Discussiones Mathematicae
General Algebra and Applications 26 (2006) 233–251

BINARY RELATIONS ON THE MONOID OF V-PROPER HYPERSUBSTITUTIONS

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Abstract

In this paper we consider different relations on the set $P(V)$ of all proper hypersubstitutions with respect to a given variety V and their properties. Using these relations we introduce the cardinalities of the corresponding quotient sets as degrees and determine the properties of solid varieties having given degrees. Finally, for all varieties of bands we determine their degrees.

Keywords: solid variety, degree of proper hypersubstitutions, isomorphism degree of proper hypersubstitutions.

2000 Mathematics Subject Classification: 08B15, 20M07.

1. Introduction

Let τ be a fixed type with fundamental operation symbols f_i, $i \in I$, where f_i is n_i-ary. Let $W_\tau(X)$ be the set of all terms of type τ on an alphabet $X = \{x_1, x_2, \ldots\}$. A hypersubstitution of type τ is a mapping which associates to every operation symbol f_i a term $\sigma(f_i)$ of the same arity as f_i. Any hypersubstitution σ can be uniquely extended to a map $\hat{\sigma}$ on $W_\tau(X)$ which is inductively defined as follows:

(i) If $t = x_j$ for some $j \geq 1$, then $\hat{\sigma}[t] := x_j$.

(ii) If $t = f_i(t_1, \ldots, t_{n_i})$ for some n_i-ary operation symbol f_i and some terms $t_1, \ldots, t_{n_i}$, then $\hat{\sigma}[t] := \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$.

The left hand side of (ii) means the composition of the term $\sigma(f_i)$ and the terms $\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]$. We can define a binary operation $\circ_h$ on the set $Hyp(\tau)$ of all hypersubstitutions of type τ by letting $\sigma_1 \circ_h \sigma_2$ be the hypersubstitution which maps each fundamental operation symbol f_i to the term $\hat{\sigma_1}[\sigma_2(f_i)]$. The set $Hyp(\tau)$ forms a monoid since the operation $\circ_h$ is associative and the identity hypersubstitution σ_{id} which maps every f_i to $f_i(x_1, \ldots, x_{n_i})$ acts as an identity element.

Hypersubstitutions can be applied to equations as well as to algebras.

Let $\mathcal{A} = (A; (f_i{}^{\mathcal{A}})_{i \in I})$ be an algebra of type τ with n_i-ary fundamental operations f_i, $i \in I$. For a hypersubstitution $\sigma \in Hyp(\tau)$ we denote by $\sigma(\mathcal{A}) = (A; (\sigma(f_i)^{\mathcal{A}})_{i \in I})$ the derived algebra, where the fundamental operation $f_i^{\sigma(\mathcal{A})}$ of the derived algebra is given by $f_i^{\sigma(\mathcal{A})} = \sigma(f_i)^{\mathcal{A}}$ for every $i \in I$. From this equation one gets $t^{\sigma(\mathcal{A})} = \sigma(t)^{\mathcal{A}}$ for all $t \in W_\tau(X)$ by induction on the complexity of terms. If K is a class of algebras of the same type and if $\sigma \in Hyp(\tau)$, then we define $\sigma(K) = \{\sigma(\mathcal{A}) \mid \mathcal{A} \in K\}$. If V is a variety of algebras of type τ, then $\sigma(V)$ is in general not a variety. Let $v_\sigma(V)$ be the variety generated by $\sigma(V)$. The variety $v_\sigma(V)$ is called the *derived* variety from V by σ. One can ask for varieties V containing any derived variety as subvariety. Those varieties can be characterized by hyperidentities. Let $s \approx t$ be an identity satisfied in a variety V of algebras of type τ. We write $V \models s \approx t$. Then $s \approx t$ is called a *hyperidentity* satisfied in V if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity in V for all $\sigma \in Hyp(\tau)$. If in a variety V every identity is satisfied as a hyperidentity, then V is called *solid.* For a submonoid $M \subseteq Hyp(\tau)$ we speak of an M-hyperidentity and an M-solid variety, respectively. It is well-know (see [4, 10]) that a variety V satisfies $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ whenever $\sigma(V)$ satisfies $s \approx t$ and conversely. From this connection between derived classes and hyperidentities follows that a variety V is solid iff it contains all derived varieties $v_\sigma(V)$. We are interested in identities which are invariant under applications of all hypersubstitutions. Conversely one can look for all hypersubstitutions which preserve all identities of a given variety V. Those hypersubstitutions are called V*-proper* ([11]). Let $P(V)$ be the set of all V-proper hypersubstitutions for a variety V. Since every equation is invariant under the application of σ_{id}, the set $P(V)$ contains at least σ_{id}. $P(V)$ is equal to $Hyp(\tau)$ if and only if V is solid.

As usual we denote by IdV the set of all identities satisfied in a variety V and by $Mod\Sigma$ for a set $\Sigma \subseteq W_\tau(X)^2$ of equations of type τ the class of all algebras of type τ where any equation from Σ is satisfied as an identity.

If we want to test whether an identity $s \approx t$ is satisfied as a hyperidentity in a variety V, we have to apply all, that means infinitely many hypersubstitutions to $s \approx t$. In [11] the author introduced an equivalence relation $\sim_V$ on $Hyp(\tau)$ which allows to restrict this checking to one representative from each $\sim_V$-block. If we have a bigger relation (with respect to set inclusion), we have less blocks and checking for hypersatisfaction is less complex supposed that this relation has the property described before. One of our problems is to find the greatest binary relation having this property.

2. Binary relations on monoids of hypersubstitutions

Let $Hyp(\tau)$ be the monoid of all hypersubstitutions of type τ and let M be a submonoid. In [11] the author defined the following binary relation on $Hyp(\tau)$.

