Discussiones Mathematicae
General Algebra and Applications 27 (2007) 69–85

MAXIMAL SUBMONOIDS OF MONOIDS OF HYPERSUBSTITUTIONS

Ilinka Dimitrova and Jörg Koppitz

University of Potsdam
Institute of Mathematics
Am Neuen Palais, 14415 Potsdam, Germany

e-mail: ilinkad@abv.bg; koppitz@rz.uni-potsdam.de

Abstract

For a monoid M of hypersubstitutions, the collection of all M-solid varieties forms a complete sublattice of the lattice $\mathcal{L}(\tau)$ of all varieties of a given type τ. Therefore, by the study of monoids of hypersubstitutions one can get more insight into the structure of the lattice $\mathcal{L}(\tau)$. In particular, monoids of hypersubstitutions were studied in [9] as well as in [5]. We will give a complete characterization of all maximal submonoids of the monoid $Reg(n)$ of all regular hypersubstitutions of type $\tau = (n)$ (introduced in [4]). The concept of a transformation hypersubstitution, introduced in [1], gives a relationship between monoids of hypersubstitutions and transformation semigroups. In the present paper, we apply the recent results about transformation semigroups by I. Guydzenov and I. Dimitrova ([11], [12]) to describe monoids of transformation hypersubstitutions.

Keywords: regular hypersubstitutions, maximal monoids of hypersubstitutions, transformation semigroups.

2000 Mathematics Subject Classification: 08B05, 08B15, 20M07, 16Y60.

1. Introduction

A number of fairly natural examples of submonoids of the monoid $Hyp(\tau)$ of all hypersubstitutions of a given type τ is listed in [9]. In particular, the monoid $Reg(n)$ of all so-called regular hypersubstitutions of type (n),

$1 \leq n \in \mathbb{N}$, is studied. This monoid was first introduced by K. Denecke and J. Koppitz in [4] (see also [5], [6] and [9]). Properties of several monoids of hypersubstitutions of type (n) are studied by Th. Changphas ([2], [3]). For example, the monoid of all so-called full hypersubstitutions of type (n), i.e. hypersubstitutions σ where $\sigma(f)$ is a full term, is considered in [3]. The concept of a full term was introduced in [7]. On the other hand one can consider transformation hypersubstitutions ([1], [2]). A hypersubstitution σ of type $\tau = (n)$ is called a transformation hypersubstitution if $\sigma(f) = f(x_{s(1)}, \ldots, x_{s(n)})$ for some mapping $s : \overline{n} \to \overline{n}$, where $\overline{n} := \{1, \ldots, n\}$ ([1]). In the present paper, we will introduce particular submonoids of the monoid $TR(n)$ of all transformation hypersubstitutions of type $\tau = (n)$. K. Denecke and M. Reichel established a Galois-connection between monoids of hypersubstitutions of a given type τ and varieties of the same type, showing that for any monoid M of hypersubstitutions of type τ, the collection of all M-solid varieties of type τ forms a complete sublattice of the lattice of all varieties of type τ ([8]). It is a general goal of research in this area to study monoids of hypersubstitutions of a given type τ. In particular, it is of some interest to know what a monoid of hypersubstitutions looks like. In the present paper, we want to give a contribution to the research on monoids of hypersubstitutions. We will describe the monoid $Reg(n)$, $2 \leq n \in \mathbb{N}$, by characterization of its maximal submonoids. On the other hand we will consider submonoids of $TR(n)$ based on transformation semigroups. Using the recent results about isotone transformations with defect ≥ 2 ([12]) and monotone transformations ([13]), we are able to describe the appropriate monoids of transformation hypersubstitutions by characterization of their maximal submonoids.

In Section 2 we set out some notations concerning hypersubstitutions and introduce our new definitions. Section 3 works out all maximal submonoids of $Reg(n)$, $2 \leq n \in \mathbb{N}$. In Section 4 we describe particular submonoids of $TR(n)$ by characterization of their maximal submonoids.

2. Hypersubstitutions, terms and transformations

We fix a natural number $n \geq 1$ and an n-ary operation symbol f. Let $W_n(X)$ be the set of all terms of type (n) over some fixed alphabet $X = \{x_1, x_2, \ldots\}$. Terms in $W_n(X_k)$ with $X_k = \{x_1, \ldots, x_k\}$, $k \geq 1$, are called k-ary. For any term $s \in W_n(X_k)$ and $t_1, \ldots, t_k \in W_n(X)$, the term $s(t_1, \ldots, t_k)$ arises by substitution of terms, i.e. in the term s one replaces the variables $x_1, \ldots, x_k$ by the terms $t_1, \ldots, t_k$, respectively. The concept of a hypersubstitution will be a crucial one. A mapping $\sigma : \{f\} \to W_n(X_n)$ which assigns to the n-ary operation symbol f an n-ary term of type (n) will be called a hypersubstitution of type $\tau = (n)$. Here we have only one n-ary operation symbol f in our type and any hypersubstitution σ is completely determined by the image $\sigma(f)$. Thus we will denote a hypersubstitution σ by σ_t if $\sigma(f) = t$. Any hypersubstitution σ can be uniquely extended to a mapping $\widehat{\sigma} : W_n(X) \to W_n(X)$, inductively as follows:

(i) $\widehat{\sigma}[w] := w$ for $w \in X$;

(ii) $\widehat{\sigma}[f(t_1, \ldots, t_n)] := \sigma(f)(\widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])$ for $t_1, \ldots, t_n \in W_n(X)$ where $\widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]$ will be assumed to be already defined.

We define a product $\circ_h$ of hypersubstitutions σ_1, σ_2 by $\sigma_1 \circ_h \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$, where $\circ$ is the usual composition of functions. Then the set $Hyp(n)$ of all hypersubstitutions of type $\tau = (n)$ forms a monoid $(Hyp(n); \circ_h, \sigma_{id})$, where σ_{id} is the identity hypersubstitution, defined by

$$\sigma_{id}(f) := f(x_1, \ldots, x_n).$$

Since $Hyp(n) = \{\sigma_t \mid t \in W_n(X_n)\}$,

$$\varphi_n : Hyp(n) \to W_n(X_n) \text{ with } \varphi_n(\sigma) = \sigma(f)$$

is a bijection. Let us define a binary operation $\diamond$ on $W_n(X_n)$ by setting

$$s \diamond t := \widehat{\sigma}_s[t].$$

Then one can verify that $(W_n(X_n); \diamond, \sigma_{id}(f))$ forms a monoid which is isomorphic to $(Hyp(n); \circ_h, \sigma_{id})$. If $\emptyset \neq X \subseteq W_n(X_n)$ then the carry set of the subsemigroup of $(W_n(X_n); \diamond)$ generated by X is denoted by $\langle X \rangle$.

