is positive then Conjecture 16 is true.

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