Definition 2.1. Let $\sigma_1,\ \sigma_2 \in Hyp(\tau)$ and let V be a variety of type τ. Then $\sigma_1 \sim_V \sigma_2$ iff $\sigma_1(f_i) \approx \sigma_2(f_i) \in IdV$ for all $i \in I$.

It is clear that $\sim_V$ is an equivalence relation on $Hyp(\tau)$. The relation $\sim_V$ can be restricted to submonoids of $Hyp(\tau)$ and the restricted relations $\sim_V|M$ are equivalence relations on M. From the definition of $\sim_V$ one obtains $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t] \in IdV$ for any term $t \in W_\tau(X)$ whenever $\sigma_1 \sim_V \sigma_2$. Further, it is quite easy to see ([11]) that the monoid $P(V)$ of all V-proper hypersubstitutions is *saturated* with respect to $\sim_V$. This means that $P(V)$ consists of full blocks with respect to $\sim_V$, i.e. if $\sigma_1 \sim_V \sigma_2$ and $\sigma_1 \in P(V)$, then $\sigma_2 \in P(V)$. This can also be expressed by:

$$\sigma_1 \sim_V \sigma_2 \wedge \forall s \approx t \in IdV\ (\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV \Rightarrow \hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV).$$

This implication makes clear that the relation $\sim_V$ has the desired property: checking for hyperidentities we can consider the quotient set $Hyp(\tau)/{\sim_V}$ and select one representative from each $\sim_V$-block for checking. Since $\sim_V$ in general is not a congruence relation on the monoid $Hyp(\tau)$, the quotient set $Hyp(\tau)/{\sim_V}$ is in general not a monoid. Since for a variety V and for any hypersubstitution $\sigma \in Hyp(\tau)$ we have $\hat{\sigma}_1[\sigma(f_i)] \approx \hat{\sigma}_2[\sigma(f_i)] \in IdV$ for all $i \in I$ whenever $\sigma_1 \sim_V \sigma_2$, the relation $\sim_V$ is a right-, but it in general not a left congruence. But the restriction $\sim_V|P(V)$ is a congruence on $P(V)$. Another interesting property of $\sim_V$ was proved in [3]. For any set

$\Sigma \subseteq W_\tau(X)^2$, we let $< \Sigma >$ denote the deductive closure of Σ (see e.g. [1], p.94), i.e. the set $IdMod\Sigma$ which can be obtained from Σ by application of the five rules of algebraic derivation. Let $M \subseteq Hyp(\tau)$ be a submonoid. Binary relations on monoids of hypersubstitutions were studied in [3]. We want to recall the following results. For a binary relation $r \subseteq M^2$ we define $e(r) := \{\sigma_1(f_i) \approx \sigma_2(f_i) \mid (\sigma_1, \sigma_2) \in r, i \in I\}$. Then in [3] was proved:

Proposition 2.2. *Let $M \subseteq Hyp(\tau)$ and $r \subseteq (Hyp(\tau))^2$.*

(i) *There exists a variety V of type τ such that $r = \sim_V$ iff r is deductively closed on $Hyp(\tau)$.*

(ii) *There exists an M-solid variety V of type τ such that $r = \sim_V$ iff r is deductively closed on $Hyp(\tau)$ and $\{(\sigma \circ_h \sigma_1, \sigma \circ_h \sigma_2) \mid \sigma \in M, (\sigma_1, \sigma_2) \in r\} \subseteq r$.*

(iii) *If $r \subseteq M^2$ then there exists an M-solid variety V of type τ such that $r = \sim_V|M$ iff r is deductively closed on M and r is a congruence on M.*

In [6] we defined the following binary relation on $Hyp(\tau)$:

Definition 2.3. Let $\sigma_1, \sigma_2 \in Hyp(\tau)$ and let V be a variety of type τ. Then $\sigma_1 \sim_{V-iso} \sigma_2$ iff for all algebras $\mathcal{A}$ in V we have $\sigma_1(\mathcal{A}) \cong \sigma_2(\mathcal{A})$.

The relation $\sim_{V-iso}$ is also an equivalence relation on $Hyp(\tau)$. In [6] was proved that $P(V)$ is saturated with respect to $\sim_{V-iso}$. One moment's reflection gives that $\sim_{V-iso}$ contains $\sim_V$ as a subrelation. Indeed, if $\sigma_1 \sim_V \sigma_2$, then $\sigma_1(f_i) \approx \sigma_2(f_i) \in IdV$ for all $i \in I$ and then for all algebras $\mathcal{A} \in V$ we have $\sigma_1(f_i)^{\mathcal{A}} = \sigma_2(f_i)^{\mathcal{A}}$ for the term operations on $\mathcal{A}$ induced by $\sigma_1(f_i)$ and $\sigma_2(f_i)$. But then $\sigma_1(\mathcal{A}) = \sigma_2(\mathcal{A})$ for all algebras $\mathcal{A} \in V$ and therefore $\sigma_1 \sim_{V-iso} \sigma_2$.

Moreover we prove:

Proposition 2.4. *Let V be a variety of type τ. The relation $\sim_{V-iso}|P(V)$ is a congruence on the monoid $P(V)$ of all V-proper hypersubstitutions.*