Proposition 1. *Let $1 \leq n \in \mathbb{N}$. Then the monoid $(Hyp(n); \circ_h, \sigma_{id})$ is isomorphic to $(W_n(X_n); \diamond, \sigma_{id}(f))$.*

Proof. We want to show that the bijection φ_n is an isomorphism. Let us mention that $\varphi_n(\sigma_{id}) = \sigma_{id}(f)$ by definition of φ_n. Moreover, for $\sigma_s, \sigma_t \in Hyp(n)$ it holds $\varphi_n(\sigma_s \circ_h \sigma_t) = (\sigma_s \circ_h \sigma_t)(f) = \widehat{\sigma}_s[\sigma_t(f)] = s \diamond \sigma_t(f) = \sigma_s(f) \diamond \sigma_t(f) = \varphi_n(\sigma_s) \diamond \varphi_n(\sigma_t)$. ■

Thus $(W_n(X_n); \diamond, \sigma_{id}(f))$ forms a monoid, which is isomorphic to the monoid of all hypersubstitutions of type $\tau = (n)$. This suggests the idea to study properties of the monoid $W_n(X_n)$ and its submonoids. We will use the following concepts and notation in the next statements and their proofs. For a term $t \in W_n(X_n)$ we put

$vb(t)$ – the total number of occurrences of variables in t(including multiplicities)

$op(t)$ – the number of occurrence of the operation symbol f in t

$vb_i(t)$ – the number of occurrence of x_i in t for $i \in \overline{n}$

$var(t)$ – the set of all variables occurring in t.

Notation 2. Let $1 \leq n \in \mathbb{N}$. Then we put $W_n^{reg} := \{t \mid t \in W_n(X_n)$ and $var(t) = X_n\}$.

The set W_n^{reg} corresponds to the set $Reg(n) = \{\sigma \mid \sigma \in Hyp(n)$ and $\sigma(f) \in W_n^{reg}\}$ of all regular hypersubstitutions of type $\tau = (n)$ which forms a monoid (see [5]). Clearly, $\sigma_{id}(f) \in W_n^{reg}$ and the monoid $(Reg(n); \circ_h, \sigma_{id})$ is isomorphic to $(W_n^{reg}; \diamond, \sigma_{id}(f))$ by the isomorphism φ_n restricted to $Reg(n)$, i.e. W_n^{reg} forms a monoid which is isomorphic to the monoid of all regular hypersubstitutions of type $\tau = (n)$. Our first aim is to determine all maximal submonoids of $(W_n^{reg}; \diamond, \sigma_{id}(f))$. This will be done in the next section.

A second kind of terms is determined by transformations on the set $\overline{n}$. Let T_n be the set of all transformations on the set $\overline{n}$, i.e. T_n is the set of all mappings $h : \overline{n} \to \overline{n}$. Then one gets a monoid $(T_n; \circ, \varepsilon_n)$, where $\circ$ is the usual composition of functions and ε_n is the identity mapping on $\overline{n}$. The natural number $n_h := n - |\{h(a) \mid a \in \overline{n}\}|$ is called the defect of a given transformation h. A transformation h is called isotone if the following implication holds for all $a, b \in \overline{n}$:

$$a \leq b \Rightarrow h(a) \leq h(b).$$

For $1 \leq k < n$ let $I_{n,k}$ be the set of all isotone transformations on $\overline{n}$ with defect $\geq k$. The set $O_n := I_{n,1}$ forms a semigroup and each of the set $I_{n,k}$, $2 \leq k < n$, forms an ideal of O_n ([14]). A transformation h is called antitone if the following implication holds for all $a, b \in \overline{n}$:

$$a \leq b \Rightarrow h(a) \geq h(b).$$

A transformation is called monotone if it is isotone or antitone. The set M_n of all monotone transformations on $\overline{n}$ with defect ≥ 1 forms a semigroup, too ([11]). Moreover, it is well-known that the set S_n of all bijective transformations on $\overline{n}$, i.e. permutations on $\overline{n}$, forms a subgroup of T_n.

For any transformation h on $\overline{n}$, we can consider the term $f(x_{h(1)}, \dots, x_{h(n)})$. So we get terms defined by transformations and it is very natural to define hypersubstitutions by transformations. For any transformation h, we will denote the hypersubstitution σ with $\sigma(f) = f(x_{h(1)}, \dots, x_{h(n)})$ by σ_h. For any set $A \subseteq T_n$, we put $A^{hyp} := \{\sigma_h \mid h \in A\}$ and $W_A := \{f(x_{h(1)}, \dots, x_{h(n)}) \mid h \in A\}$. In particular, we put $P_n := W_{S_n}$. Clearly, the mapping $\rho_n : T_n \to T_n^{hyp}$ defined by

$$\rho_n(h) := \sigma_h$$

is a bijection. In particular, ρ_n is an anti-isomorphism (dual isomorphism).

Proposition 3. *Let* $1 \leq n \in \mathbb{N}$. *Then* $(T_n; \circ, \varepsilon_n)$ *is anti-isomorphic to* $(T_n^{hyp}; \circ_h, \sigma_{id})$.

Proof. We have to show that $\rho_n(\varphi \circ \pi) = \rho_n(\pi) \circ_h \rho_n(\varphi)$ for all $\varphi, \pi \in A$. Let $\varphi, \pi \in A$. Then we have $\rho_n(\pi) \circ_h \rho_n(\varphi) = \sigma_\pi \circ_h \sigma_\varphi = \widehat{\sigma}_\pi \circ \sigma_\varphi$ and $\rho_n(\varphi \circ \pi) = \sigma_{\varphi \circ \pi}$. Further we have

$$\begin{aligned}
\sigma_{\varphi\circ\pi}(f) &= f(x_{(\varphi\circ\pi)(1)}, \dots, x_{(\varphi\circ\pi)(n)}) \\
&= f(x_{\pi(1)}, \dots, x_{\pi(n)})(x_{\varphi(1)}, \dots, x_{\varphi(n)}) \\
&= \widehat{\sigma}_{f(x_{\pi(1)},\dots,x_{\pi(n)})}[f(x_{\varphi(1)}, \dots, x_{\varphi(n)})] \\
&= \widehat{\sigma}_\pi[f(x_{\varphi(1)}, \dots, x_{\varphi(n)})] \\
&= \widehat{\sigma}_\pi[\sigma_\varphi(f)] \\
&= (\widehat{\sigma}_\pi \circ \sigma_\varphi)(f).
\end{aligned}$$

This shows that $\rho_n(\pi) \circ_h \rho_n(\varphi) = \rho_n(\varphi \circ \pi)$. ■

In particular, then each of the sets O_n^{hyp}, M_n^{hyp} and $I_{n,2}^{hyp}$ forms a semigroup. We will consider these semigroups in Section 4. Moreover, the mapping $\gamma_n : T_n \to W_{T_n}$ defined by $\gamma_n(h) := f(x_{h(1)}, \dots, x_{h(n)})$ is evidently an anti-isomorphism by Proposition 1 and Proposition 3. This gives:

Corollary 4. *Let* $1 \leq n \in \mathbb{N}$. *Then* $(T_n; \circ, \varepsilon_n)$ *is anti-isomorphic to* $(W_{T_n}; \diamond, \sigma_{id}(f))$.