Proof. We prove that $\sim_{V-iso}|P(V)$ is a left and a right congruence on $P(V)$. Assume that $\sigma_1 \sim_{V-iso}|P(V)\, \sigma_2$ and that $\sigma \in P(V)$. Since $\sigma(\mathcal{A}) \in V$ we have $\sigma_1(\sigma(\mathcal{A})) \cong \sigma_2(\sigma(\mathcal{A}))$ for all $\mathcal{A} \in V$. We mentioned earlier the equation $f_i^{\sigma(\mathcal{A})} = \sigma(f_i)^{\mathcal{A}}$ for all $i \in I$. These equations give $f_i^{\sigma_1(\sigma(\mathcal{A}))} = \sigma_1(f_i)^{\sigma(\mathcal{A})} = \hat{\sigma}[\sigma_1(f_i)]^{\mathcal{A}} = (\sigma \circ_h \sigma_1)(f_i)^{\mathcal{A}}$ and thus $\sigma_1(\sigma(\mathcal{A})) = (\sigma \circ_h \sigma_1)(\mathcal{A})$

and then $\sigma \circ_h \sigma_1 \sim_{V-iso}|P(V)\ \sigma \circ_h \sigma_2$. Since isomorphic algebras have isomorphic derived algebras, from $\sigma_1(\mathcal{A}) \cong \sigma_2(\mathcal{A})$ there follows $\sigma(\sigma_1(\mathcal{A})) \cong \sigma(\sigma_2(\mathcal{A}))$ for all $\mathcal{A} \in V$ and thus $\sigma_1 \circ_h \sigma \sim_{V-iso}|P(V)\ \sigma_2 \circ_h \sigma$. ■

We mention that both parts of the proof need that the derived algebras belong to V and this is only guaranteed when σ, σ_1 and $\sigma_2 \in P(V)$. Therefore the relation $\sim_{V-iso}$ is not a congruence on $Hyp(\tau)$. But for a solid variety V we have $P(V) = Hyp(\tau)$ and then $\sim_{V-iso}$ is a congruence on $Hyp(\tau)$.

The third relation which we want to consider is defined by:

Definition 2.5. Let $\sigma_1, \sigma_2 \in Hyp(\tau)$ and let V be a variety of type τ. Then $\sigma_1 \approx_V^j \sigma_2$ iff $\upsilon_{\sigma_1}(V) \vee V = \upsilon_{\sigma_2}(V) \vee V$.

Again we have an equivalence relation on $Hyp(\tau)$ and we prove

Lemma 2.6. *Let V be a variety of type τ. Then $P(V)$ is saturated with respect to $\approx_V^j$.*

Proof. Let $\sigma_1 \approx_V^j \sigma_2$ and let $\hat{\sigma_1}[s] \approx \hat{\sigma_1}[t] \in IdV$ for all $s \approx t \in IdV$. Then $Id(\upsilon_{\sigma_1}(V) \vee V) = Id(\upsilon_{\sigma_2}(V) \vee V)$ we get $Id\upsilon_{\sigma_1}(V) \cap IdV = Id\upsilon_{\sigma_2}(V) \cap IdV$. Since from $\hat{\sigma_1}[s] \approx \hat{\sigma_1}[t] \in IdV$ there follows $s \approx t \in Id\sigma_1(V)$ we have $s \approx t \in Id\sigma_1(V) \cap IdV$, so $s \approx t \in Id\upsilon_{\sigma_2}(V) \cap IdV$ implies $\hat{\sigma_2}[s] \approx \hat{\sigma_2}[t] \in IdV$. ■

Proposition 2.7. *Let V be a variety of type τ. Then the cardinality of the quotient set $P(V)/ \approx_V^j |P(V)$ is 1.*

Proof. Let $\sigma_1, \sigma_2 \in P(V)$. We want to show that $\sigma_1 \approx_V^j |P(V)\sigma_2$. Let $s \approx t \in Id(\upsilon_{\sigma_1}(V) \vee V) = Id\upsilon_{\sigma_1}(V) \cap IdV$. Then from $s \approx t \in IdV$ there follows $\hat{\sigma_2}[s] \approx \hat{\sigma_2}[t] \in IdV$ since $\sigma_2 \in P(V)$. By using the conjugate property, we obtain that $s \approx t \in Id\sigma_2(V) = Id\upsilon_{\sigma_2}(V)$. Then there follows $s \approx t \in Id\upsilon_{\sigma_2}(V) \cap IdV = Id(\upsilon_{\sigma_2}(V) \vee V)$. So $Id(\upsilon_{\sigma_1}(V) \vee V) \subseteq Id(\upsilon_{\sigma_2}(V) \vee V)$. Similarly we can show that $Id(\upsilon_{\sigma_2}(V) \vee V) \subseteq Id(\upsilon_{\sigma_1}(V) \vee V)$. Therefore $\sigma_1 \approx_V^j |P(V)\sigma_2$. This shows that $|P(V)/ \approx_V^j |P(V)| = 1$. ■

Since $P(V)/ \approx_V^j |P(V)$ consists of precisely one block, the relation $\approx_V^j$ is the greatest equivalence relation on $Hyp(\tau)$ such that $P(V)$ is saturated with respect to this relation.

Theorem 2.8. *Let V be a variety of type τ and let $r \subseteq Hyp(\tau)^2$ be an equivalence relation. Then $P(V)$ is saturated with respect to r iff $r \subseteq \approx^j_V$.*

Proof. The first direction is clear because of the previous remark. Conversely, assume that $r \subseteq \approx^j_V$, then from $(\sigma_1, \sigma_2) \in r$, there follows $\sigma_1 \approx^j_V \sigma_2$ and then $Id\sigma_1(V) \cap IdV = Id\sigma_2(V) \cap IdV$. This means that for all $s \approx t \in IdV$ there holds: if $\hat{\sigma_1}[s] \approx \hat{\sigma_1}[t] \in IdV$, then $\hat{\sigma_2}[s] \approx \hat{\sigma_2}[t] \in IdV$ and therefore $P(V)$ is saturated with respect to r. ■

The forth relation which we want to consider is defined by:

Definition 2.9. Let σ_1, $\sigma_2 \in Hyp(\tau)$ and let V be a variety of type τ. Then $\sigma_1 \approx_V \sigma_2$ iff $\upsilon_{\sigma_1}(V) = \upsilon_{\sigma_2}(V)$, i.e. if the derived varieties are equal.