3. The maximal submonoids of $Reg(n)$

To characterize all maximal subsemigroups of $(W_n^{reg}; \diamond)$ for a given natural number $n \geq 2$, we need some technical lemmas. The following facts were proved in [10]:

Lemma 5. *Let* $s \in W_n^{reg}$ *and* $t, t_1, \ldots, t_n \in W_n(X_n)$ *with* $t = f(t_1, \ldots, t_n)$. *Then*

(a) $vb(s \diamond t) \geq vb(t)$;

(b) $vb(s \diamond t) = \sum_n^{i=1} vb_i(s) vb(s \diamond t_i)$;

(c) $vb(s(t_1, \ldots, t_n)) = \sum_{i=1}^n vb_i(s) vb(t_i)$.

Corollary 6. *For* $s, t \in W_n^{reg}$ *it holds:*

(a) *If* $s \notin P_n$ *then* $vb(t) < vb(s \diamond t)$.

(b) *If* $t \notin P_n$ *then* $vb(s) < vb(s \diamond t)$.

Proof. We have $vb(s \diamond t) \geq \sum_{i=1}^n vb_i(s) vb(t_i)$ by Lemma 5 and $vb_i(s) \neq 0$ for $i \in \overline{n}$ since $var(s) = X_n$.

(a) If $s \notin P_n$ then there is a $j \in \overline{n}$ with $vb_j(s) \geq 2$ and thus

$$\sum_{i=1}^n vb_i(s) vb(t_i) \geq vb(t_j) + \sum_{i=1}^n vb(t_i) > \sum_{i=1}^n vb(t_i) = vb(t).$$

(b) If $t \notin P_n$ then there is a $j \in \overline{n}$ with $vb(t_j) \geq 2$ and thus

$$\sum_{i=1}^n vb_i(s) vb(t_i) \geq 1 + \sum_{i=1}^n vb_i(s) > \sum_{i=1}^n vb_i(s) = vb(s).$$

■

Lemma 7. *For $s \in W_n^{reg}$ and $t \in P_n$ we have $vb(s \diamond t) = vb(t \diamond s) = vb(s)$.*

Proof. There is a $\pi \in S_n$ such that $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$. Further, there are $s_1, \ldots, s_n \in W_n(X_n)$ such that $s = f(s_1, \ldots, s_n)$. Then we have $vb(s \diamond t) = vb(\widehat{\sigma}_s[t]) = vb(s(x_{\pi(1)}, \ldots, x_{\pi(n)})) = \sum_{i=1}^n vb_i(s) vb(x_{\pi(i)}) = \sum_{i=1}^n vb_i(s) = vb(s)$ by Lemma 5c). We show now by induction that $vb(\widehat{\sigma}_t[r]) = vb(r)$ for all $r \in W_n(X_n)$. Clearly, $\widehat{\sigma}_t[r] = r$ for $r \in X_n$. Let $r = f(r_1, \ldots, r_n)$, $r_1, \ldots, r_n \in W_n(X_n)$, and suppose that $vb(\widehat{\sigma}_t[r_i]) = vb(r_i)$ for $i \in \overline{n}$. Then

$$\begin{aligned} vb(\widehat{\sigma}_t[r]) &= vb(f(x_{\pi(1)}, \ldots, x_{\pi(n)})(\widehat{\sigma}_t[r_1], \ldots, \widehat{\sigma}_t[r_n])) \\ &= vb(f(\widehat{\sigma}_t[r_{\pi(1)}], \ldots, \widehat{\sigma}_t[r_{\pi(n)}])) = \sum_{i=1}^n vb(\widehat{\sigma}_t[r_i]) = \sum_{i=1}^n vb(r_i) = vb(r). \end{aligned}$$

In particular, $vb(t \diamond s) = vb(\widehat{\sigma}_t[s]) = vb(s)$. ∎

Now we consider the Green‘s relation $\mathcal{J}$ on the semigroup $(W_n^{reg}; \diamond)$ which is defined by $s\mathcal{J}t$ if there are $s_1, s_2, t_1, t_2 \in W_n^{reg}$ such that $s = t_1 \diamond t \diamond t_2$ and $t = s_1 \diamond s \diamond s_2$. For $t \in W_n^{reg}$, we denote the J-class containing t by J_t, i.e.

$$J_t := \{s \mid s \in W_n^{reg} \text{ and } s\mathcal{J}t\}.$$

The relation $\mathcal{J}$ on W_n^{reg} can be characterized as follows:

Lemma 8. *Let $s, t \in W_n^{reg}$. Then there holds $s\mathcal{J}t$ iff there are $s_1, s_2, t_1, t_2 \in P_n$ such that $s = t_1 \diamond t \diamond t_2$ and $t = s_1 \diamond s \diamond s_2$.*

Proof. One direction is clear. Conversely, let $s\mathcal{J}t$. Then there are $s_1, s_2, t_1, t_2 \in W_n^{reg}$ such that $s = t_1 \diamond t \diamond t_2$ and $t = s_1 \diamond s \diamond s_2$. We will show that $s_1, s_2, t_1, t_2 \in P_n$. We have $s = (t_1 \diamond s_1) \diamond s \diamond (s_2 \diamond t_2)$. Assume that $(t_1 \diamond s_1) \notin P_n$. Then $vb(s) < vb((t_1 \diamond s_1) \diamond s)$ by Corollary 6. Further, we have $vb((t_1 \diamond s_1) \diamond s)) \leq vb((t_1 \diamond s_1) \diamond s \diamond (s_2 \diamond t_2))$ by Corollary 6 and Lemma 7, respectively. This gives $vb(s) < vb(s)$, a contradiction. Assume that $(s_2 \diamond t_2) \notin P_n$. Then $vb((t_1 \diamond s_1) \diamond s) < vb((t_1 \diamond s_1) \diamond s \diamond (s_2 \diamond t_2)) = vb(s)$ by Corollary 6. But since $(t_1 \diamond s_1) \in P_n$ we have $vb((t_1 \diamond s_1) \diamond s) = vb(s)$ by Lemma 7. This gives $vb(s) < vb(s)$, a contradiction. So, both terms $(t_1 \diamond s_1)$ and $(s_2 \diamond t_2)$ belong to P_n. This provides $vb(s_1), vb(t_1) \geq vb(t_1 \diamond s_1) = 1$ and $vb(s_2), vb(t_2) \geq vb(t_2 \diamond s_2) = 1$. But this is only possible if $s_1, s_2, t_1, t_2 \in P_n$, by Corollary 6. ∎

Proposition 9. *Let $s, t \in W_n^{reg}$. Then there holds $s\mathcal{J}t$ iff there are $s_1, s_2 \in P_n$ such that $t = s_1 \diamond s \diamond s_2$.*

Proof. One direction is clear by Lemma 8. Conversely, let $s_1, s_2 \in P_n$ such that $t = s_1 \diamond s \diamond s_2$. Then there are $\rho, \pi \in S_n$ with $s_1 = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ and $s_2 = f(x_{\rho(1)}, \ldots, x_{\rho(n)})$. Then there are $\rho^{-1}, \pi^{-1} \in S_n$ with $\rho^{-1} \circ \rho = \pi \circ \pi^{-1} = \varepsilon_n$ and we get $f(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}) \diamond t \diamond f(x_{\rho^{-1}(1)}, \ldots, x_{\rho^{-1}(n)}) = f(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}) \diamond f(x_{\pi(1)}, \ldots, x_{\pi(n)}) \diamond s \diamond f(x_{\rho(1)}, \ldots, x_{\rho(n)}) \diamond f(x_{\rho^{-1}(1)}, \ldots, x_{\rho^{-1}(n)})$. Then Proposition 1 provides $\sigma_{\pi^{-1}} \circ_h \sigma_t \circ_h \sigma_{\rho^{-1}} = \sigma_{\pi^{-1}} \circ_h \sigma_\pi \circ_h \sigma_s \circ_h \sigma_\rho \circ_h \sigma_{\rho^{-1}}$ where $\sigma_{\pi^{-1}} \circ_h \sigma_\pi \circ_h \sigma_s \circ_h \sigma_\rho \circ_h \sigma_{\rho^{-1}} = \sigma_{(\pi \circ \pi^{-1})} \circ_h \sigma_s \circ_h \sigma_{(\rho^{-1} \circ \rho)} = \sigma_{\varepsilon_n} \circ_h \sigma_s \circ_h \sigma_{\varepsilon_n} = \sigma_s$ by Proposition 3. This gives $f(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}) \diamond t \diamond f(x_{\rho^{-1}(1)}, \ldots, x_{\rho^{-1}(n)}) = s$ again by Proposition 1. Altogether, this shows that $s\mathcal{J}t$ using Lemma 8. ∎

Notation 10. A term $t \in W_n^{reg}$ is called a proper $\diamond$-product if there are $r, s \in W_n^{reg} \setminus P_n$ such that $t = r \diamond s$. Let W_n^{dec} denote the set of all proper $\diamond$-products of W_n^{reg}.

Now we are able to characterize all maximal subsemigroups of $(W_n^{reg}; \diamond)$, i.e. all subsets $W \subseteq W_n^{reg}$ with $\langle W \cup \{a\} \rangle = W_n^{reg}$ for all $a \in W_n^{reg} \setminus W$.

Theorem 11. *A set $W \subseteq W_n^{reg}$ forms a maximal subsemigroup of $(W_n^{reg}; \diamond)$ iff one of the following statements is satisfied:*

(i) *There is a $t \in W_n^{reg} \setminus (W_n^{dec} \cup P_n)$ such that $W = W_n^{reg} \setminus J_t$.*

(ii) *There is a maximal subgroup S of S_n such that $W = (W_n^{reg} \setminus P_n) \cup W_S$.*

Proof. Suppose that (i) is satisfied, i.e. there is a $t \in W_n^{reg} \setminus (W_n^{dec} \cup P_n)$ such that $W = W_n^{reg} \setminus J_t$. We show that W forms a subsemigroup of $(W_n^{reg}; \diamond)$. For this let $a, b \in W_n^{reg} \setminus J_t$. Then $a \diamond b \in W_n^{reg}$. Assume that $a \diamond b \in J_t$. Then there are $s_1, s_2 \in P_n$ such that $t = (s_1 \diamond a) \diamond (b \diamond s_2)$ by Proposition 9. Since $t \notin W_n^{dec}$ we have $(s_1 \diamond a) \in P_n$ or $(b \diamond s_2) \in P_n$. Without loss of generality let $(s_1 \diamond a) \in P_n$. Then we get $b \in J_t$ by Proposition 9, a contradiction. Thus $a \diamond b \in W_n^{reg} \setminus J_t = W$. This shows that $(W; \diamond)$ is a subsemigroup of $(W_n^{reg}; \diamond)$. Now we show that W is maximal. First, we show that $P_n \subseteq W$. Assume that $P_n \nsubseteq W$. Then there are an $s \in P_n \cap J_t$ and $s_1, s_2 \in P_n$ such that $t = s_1 \diamond s \diamond s_2$. Then Lemma 7 provides $vb(t) = vb(s) = n$, i.e. $t \in P_n$, a contradiction. Hence $P_n \subseteq W$.

Let now $s \in W_n^{reg} \setminus W$, i.e. $s \in J_t$. Then there are $s_1, s_2 \in P_n$ such that $t = s_1 \diamond s \diamond s_2$. Hence $t \in \langle W \cup \{s\} \rangle$ and thus $\{s_1 \diamond t \diamond s_2 \mid s_1, s_2 \in P_n\} \subseteq \langle W \cup \{s\} \rangle$. Further, we have $J_t = \{s_1 \diamond t \diamond s_2 \mid s_1, s_2 \in P_n\}$ by Proposition 9, hence $J_t \subseteq \langle W \cup \{s\} \rangle$. This shows that W is maximal.

Suppose that (ii) is satisfied, i.e. there is a maximal subgroup S of S_n such that $W = (W_n^{reg} \setminus P_n) \cup W_S$. We show that W forms a subsemigroup of $(W_n^{reg}; \diamond)$. For this let $a, b \in W$. If $a \notin P_n$ or $b \notin P_n$ then $vb(b) < vb(a \diamond b)$ and $vb(a) < vb(a \diamond b)$, respectively, by Corollary 6. Thus $vb(a \diamond b) > 1$. This shows that $a \diamond b \notin P_n$, i.e. $a \diamond b \in W$. We consider now the case that $a, b \in P_n$, i.e. $a, b \in W_S$. Since S is a subgroup of S_n, we have $a \diamond b \in W_S$ by Corollary 4. This shows that $(W; \diamond)$ is a subsemigroup of $(W_n^{reg}; \diamond)$. Now we conclude that $(W_n^{reg} \setminus P_n) \cup W_S$ is maximal since $(W_S; \diamond)$ is a maximal subgroup of $(P_n; \diamond)$ by Corollary 4.