Clearly $\approx_V$ is an equivalence relation on $Hyp(\tau)$ and $\approx_V \subseteq \approx^j_V$. Then from Theorem 2.8 we obtain

Lemma 2.10. *Let V be a variety of type τ. Then $P(V)$ is saturated with respect to $\approx_V$.*

If $\sigma_1 \sim_{V-iso} \sigma_2$, i.e. if for all $\mathcal{A} \in V$ we have $\sigma_1(\mathcal{A}) \cong \sigma_2(\mathcal{A})$, then $\upsilon_{\sigma_1}(V) = \upsilon_{\sigma_2}(V)$ and thus $\sigma_1 \approx_V \sigma_2$ and this means $\sim_{V-iso} \subseteq \approx_V$.

Since $\sigma_1(\sigma_2(\mathcal{A})) = (\sigma_2 \circ_h \sigma_1)(\mathcal{A})$ we have $\sigma_1(\sigma_2(V)) = (\sigma_2 \circ_h \sigma_1)(V)$ and then $\upsilon_{\sigma_2 \circ_h \sigma_1}(V) = ModId(\sigma_2 \circ_h \sigma_1)(V) = ModId\sigma_1(ModId\sigma_2(V)) = \upsilon_{\sigma_1}(\upsilon_{\sigma_2}(V))$ since

$\sigma_1(ModId\sigma_2(V)) \models s \approx t$

$\Leftrightarrow$ $ModId\sigma_2(V) \models \hat{\sigma_1}[s] \approx \hat{\sigma_1}[t]$ by the conjugate property

$\Leftrightarrow$ $\hat{\sigma_1}[s] \approx \hat{\sigma_1}[t] \in IdModId\sigma_2(V)$ by a property of the Galois connection (Mod, Id)

$\Leftrightarrow$ $\hat{\sigma_1}[s] \approx \hat{\sigma_1}[t] \in Id\sigma_2(V)$

$\Leftrightarrow$ $\sigma_2(V) \models \hat{\sigma_1}[s] \approx \hat{\sigma_1}[t]$

$\Leftrightarrow$ $V \models (\sigma_2 \circ_h \sigma_1)\hat{}[s] \approx (\sigma_2 \circ_h \sigma_1)\hat{}[t]$ by the conjugate property

$\Leftrightarrow$ $(\sigma_2 \circ_h \sigma_1)(V) \models s \approx t$.

This means $Id\sigma_1(ModId\sigma_2(V)) = Id(\sigma_2 \circ_h \sigma_1)(V)$ and therefore $ModId\sigma_1(ModId\sigma_2(V)) = ModId(\sigma_2 \circ_h \sigma_1)(V)$ and thus $\upsilon_{\sigma_1}(\upsilon_{\sigma_2}(V)) = \upsilon_{\sigma_2 \circ_h \sigma_1}(V)$.

Using this property we are able to prove:

Proposition 2.11. *Let V be a variety of type τ. The relation $\approx_V$ is a right congruence on $Hyp(\tau)$.*

Proof. If $\sigma_1 \approx_V \sigma_2$, then $v_{\sigma_1}(V) = v_{\sigma_2}(V)$ and thus $v_\sigma(v_{\sigma_1}(V)) = v_\sigma(v_{\sigma_2}(V))$ and then $v_{\sigma_1 \circ_h \sigma}(V) = v_{\sigma_2 \circ_h \sigma}(V)$ and this means $\sigma_1 \circ_h \sigma \approx_V \sigma_2 \circ_h \sigma$. Therefore $\approx_V$ is a right congruence on $Hyp(\tau)$. ■

3. THE DEGREE OF PROPER HYPERSUBSTITUTIONS

In [6] for any variety V the cardinals $d_p(V) := |P(V)/\sim_V \ |P(V)|$ and $isd_p(V) := |P(V)/\sim_{V-iso} \ |P(V)|$ were introduced. The inclusion $\sim_V \subseteq \sim_{V-iso}$ implies $d_p(V) \geq isd_p(V)$. Now we define $id_p(V) := |P(V)/\approx_V \ |P(V)|$. In [9] the author introduced the dimension of a variety V as the cardinality of the set of all proper derived varieties $v_\sigma(V)$ of V. Clearly, $dim(V)+1 = id_p(V)$. Since $\sim_{V-iso} \subseteq \approx_V$ we have $d_p(V) \geq isd_p(V) \geq id_p(V)$. In [6] was proved that for a non-trivial solid variety of type $\tau = (n_i)_{i\in I}$ such that $n := max\{n_i \mid i \in I\}$ exists we have $d_p(V) \geq \prod_{i\in I} n_i + n^n - n$. Here we want to prove a similar result for $id_p(V)$. But first we prove two propositions for projection hypersubstitutions, i.e. hypersubstitutions which map any operation symbol to a variable.

Proposition 3.1. *Let V be a non-trivial variety of type $\tau = (n_i)_{i\in I}$ which has at least one operation symbol with an arity greater than 1 and assume that σ_1, σ_2 are different projection hypersubstitutions. Then $\sigma_1 \not\approx_V \sigma_2$.*

Proof. If σ_1, σ_2 are different projection hypersubstitutions of type τ, then there is an element $j \in I$ with $\sigma_1(f_j) = x_{k(j)} \neq x_{l(j)} = \sigma_2(f_j)$ where $k(j)$, $l(j) \in \{1, \ldots, n_j\}$. Suppose that $\sigma_1 \approx_V \sigma_2$. Then $Id\sigma_1(V) = Id\sigma_2(V)$. For all $\mathcal{A} \in V$ the derived algebras $\sigma_1(\mathcal{A})$ satisfy the identity $f_j(x_1, \ldots, x_{n_j}) \approx x_{k(j)}$. Therefore $f_j(x_1, \ldots, x_{n_j}) \approx x_{k(j)} \in Id\sigma_1(V) = Id\sigma_2(V)$ and by the conjugate property $V \models \hat{\sigma_2}[f_j(x_1, \ldots, x_{n_j})] \approx x_{k(j)}$ and thus $V \models x_{l(j)} \approx x_{k(j)}$, a contradiction. ■