Conversely, let $(W; \diamond)$ be a maximal subsemigroup of $(W_n^{reg}; \diamond)$. We put $M := W_n^{reg} \setminus W$. Then we have $M \cap P_n = \emptyset$ or $M \cap P_n \neq \emptyset$. Suppose that $M \cap P_n \neq \emptyset$. Let us consider the set $S := \{\pi \in S_n \mid f(x_{\pi(1)}, \ldots, x_{\pi(n)}) \in W\}$. Then we have $W \cap P_n = W_S$. Since both sets W and P_n form semigroups, $W \cap P_n = W_S$ forms a subsemigroup of $(P_n; \diamond)$. Then S is a proper subsemigroup of S_n by Proposition 3. Since S_n is finite, each subsemigroup of S_n is a group. Hence there is a maximal subgroup T of S_n containing S and we have $W \subseteq (W_n^{reg} \setminus P_n) \cup W_T$ where $(W_n^{reg} \setminus P_n) \cup W_T$ forms a subsemigroup of $(W_n^{reg}; \diamond)$ by the previous considerations. Since $(W; \diamond)$ is a maximal subsemigroup of $(W_n^{reg}; \diamond)$, we can conclude that $W = (W_n^{reg} \setminus P_n) \cup W_T$.

Suppose that $M \cap P_n = \emptyset$. Let $t \in M$. Then $t \notin P_n$. Assume that $t \in W_n^{dec}$. Then there are $t_1, t_2 \in W_n^{reg} \setminus P_n$ such that $t = t_1 \diamond t_2$. Since $t \notin W$, one of the terms t_1 and t_2 does not belong to W. Without loss of generality let $t_1 \notin W$. Then Corollary 6 implies $vb(t_1) < vb(t_1 \diamond t_2) = vb(t)$. Let $a_1, \ldots, a_k \in W_n^{reg}$ for some natural number $k > 0$ with $a_j = t$ for some $j \in \{1, \ldots, k\}$. Then $vb(a_1 \diamond \ldots \diamond a_k) \geq vb(t)$ by Corollary 6 and Lemma 7. Thus $t_1 \notin \langle W \cup \{t\} \rangle$, i.e. $\langle W \cup \{t\} \rangle \neq W_n^{reg}$. Because of the maximality of $(W; \diamond)$ we get $t \in W$, a contradiction. Hence $t \notin W_n^{dec}$ and altogether $t \in W_n^{reg} \setminus (W_n^{dec} \cup P_n)$. Now we show $J_t \subseteq M$. Otherwise there is an $s \in J_t$ with $s \in W$. Then there are $s_1, s_2 \in P_n$ such that $t = s_1 \diamond s \diamond s_2$. Since $P_n \subseteq W$ we get $t \in W$, a contradiction. Now we have $W \subseteq W_n^{reg} \setminus J_t$, where $W_n^{reg} \setminus J_t$ forms a semigroup (see the previous considerations). Since $(W; \diamond)$ is a maximal subsemigroup of $(W_n^{reg}; \diamond)$, we obtain $W = W_n^{reg} \setminus J_t$. $\blacksquare$

Remark 12. Clearly, each maximal submonoid of $(W_n^{reg};\diamond)$ contains the identity element $\sigma_{id}(f)$. Therefore, Theorem 11 characterizes all maximal submonoids of $(W_n^{reg};\diamond,\sigma_{id}(f))$.

Since the only maximal subgroup of S_2 as well as the four maximal subgroups of S_3 are well known, we can formulate Theorem 11 for $n = 2$ and $n = 3$, respectively, in the following way.

Proposition 13. *A set $W \subseteq W_2^{reg}$ forms a maximal subsemigroup of $(W_2^{reg};\diamond)$ iff $W = W_2^{reg} \setminus \{f(x_2,x_1)\}$ or $W = W_2^{reg} \setminus J_t$ for some $t \in W_2^{reg} \setminus (W_2^{dec} \cup P_2)$.*

Proposition 14. *A set $W \subseteq W_3^{reg}$ forms a maximal subsemigroup of $(W_3^{reg};\diamond)$ iff $W = W_3^{reg} \setminus J_t$ for some $t \in W_3^{reg} \setminus (W_3^{dec} \cup P_3)$ or W coincides with one of the following four sets:*

(a) $W_3^{reg} \setminus \{f(x_1,x_3,x_2), f(x_3,x_2,x_1), f(x_2,x_1,x_3)\}$

(b) $W_3^{reg} \setminus \{f(x_1,x_3,x_2), f(x_3,x_2,x_1), f(x_2,x_3,x_1), f(x_3,x_1,x_2)\}$

(c) $W_3^{reg} \setminus \{f(x_1,x_3,x_2), f(x_2,x_1,x_3), f(x_2,x_3,x_1), f(x_3,x_1,x_2)\}$

(d) $W_3^{reg} \setminus \{f(x_2,x_1,x_3), f(x_3,x_2,x_1), f(x_2,x_3,x_1), f(x_3,x_1,x_2)\}$.

To determine all maximal subsemigroups of W_n^{reg} we need the knowledge of all maximal subgroups of S_n and of all elements of the set $W_n^{reg}\setminus(W_n^{dec}\cup P_n)$. The O'Nan Scott-Theorem gives a classification of all maximal subgroups of S_n (e.g. [15]). But the characterization of all proper $\diamond$-products seems to be a too complex problem. Therefore we restrict ourselves to study only necessary or only sufficient properties of the elements of W_n^{dec}. First, we show that a term $t \in W_n^{reg}$ does not belong to W_n^{dec} if $op(t)$ is a prime number.

Lemma 15. *Let $s \in W_n^{reg}$ and $t \in W_n(X_n)$. Then there is a natural number $k \geq op(t)$ such that $op(s \diamond t) = k \cdot op(s)$.*

Proof. If $t \in X_n$ then $op(t) = 0$ and thus $op(s \diamond t) = op(\widehat{\sigma}_s[t]) = op(t) = 0 = 0 \cdot op(s)$. Suppose that $t = f(t_1,\dots,t_n)$ with $t_1,\dots,t_n \in W_n(X_n)$ and $op(s \diamond t_i) = k_i \cdot op(s)$ with $k_i \geq op(t_i)$ for $i \in \overline{n}$. We have $op(s \diamond t) = op(s) + \sum_{i=1}^{n} vb_i(s) \cdot op(\widehat{\sigma}_s[t_i])$ (see [10]). Further, it holds

$$op(s) + \sum_{i=1}^{n} vb_i(s) \cdot op(\widehat{\sigma}_s[t_i])$$

$$= op(s) + \sum_{i=1}^{n} vb_i(s) \cdot k_i \cdot op(s)$$

$$= op(s) \cdot \left(1 + \sum_{i=1}^{n} vb_i(s) \cdot k_i\right).$$

Since $vb_i(s) \neq 0$ (because of $var(s) = X_n$) and $k_i \geq op(t_i)$ for $i \in \overline{n}$ we have $\sum_{i=1}^{n} vb_i(s) \cdot k_i \geq \sum_{i=1}^{n} vb_i(s) \cdot op(t_i) \geq \sum_{i=1}^{n} op(t_i) = op(t) - 1$, i.e. $1 + \sum_{i=1}^{n} vb_i(s) \cdot k_i \geq op(t)$. ∎

Proposition 16. *If $t \in W_n^{reg}$ such that $op(t)$ is a prime number then $t \notin W_n^{dec} \cup P_n$.*

Proof. Since $op(t)$ is a prime number, $op(t) \geq 2$. Thus $t \notin P_n$. Assume that $t \in W_n^{dec}$. Then there are $r, s \in W_n^{reg} \setminus P_n$ such that $t = r \diamond s$. This provides $op(t) = op(r \diamond s) = k \cdot op(r)$ for some $k \geq op(s)$ by Lemma 15. Since $r, s \notin P_n$ we have $op(r)$, $op(s) \geq 2$ and thus $k \geq op(s) \geq 2$. Hence $op(t) = k \cdot op(r)$ is not a prime number, a contradiction. This shows that $t \notin W_n^{dec}$. ∎

An element of W_n^{dec} has the following structure:

Proposition 17. *For any $t \in W_n^{dec}$ there are an $s \in W_n^{reg} \setminus P_n$ and $t_1, \ldots, t_n \in W_n(X_n)$ with $t_j \notin X_n$ for some $j \in \overline{n}$ such that $t = s(s \diamond t_1, \ldots, s \diamond t_n)$.*

Proof. Since $t \in W_n^{dec}$ there are $r, s \in W_n^{reg} \setminus P_n$ such that $t = r \diamond s = \widehat{\sigma}_r[s]$. Further, there are $s_1, \ldots, s_n \in W_n(X_n)$ such that $s = f(s_1, \ldots, s_n)$ and we obtain $\widehat{\sigma}_r[s] = r(\widehat{\sigma}_r[s_1], \ldots, \widehat{\sigma}_r[s_n]) = r(r \diamond s_1, \ldots, r \diamond s_n)$. Since $s \notin P_n$ there is a $j \in \overline{n}$ with $s_j \notin X_n$. ∎

Example 18. We consider the case $n = (3)$ and the term

$$t = f(x_1, f(x_1, x_1, x_1), f(x_1, x_1, f(x_1, x_2, x_3))).$$

Although $op(t)$ is not a prime number, t does not belong to W_3^{dec}. Indeed, assume that there are an $s \in W_3^{reg} \setminus P_3$ and $t_1, t_2, t_3 \in W_3(X_3)$ with $t_i \notin X_3$ for some $i \in \{1, 2, 3\}$ such that $t = s(s \diamond t_1, s \diamond t_2, s \diamond t_3)$. Then $op(s) \geq 2$ and $op(s \diamond t_i) \geq 4$ by Lemma 15. This provides that $op(t) \geq 4 + op(s) > 4$, a contradiction.

4. Transformation hypersubstitutions

A list of all maximal subsemigroups of the ideal O_n of all isotone transformations on $\overline{n}$ with defect ≥ 1 is given in [16]. I. Guydzenov and I. Dimitrova have determined all maximal subsemigroups of M_n as well as of the ideal $I_{n,2}$ of all isotone transformations on $\overline{n}$ with defect ≥ 2, see [12] and [13], respectively. These results can be regarded as generalizations of the results in [16] concerning O_n. We want to use the mentioned results to characterize the maximal submonoids of particular monoids of transformation hypersubstitutions. It is easy to verify that each of the sets $O_n^{hyp} \cup \{\sigma_{id}\}$, $M_n^{hyp} \cup \{\sigma_{id}\}$ and $I_{n,2}^{hyp} \cup \{\sigma_{id}\}$ forms a submonoid of $TR(n)$. We can use Proposition 3 to characterize the maximal submonoids of each of these monoids.

Lemma 19. *Let $1 \leq n \in \mathbb{N}$ and let $(A; \circ)$ be a transformation semigroup on $\overline{n}$ with $\varepsilon_n \notin A$. Then a set $M \subseteq TR(n)$ forms a maximal submonoid of $(A^{hyp} \cup \{\sigma_{id}\}; \circ_h, \sigma_{id})$ iff there is a maximal subsemigroup $(B; \circ)$ of $(A; \circ)$ such that $M = B^{hyp} \cup \{\sigma_{id}\}$.*

Proof. Suppose that $(M; \circ_h, \sigma_{id})$ is a maximal submonoid of $(A^{hyp} \cup \{\sigma_{id}\}; \circ_h, \sigma_{id})$. Then there is a set $B \subseteq A$ such that $B^{hyp} \cup \{\sigma_{id}\} = M$. Since $\varepsilon_n \notin A$, $\sigma_{id} \notin A^{hyp}$ and thus $(A^{hyp}; \circ_h)$ forms a semigroup. Hence $M \setminus \{\sigma_{id}\}$ forms a semigroup, in particular, $(M \setminus \{\sigma_{id}\}; \circ_h)$ is a maximal subsemigroup of $(A^{hyp}; \circ_h)$. This implies that $(B; \circ)$ is a maximal subsemigroup of $(A; \circ)$ by Proposition 3. Conversely, suppose that $(B; \circ)$ is a maximal subsemigroup of $(A; \circ)$ such that $M = B^{hyp} \cup \{\sigma_{id}\}$. Then $(B^{hyp}; \circ_h)$ is a maximal subsemigroup of $(A^{hyp}; \circ_h)$ by Proposition 3. Thus $\langle M \cup \{\sigma\} \rangle = \langle B^{hyp} \cup \{\sigma, \sigma_{id}\} \rangle = \langle B^{hyp} \cup \{\sigma\} \rangle \cup \{\sigma_{id}\} = A^{hyp} \cup \{\sigma_{id}\}$ for all $\sigma \in (A^{hyp} \cup \{\sigma_{id}\}) \setminus M$. This shows that $(M; \circ_h, \sigma_{id})$ is a maximal submonoid of $(A^{hyp} \cup \{\sigma_{id}\}; \circ_h, \sigma_{id})$. ■

For the set $TR_2(n) := I_{n,2}^{hyp} \cup \{\sigma_{id}\}$ we have

Corollary 20. *Let $1 \leq n \in \mathbb{N}$. Then the following monoids are all maximal submonoids of $(TR_2(n); \circ_h, \sigma_{id})$: $(A^{hyp} \cup \{\sigma_{id}\}; \circ_h, \sigma_{id})$, where $(A; \circ)$ is a maximal subsemigroup of $(I_{n,2}; \circ)$.*

The maximal subsemigroups of $(I_{n,2};\circ)$ are listed in [12]. For example, let us consider the case $n = 4$.