Proposition 3.2. *Let V be a non-trivial solid variety of type $\tau = (n_i)_{i\in I}$ with $n_i > 0$ for all $i \in I$ which has at least one operation symbol with an arity greater than 1 and assume that σ is a projection hypersubstitution of type τ. Then $\sigma \not\approx_V \sigma_{id}$.*

Proof. Let σ be the projection hypersubstitution of type τ defined by $\sigma(f_i) = x_{k(i)}$ for all $i \in I$. Suppose that $\sigma \approx_V \sigma_{id}$. Then $Id\sigma(V) = Id\sigma_{id}(V) = IdV$. Since the type contains at least one operation symbol with arity greater than 1, there is a projection hypersubstitution σ' which is different from σ, i.e. there is a $j \in I$ with $\sigma(f_j) = x_{k(j)} \neq x_{l(j)} = \sigma'(f_j)$. Since V is solid, we have $\sigma' \in P(V)$. Clearly, $f_j(x_1, \ldots, x_{n_j}) \approx x_{k(j)} \in Id\sigma(V) = IdV$. So $\hat{\sigma'}[f_j(x_1, \ldots, x_{n_j})] = x_{l(j)} \approx x_{k(j)} = \hat{\sigma'}[x_{k(j)}] \in IdV$ and V is trivial, a contradiction. ∎

Proposition 3.3. *A non-trivial variety V of type $\tau = (n_i)_{i \in I}$ is solid and $id_p(V) = 1$ iff V is of type $\tau = (1, 1, \ldots)$ and $V = Mod\{f_i(x) \approx x \mid i \in I\}$.*

Proof. Let V be a non-trivial solid variety with $id_p(V) = 1$. Since V is solid, we have $RA_\tau \subseteq V$ where RA_τ is the variety of rectangular algebras of type τ (see [4]). Since σ_{x_1} defined by $\sigma_{x_1}(f_i) = x_1$ for all $i \in I$ and $\sigma_{x_{n_i}}$ defined by $\sigma_{x_{n_i}}(f_i) = x_{n_i}$ for all $i \in I$ are elements of $P(V)$ and since $id_p(V) = 1$ the identities $f_i(x_1, \ldots, x_{n_i}) \approx x_1$ and $f_i(x_1, \ldots, x_{n_i}) \approx x_{n_i}$ are satisfied in V and there follows $x_1 \approx x_{n_i} \in IdV$. Since V is non-trivial, we get $x_{n_i} = x_1$ for all $i \in I$. Since $f_i(x) \approx x \in Id\sigma_x(V)$ for all $i \in I$ where σ_x is the hypersubstitution mapping each operation symbol f_i to x and since $\upsilon_\sigma(V) = V$ we get $V = Mod\{f_i(x) \approx x \mid i \in I\}$. The other direction follows from Proposition 2.6 in [2]. ∎

Our aim is to show that for some solid varieties the degree $id_P(V)$ (and a generalization which will introduced later on) has a non-trivial lower bound which depends on the type of the variety. The way to show this fact is proving that we have enough proper hypersubstitutions which are pairwise non-related to each other. We can find such hypersubstitutions under the projection hypersubstitutions and sometimes under bijection hypersubstitutions. Later on we need the following lemma about bijection hypersubstitutions.

Lemma 3.4. *Let V be a variety of type τ and let σ be a hypersubstitution of this type whose extension $\hat{\sigma}$ is bijective. Then $\sigma \in P(V)$ iff $\sigma \approx_V \sigma_{id}$.*

Proof. We remark that hypersubstitutions σ such that $\hat{\sigma}$ are bijective were characterized in [10], Theorem 6.2.7. If $\sigma \approx_V \sigma_{id}$, then $\upsilon_\sigma(V) = V$ for the derived variety and then $\sigma(V) \subseteq V$, i.e. σ is V-proper. If conversely $\sigma \in P(V)$, then the cyclic group $< \hat{\sigma} >$ is a subgroup of the semigroup $(\widehat{P(V)}; \circ)$

with $\widehat{P(V)} := \{\hat{\sigma'} \mid \sigma' \in P(V)\}$. Therefore the inverse $\hat{\sigma}^{-1}$ of the extension of the bijective hypersubstitution $\hat{\sigma}$ belongs to $\widehat{P(V)}$. If $s \approx t \in Id\sigma(V)$, then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$ and then also $(\hat{\sigma}^{-1})[\hat{\sigma}[s]] \approx (\hat{\sigma}^{-1})[\hat{\sigma}[t]] \in IdV$, i.e. $s \approx t \in IdV$ and then $Id\sigma(V) \subseteq IdV$ which implies $V \subseteq \upsilon_\sigma(V)$. The converse inclusion is clear since $\sigma \in P(V)$. Altogether we have $\upsilon_\sigma(V) = V$ and $\sigma \approx_V \sigma_{id}$. ∎

Let H_n be the full transformation monoid of all transformations on $\{1, \ldots, n\}$. Green's equivalence $\mathcal{L}$ is defined on H_n by

$$f\mathcal{L}g :\Leftrightarrow \exists h, l \in H_n \ (f = h \circ g \text{ and } g = l \circ f).$$

It is well-know that for two transformations f, g we have $f\mathcal{L}g$ iff $Imf = Img$. We define $n^* = |H_n/\mathcal{L}| - n$.

For $s \in H_n$ we define the hypersubstitution σ_s^j mapping f_j to $f_j(x_{s(1)}, \cdots, x_{s(n)}), s \in H_n$ and f_i to $f_i(x_1, \cdots, x_{n_i})$ for any $i \neq j, i \in I$.

Lemma 3.5. *Let V be a non-trivial solid variety of type $\tau = (n_i)_{i\in I}$ with $n_i > 0$ for all $i \in I$ such that $n := max\{n_i \mid i \in I\}$ exists and let $n = n_j$. Then for all s_1, $s_2 \in H_n$ we have*

$$s_1 \not\mathcal{L} s_2 \Longrightarrow \sigma_{s_1}^j \not\approx_V \sigma_{s_2}^j.$$

Proof. Suppose that there are mappings s_1, $s_2 \in H_n$ with $s_1 \not\mathcal{L} s_2$, but $\sigma_{s_1}^j \approx_V \sigma_{s_2}^j$. From $s_1 \not\mathcal{L} s_2$ there follows $Ims_1 \neq Ims_2$, i.e. there is an element $k \in Ims_2$ and $k \notin Ims_1$ or conversely. Without loss of generality we assume that $k \in Ims_2$ and $k \notin Ims_1$. Then $s_1(i) \neq k$ for all $i \in \{1, \ldots, n\}$. Let $j := maxs_2^{-1}(k)$. We define a mapping s on $\{1, \ldots, n\}$ by

$$s(i) := \begin{cases} s_1(j) & \text{if} \quad i = k \\ i & \text{otherwise.} \end{cases}$$

Clearly, s is not the identity mapping since $s(k) = s_1(j)$, but $s_1(j) \neq k$. Now we show that $s \circ s_1 = s_1$ and $s \circ s_2 \neq s_2$. From $k \notin Ims_1$ we get $(s \circ s_1)(i) = s_1(i)$ for every $i \in \{1, \ldots, n\}$ and thus $s \circ s_1 = s_1$. Further, $(s \circ s_2)(j) = s(k) = s_1(j) \neq k = s_2(j)$ and then $s \circ s_2 \neq s_2$. Now we prove that $f_j(x_{s(1)}, \ldots, x_{s(n)}) \approx f_j(x_1, \ldots, x_n) \in Id\sigma_{s_1}^j(V)$. Since $\hat{\sigma}_{s_1}^j[f_j(x_{s(1)}, \ldots, x_{s(n)})] = f_j(x_{(s\circ s_1)(1)}, \ldots, x_{(s\circ s_1)(n)})= f_j(x_{s_1(1)}, \ldots, x_{s_1(n)})$ and $\hat{\sigma}_{s_1}^j[f_j(x_1, \ldots, x_n)]= f_j(x_{s_1(1)}, \ldots, x_{s_1(n)})$ we have

$\hat{\sigma}^j_{s_1}[f_j(x_{s(1)},\ldots,x_{s(n)})] \approx \hat{\sigma}^j_{s_1}[f_j(x_1,\ldots,x_n)] \in IdV$. This implies $f_j(x_{s(1)},\ldots,x_{s(n)}) \approx f_j(x_1,\ldots,x_n) \in Id\sigma^j_{s_1}(V) = Id\sigma^j_{s_2}(V)$ and thus

$$\hat{\sigma}^j_{s_2}[f_j(x_{s(1)},\ldots,x_{s(n)})] \approx \hat{\sigma}^j_{s_2}[f_j(x_1,\ldots,x_n)] \in IdV.$$

The last identity implies

$$f_j(x_{(s\circ s_2)(1)},\ldots,x_{(s\circ s_2)(n)}) \approx f_j(x_{s_2(1)},\ldots,x_{s_2(n)}) \in IdV$$

with $s \circ s_2 \neq s_2$. By the claim in the proof of Lemma 3.3 in [6] we have

$$f_j(x_{(s\circ s_2)(1)},\ldots,x_{(s\circ s_2)(n)}) \approx f_j(x_{s_2(1)},\ldots,x_{s_2(n)}) \notin IdV,$$

a contradiction and therefore Lemma 3.5 is proved. ■

Now we prove that no projection hypersubstitution can collapse with respect to $\approx_V$ with one of the σ^j_s's where s is non-constant.

Lemma 3.6. *Let V be a non-trivial solid variety of type $\tau = (n_i)_{i\in I}$ with $n_i > 0$ for all $i \in I$ such that $n := max\{n_i \mid i \in I\}$ exists and $n = n_j$. Then for all $s \in H_n$ such that $|Ims| > 1$, for any hypersubstitution of the form σ^j_s and for any projection hypersubstitution σ we have $\sigma \not\approx_V \sigma^j_s$.*

Proof. Assume that $\sigma \approx_V \sigma^j_s$. Because of $Id\sigma(V) = Id\sigma^j_s(V)$ from $f_j(x_1,\ldots,x_n) \approx x_{j_l} \in Id\sigma(V)$ where $\sigma(f_j) = x_{j_l}$, $1 \leq j_l \leq n$, there follows $f_j(x_1,\ldots,x_n) \approx x_{j_l} \in Id\sigma^j_s(V)$ and then $\hat{\sigma}^j_s[f_j(x_1,\ldots,x_n)] = f_j(x_{s(1)},\ldots,x_{s(n)}) \approx x_{j_l} = \hat{\sigma}^j_s[x_{j_l}] \in IdV$. Since $|Ims| > 1$, there is a $k \in \{1,\ldots,n\}$ with $s(k) \neq j_l$. Let σ' be a projection hypersubstitution with $\sigma'(f_j) = x_{s(k)}$. Then $\hat{\sigma'}[f_j(x_{s(1)},\ldots,x_{s(n)})] = x_{s(k)} \approx x_{j_l} = \hat{\sigma'}[x_{j_l}] \in IdV$ implies that V is trivial, a contradiction. ■

Theorem 3.7. *Let V be a non-trivial solid variety of type $\tau = (n_i)_{i\in I}$ with $n_i > 0$ for all $i \in I$ such that $n := max\{n_i \mid i \in I\}$ exists. Then $id_p(V) \geq \prod_{i\in I} n_i + n^*$.*

Proof. We consider the cases $n > 1$ and $n = 1$.