Example 21. Let $n = 4$. Then we have $TR_2(4) = \{\sigma_{f(x_i,x_i,x_i,x_i)} \mid 1 \leq i \leq 4\} \cup \{\sigma_{f(x_i,x_i,x_i,x_j)} \mid 1 \leq i < j \leq 4\} \cup \{\sigma_{f(x_i,x_i,x_j,x_j)} \mid 1 \leq i < j \leq 4\} \cup \{\sigma_{f(x_i,x_j,x_j,x_j)} \mid 1 \leq i < j \leq 4\} \cup \{\sigma_{id}\}$. There are eleven maximal submonoids of $(TR_2(4);\circ_h,\sigma_{id})$, namely

$$A_{i,j} = TR_2(4)\backslash\{\sigma_{f(x_i,x_i,x_i,x_j)},\sigma_{f(x_i,x_i,x_j,x_j)},\sigma_{f(x_i,x_j,x_j,x_j)}\} \text{ for } 1\leq i < j\leq 4$$

$$A_1 = TR_2(4)\backslash\{\sigma_{f(x_i,x_j,x_j,x_j)} \mid 1 \leq i < j \leq 4 \text{ and } i+j \neq 3\}$$

$$A_2 = TR_2(4)\backslash\{\sigma_{f(x_i,x_i,x_i,x_j)} \mid 1 \leq i < j \leq 4 \text{ and } i+j \neq 7\}$$

$$A_3 = TR_2(4)\backslash(\{\sigma_{f(x_1,x_1,x_1,x_j)} \mid 2 \leq j \leq 4\} \cup \{\sigma_{f(x_1,x_1,x_j,x_j)} \mid 2 \leq j \leq 4\})$$

$$A_4 = TR_2(4)\backslash(\{\sigma_{f(x_i,x_j,x_j,x_j)} \mid 1 \leq i < j \leq 4 \text{ and } i+j \neq 3,7\}) \cup$$

$$\{\sigma_{f(x_i,x_i,x_i,x_j)} \mid 1 \leq i < j \leq 4 \text{ and } i+j \neq 3,7\}$$

$$A_5 = TR_2(4)\backslash\{\sigma_{f(x_i,x_i,x_j,x_j)} \mid 1 \leq i < j \leq 4 \text{ and } ij \neq 6\}.$$

For the set $TR_1(n) := O_n^{hyp} \cup \{\sigma_{id}\}$ we get

Corollary 22. *Let $1 \leq n \in \mathbb{N}$. Then the following monoids are the maximal submonoids of $(TR_1(n);\circ_h,\sigma_{id})$: $(A^{hyp} \cup \{\sigma_{id}\};\circ_h,\sigma_{id})$, where $(A;\circ)$ is a maximal subsemigroup of $(O_n;\circ)$.*

The maximal subsemigroups of $(O_n;\circ)$ are listed in [16]. For example, let us consider the case $n = 4$.

Example 23. Let $n = 4$. Then we have $TR_1(4) = TR_2(4) \cup \{\sigma_{f(x_i,x_i,x_j,x_l)} \mid 1 \leq i < j < l \leq 4\} \cup \{\sigma_{f(x_i,x_j,x_j,x_l)} \mid 1 \leq i < j < l \leq 4\} \cup \{\sigma_{f(x_i,x_j,x_l,x_l)} \mid 1 \leq i < j < l \leq 4\}$. There are ten maximal submonoids of $(TR_1(4);\circ_h,\sigma_{id})$, namely

$$B_{i,j,l} = TR_1(4)\backslash\{\sigma_{f(x_i,x_i,x_j,x_l)}, \sigma_{f(x_i,x_j,x_j,x_l)}, \sigma_{f(x_i,x_j,x_l,x_l)}\} \text{ for } 1 \le i < j < l \le 4$$

$$B_1 = TR_1(4)\backslash\{\sigma_{f(x_i,x_j,x_j,x_l)} \mid 1 \le i < j < l \le 4\}$$

$$B_2 = TR_1(4)\backslash\{\sigma_{f(x_1,x_3,x_3,x_4)}, \sigma_{f(x_2,x_3,x_3,x_4)}, \sigma_{f(x_1,x_3,x_4,x_4)}, \sigma_{f(x_2,x_3,x_4,x_4)}\}$$

$$B_3 = TR_1(4)\backslash\{\sigma_{f(x_i,x_j,x_4,x_4)} \mid 1 \le i < j \le 3\}$$

$$B_4 = TR_1(4)\backslash\{\sigma_{f(x_1,x_1,x_i,x_j)} \mid 2 \le i < j \le 4\}$$

$$B_5 = TR_1(4)\backslash\{\sigma_{f(x_1,x_1,x_2,x_3)}, \sigma_{f(x_1,x_1,x_2,x_4)}, \sigma_{f(x_1,x_2,x_2,x_3)}, \sigma_{f(x_1,x_2,x_2,x_4)}\}$$

$$B_6 = TR_1(4)\backslash\{\sigma_{f(x_1,x_1,x_2,x_4)}, \sigma_{f(x_1,x_1,x_3,x_4)}, \sigma_{f(x_1,x_2,x_4,x_4)}, \sigma_{f(x_1,x_3,x_4,x_4)}\}.$$

For the set $TR_{mon}(n) := M_n^{hyp} \cup \{\sigma_{id}\}$ we have

Corollary 24. *Let $1 \le n \in \mathbb{N}$. Then the following monoids are all maximal submonoids of $(TR_{mon}(n); \circ_h, \sigma_{id})$: $(A^{hyp} \cup \{\sigma_{id}\}; \circ_h, \sigma_{id})$, where $(A; \circ)$ is a maximal subsemigroup of $(M_n; \circ)$.*

The maximal subsemigroups of $(M_n; \circ)$ are listed in [13]. For example, let us consider the case $n = 4$.

Example 25. Let $n = 4$. Then we have $TR_{mon}(4) = TR_1(4) \cup \{\sigma_{f(x_i,x_i,x_i,x_j)} \mid 1 \le j < i \le 4\} \cup \{\sigma_{f(x_i,x_i,x_j,x_j)} \mid 1 \le j < i \le 4\} \cup \{\sigma_{f(x_i,x_j,x_j,x_j)} \mid 1 \le j < i \le 4\} \cup \{\sigma_{f(x_i,x_i,x_j,x_l)} \mid 1 \le l < j < i \le 4\} \cup \{\sigma_{f(x_i,x_j,x_j,x_l)} \mid 1 \le l < j < i \le 4\} \cup \{\sigma_{f(x_i,x_j,x_l,x_l)} \mid 1 \le l < j < i \le 4\}$. There are eleven maximal submonoids of $(TR_{mon}(4); \circ_h, \sigma_{id})$, namely

$$\begin{aligned}
C_{i,j,l} &= TR_{mon}(4)\backslash\{\sigma_{f(x_i,x_i,x_j,x_l)}, \sigma_{f(x_i,x_j,x_j,x_l)}, \sigma_{f(x_i,x_j,x_l,x_l)}, \sigma_{f(x_l,x_l,x_j,x_i)},\\
&\qquad \sigma_{f(x_l,x_j,x_j,x_i)}, \sigma_{f(x_l,x_j,x_i,x_i)}\} \text{ for } 1 \leq i < j < l \leq 4\\
C_1 &= TR_{mon}(4)\backslash(\{\sigma_{f(x_l,x_l,x_j,x_i)} \mid 1 \leq i < j < l \leq 4\} \cup\\
&\qquad \{\sigma_{f(x_l,x_j,x_j,x_i)} \mid 1 \leq i < j < l \leq 4\} \cup\\
&\qquad \{\sigma_{f(x_l,x_j,x_i,x_i)} \mid 1 \leq i < j < l \leq 4\})\\
C_2 &= TR_{mon}(4)\backslash(\{\sigma_{f(x_i,x_j,x_j,x_l)} \mid 1 \leq i < j < l \leq 4\} \cup \{\sigma_{f(x_l,x_j,x_j,x_i)} \mid\\
&\qquad 1 \leq i < j < l \leq 4\})\\
C_3 &= TR_{mon}(4)\backslash\{\sigma_{f(x_1,x_3,x_3,x_4)}, \sigma_{f(x_2,x_3,x_3,x_4)}, \sigma_{f(x_1,x_3,x_4,x_4)}, \sigma_{f(x_2,x_3,x_4,x_4)},\\
&\qquad \sigma_{f(x_4,x_3,x_3,x_1)}, \sigma_{f(x_4,x_3,x_3,x_2)}, \sigma_{f(x_4,x_3,x_1,x_1)}, \sigma_{f(x_4,x_3,x_2,x_2)}\}\\
C_4 &= TR_{mon}(4)\backslash(\{\sigma_{f(x_i,x_j,x_4,x_4)} \mid 1 \leq i < j \leq 3\} \cup \{\sigma_{f(x_4,x_j,x_i,x_i)} \mid\\
&\qquad 1 \leq i < j \leq 3\})\\
C_5 &= TR_{mon}(4)\backslash(\{\sigma_{f(x_1,x_1,x_1,x_j)} \mid 2 \leq i < j \leq 4\} \cup \{\sigma_{f(x_j,x_j,x_i,x_1)} \mid\\
&\qquad 2 \leq i < j \leq 4\})\\
C_6 &= TR_{mon}(4)\backslash\{\sigma_{f(x_1,x_1,x_2,x_3)}, \sigma_{f(x_1,x_1,x_2,x_4)}, \sigma_{f(x_1,x_2,x_2,x_3)}, \sigma_{f(x_1,x_2,x_2,x_4)},\\
&\qquad \sigma_{f(x_3,x_3,x_2,x_1)}, \sigma_{f(x_4,x_4,x_2,x_1)}, \sigma_{f(x_3,x_2,x_2,x_1)}, \sigma_{f(x_4,x_2,x_2,x_1)}\}\\
C_7 &= TR_{mon}(4)\backslash\{\sigma_{f(x_1,x_1,x_2,x_4)}, \sigma_{f(x_1,x_1,x_3,x_4)}, \sigma_{f(x_1,x_2,x_4,x_4)}, \sigma_{f(x_1,x_3,x_4,x_4)},\\
&\qquad \sigma_{f(x_4,x_2,x_1,x_1)}, \sigma_{f(x_4,x_3,x_1,x_1)}, \sigma_{f(x_4,x_4,x_2,x_1)}, \sigma_{f(x_4,x_4,x_3,x_1)}\}.
\end{aligned}$$

References

[1] V. Budd, K. Denecke and S.L. Wismath, *Short Solid Superassociative Type* (n) *Varieties*, East-West Journal of Mathematics **2** (2) (2001), 129–145.

[2] Th. Changphas, Monoids of Hypersubstitutions, Dissertation, Universität Potsdam 2004.

[3] Th. Changphas and K. Denecke, *Full Hypersubstitutions and Full Solid Varieties of Semigroups*, East-West Journal of Mathematics **4** (1) (2002), 177–193.

[4] K. Denecke and J. Koppitz, *M-Solid Varieties of Semigroups*, Discuss. Math. **15** (1995), 23–41.

[5] K. Denecke and J. Koppitz, M-Solid Varieties of Algebras, Springer Science-Business Media 2006.

[6] K. Denecke, J. Koppitz and S. Niwczyk, *Equational theories generated by hypersubstitutions of type* (n), Int. Journal of Algebra and Computation **12** (6) (2002), 867–876.

[7] K. Denecke, J. Koppitz and S. Shtrakov, *The Depth of a Hypersubstitution*, Journal of Automata, Languages and Combinatorics **6** (3) (2001), 253–262.

[8] K. Denecke and M. Reichel, *Monoids of hypersubstitutions and M-solid varieties*, Contributions to General Algebra 9, Wien 1995, 117–126.

[9] K. Denecke and S.L. Wismath, Hyperidentities and clones, Gordon and Breach Scientific Publishers, 2000.

[10] K. Denecke and S.L. Wismath, *Complexity of Terms, Composition, and Hypersubstitution*, Int. Journal of Mathematics and Mathematical Sciences **15** (2003), 959–969.

[11] V.H. Fernandes, G.M.S. Gomes and M.M. Jesus, *Presentations for Some Monoids of Partial Transformations on a Finite Chain*, Communications in Algebra **33** (2005), 587–604.

[12] Il. Gyudzhenov and Il. Dimitrova, *On the Maximal Subsemigroups of the Semigroup of All Isotone Transformations with Defect* ≥ 2, Comptes rendus de l'Academie bulgare des Sciences **59** (3) (2006), 239–244.

[13] Il. Gyudzhenov and Il. Dimitrova, *On the Maximal Subsemigroups of the Semigroup of all Monotone Transformations*, Discuss. Math., submitted.

[14] J.M. Howie, An Introduction to Semigroup Theory, Academic Press, London 1976.

[15] M.W. Liebeck, C.E. Praeger and J. Saxl, *A Classification of the Maximal Subgroups of the Finite Alternating and Symmetric Groups*, Journal of Algebra **111** (1987), 365–383.

[16] X. Yang, *A Classification of Maximal Subsemigroups of Finite Order-Preserving Transformation Semigroups*, Communications in Algebra **28** (3) (2000), 1503–1513.

Received 23 March 2006
Revised 13 April 2006