For $n = 1$ the inequality is clearly valid. Assume that $n > 1$. There is an element $j \in I$ with $n_j = n$ and there are exactly $\prod_{i \in I} n_i$ different projection hypersubstitutions of type τ. Since V is non-trivial and $n > 1$, by Proposition 3.1 for any pair σ, σ' of different projection hypersubstitutions we have $\sigma \not\approx_V \sigma'$. Since V is solid, any projection hypersubstitution is V-proper and therefore $id_p(V) \geq \prod_{i \in I} n_i$. Now for any $s \in H_n$ we consider the hypersubstitution σ_s^j mapping the n-ary operation symbol f_j to $f_j(x_{s(1)}, \ldots, x_{s(n)})$ and f_i for $i \neq j$ to $f_i(x_1, \ldots, x_{n_i})$. By Lemma 3.5 we get that $P(V)/\approx_V$ contains $n^* + n$ pairwise different blocks. Two hypersubstitutions σ_s^j, $\sigma_{s'}^j$ with different images of s and s' generate different blocks. By Lemma 3.6 no projection hypersubstitution can collapse with respect to $\approx_V$ with one of the σ_s^j's where s is non-constant. Since $H_n/\mathcal{L}$ contains only n blocks generated by constant mappings, we get $id_p(V) \geq \prod_{i \in I} n_i + n^*$. ■

Now we are interested in properties of solid varieties which satisfy the equality $id_p(V) = \prod_{i \in I} n_i + n^*$.

Proposition 3.8. *Let V be a non-trivial solid variety of type $\tau = (n_i)_{i \in I}$ with $n_i > 0$ for all $i \in I$ such that $n := max\{n_i \mid i \in I\}$ exists. Assume that $n = n_j$. If $id_p(V) = \prod_{i \in I} n_i + n^*$, then $n_i = 1$, $f_i(x) \approx x \in IdV$ for all $i \neq j$, $i \in I$ and for all n-ary terms t one of the following conditions is satisfied:*

(i) *there exists an integer $l \in \{1, \ldots, n\}$ such that $t(x_1, \ldots, x_n) \approx x_l \in IdV$,*

(ii) *there exists a mapping $s \in H_n$ which is not bijective and $t(x_{s(1)}, \ldots, x_{s(n)}) \approx t \in IdV$,*

(iii) *$IdV = Id\sigma(V)$ for a hypersubstitution σ with $\sigma(f_j) = t$.*

Proof. We prove at first that $n_i = 1$ for all $i \in I$ with $i \neq j$. Suppose that there is an element k with $k \in I$ and $k \neq j$ such that $n_k > 1$. The idea of the proof is to show that in this case $id_p(V) > \prod_{i \in I} n_i + n^*$ which contradicts the assumption of the proposition. Therefore we have to find enough hypersubstitutions which are not related to each other with respect to $\approx_V$. Let σ_s^j be the hypersubstitution mapping the operation symbol f_j

to $f_j(x_{s(1)},\dots,x_{s(n)})$ for a mapping $s \in H_n$ which is not bijective and f_i to $f_i(x_1,\dots,x_{n_i})$ for all $i \in I$, $i \neq j$, and let σ' be the hypersubstitution which maps $\sigma'(f_j) = f_j(x_1,\dots,x_n)$ and $\sigma'(f_j) = x_{n_i}$ for all $i \in I \setminus \{j\}$. Further we need the fact that for every mapping s which is not bijective there is a non-identical mapping s' such that $s' \circ s = s$.

Fact 1. For all $s \in H_n$ which are not bijective we have $\sigma' \not\approx_V \sigma^j_s$.

Proof of the Fact. Suppose that there is a mapping $s \in H_n$ which is not a permutation such that $\sigma' \approx_V \sigma^j_s$. Let s' be a non-identical mapping from H_n with $s' \circ s = s$. Then we have that $f_j(x_{s'(1)},\dots,x_{s'(n)}) \approx f_j(x_1,\dots,x_n) \in Id\sigma^j_s(V)$. Then from $f_j(x_{s'(1)},\dots,x_{s'(n)}) \approx f_j(x_1,\dots,x_n) \in Id\sigma'(V)$ there follows $\hat{\sigma}'[f_j(x_{s'(1)},\dots,x_{s'(n)})] = f_j(x_{s'(1)},\dots,x_{s'(n)}) \approx f_j(x_1,\dots,x_n) = \hat{\sigma}'[f_j(x_1,\dots,x_n)] \in IdV$. Since s' is not the identity mapping, there is an element $m \in \{1,\dots,n\}$ such that $s'(m) \neq m$ (i.e. $x_{s'(m)} \neq x_m$). Let σ'' be a projection hypersubstitution with $\sigma''(f_j) = x_m$. Since V is solid, so σ'' is proper and $\hat{\sigma}''[f_j(x_{s'(1)},\dots,x_{s'(n)})] = x_{s'(m)} \approx x_m = \hat{\sigma}''[f_j(x_1,\dots,x_n)] \in IdV$, a contradiction since V is non-trivial.

Fact 2. $\sigma' \not\approx_V \sigma_{id}$. By definition of σ' we have $f_k(x_1,\dots,x_{n_k}) \approx x_{n_k} \in Id\sigma'(V)$. Since $n_k > 1$ and since V is solid we get $f_k(x_1,\dots,x_k) \approx x_{n_k} \notin IdV$. This implies $Id\sigma'(V) \neq IdV$, i.e. $\sigma' \not\approx_V \sigma_{id}$.

Fact 3. For each projection hypersubstitution σ we have $\sigma' \not\approx_V \sigma$.

If σ is a projection hypersubstitution, then $f_j(x_1,\dots,x_n) \approx x_m \in Id\sigma(V)$ where $\sigma(f_j) = x_m$ and $m \in \{1,\dots,n\}$. If $\sigma' \approx_V \sigma$, then $f_j(x_1,\dots,x_n) \approx x_m \in Id\sigma'(V)$, so $\hat{\sigma}'[f_j(x_1,\dots,x_n)] = f_j(x_1,\dots,x_n) \approx x_m = \hat{\sigma}'[x_m] \in IdV$, i.e $f_j(x_1,\dots,x_n) \approx x_m \in IdV$, a contradiction. Therefore $\sigma' \not\approx_V \sigma$.

Altogether, this means that $[\sigma']_{\approx_V} \notin \{[\sigma]_{\approx_V} \mid \sigma$ is a projection hypersubstitution$\} \cup \{[\sigma^j_s]_{\approx_V} \mid s \in H_n$ and $|Ims| > 1\}$ and then $id_p(V) > \prod_{i\in I} n_i + n^*$ since by Proposition 3.1, Lemma 3.5 and Lemma 3.6 the considered blocks are pairwise different. This is a contradiction and therefore $n_i = 1$ for all $i \in I$ with $i \neq j$, i.e. $\tau = (1,\dots,1,n,1,\dots,1,\dots)$. If $n = 1$, then by Proposition 3.3 $V = Mod\{f_i(x) \approx x \mid i \in I\}$ and from these identities one obtains $t(x) \approx x$ for any $t \in W_\tau(\{x_1\})$.

We assume that $n > 1$ and want to show that V satisfies $f_i(x) \approx x$ for every $i \neq j$, $i \in I$. Let σ'' be the hypersubstitution defined by

$\sigma''(f_j) = f_j(x_1, \ldots, x_n)$ and $\sigma''(f_i) = x_1$ for all $i \in I \setminus \{j\}$. Clearly $\sigma'' \not\approx_V \sigma$ for any projection hypersubstitution σ since $f_j(x_1, \ldots, x_n) \approx x_m \notin Id\sigma''(V)$ for all $1 \leq m \leq n$. Using the same arguments as in the first part of the proof we have $\sigma'' \not\approx_V \sigma_s^j$ for all $s \in H_n$ and $Im\ s \neq \{1, \ldots, n\}$. Since by $id_p(V) = \prod_{i \in I} n_i + n^*$ and by $P(V)/{\approx_V} \supseteq \{[\sigma]_{\approx_V} \mid \sigma$ is a projection hypersubstitution$\} \cup \{[\sigma_s^j]_{\approx_V} \mid s \in H_n$ and $|Ims| > 1\}$ we get $P(V)/{\approx_V} = \{[\sigma]_{\approx_V} \mid \sigma$ is a projection hypersubstitution$\} \cup \{[\sigma_s^j]_{\approx_V} \mid s \in H_n$ and $|Ims| > 1\}$ there follows $\sigma'' \approx_V \sigma_{s'}^j$ where s' is a permutation. Then $\sigma_{s'}^j \approx_V \sigma_{id}$. By transitivity we have $\sigma'' \approx_V \sigma_{id}$. Since $\hat{\sigma}''[f_i(x)] = x \approx x = \hat{\sigma}''[x] \in IdV$ implies $f_i(x) \approx x \in Id\sigma''(V)$ we have $f_i(x) \approx x \in IdV$. Let $t \in W_\tau(X_n)$ be an arbitrary n-ary term of type τ. We have to verify that (i), (ii) or (iii) is satisfied. We define the hypersubstitution σ_t by $\sigma_t(f_j) = t$ and $\sigma_t(f_i) = f_i(x)$ for all $i \in I \setminus \{j\}$. From $Hyp(\tau)/{\approx_V} = P(V)/{\approx_V} = \{[\sigma]_{\approx_V} \mid \sigma$ is a projection hypersubstitution$\} \cup \{[\sigma_s^j]_{\approx_V} \mid s \in H_n$ and $|Ims| > 1\}$ there follows that there is a projection hypersubstitution σ such that $\sigma_t \approx_V \sigma$ or there is a mapping $s' \in H_n$ which is not bijective and $|Ims'| > 1$ such that $\sigma_t \approx_V \sigma_{s'}^j$ or $\sigma_t \approx_V \sigma_{id}$. In the first case we have $f_j(x_1, \ldots, x_n) \approx x_{j_l} \in Id\sigma(V) = Id\sigma_t(V)$, so $\hat{\sigma}_t[f_j(x_1, \ldots, x_n)] \approx x_{j_l} = \hat{\sigma}_t[x_{j_l}] \in IdV$, i.e. $t(x_1, \ldots, x_n) \approx x_{j_l} \in IdV$. In the second case there is a non-bijective $s \in H_n$ with $f_j(x_{s(1)}, \ldots, x_{s(n)}) \approx f_j(x_1, \ldots, x_n) \in Id\sigma_{s'}^j(V)$. Then $\hat{\sigma}_t[f_j(x_{s(1)}, \ldots, x_{s(n)})] \approx \hat{\sigma}_t[f_j(x_1, \ldots, x_n)] \in IdV$ implies $t(x_{s(1)}, \ldots, x_{s(n)}) \approx t \in IdV$. In the last case we get $Id\sigma_t(V) = IdV$. Clearly, $\sigma_{id} \approx_V \sigma$ where σ is a hypersubstitution with $\sigma(f_j) = t$. Then $Id\sigma(V) = IdV$. ∎

4. THE ISOMORPHISM DEGREE OF PROPER HYPERSUBSTITUTIONS

Because of $id_p(V) \leq isd_p(V)$ Theorem 3.7 is also satisfied for $isd_p(V)$. The generalization of Proposition 3.1 to $isd_p(V)$ is contained in [7].

Proposition 4.1 [7]**.** *Let V be a non-trivial variety of type $\tau = (n_i)_{i \in I}$ with $n_i > 0$ for all $i \in I$ such that at least one operation symbol of arity > 1 and let σ, σ' be different projection hypersubstitutions. Then $\sigma \not\sim_{V-iso} \sigma'$.*

Under the same assumptions for any projection hypersubstitution σ we have $\sigma \not\sim_{V-iso} \sigma_{id}$ ([7]).

Now we consider hypersubstitutions of the form σ_s^j for $s \in H_n$.

Proposition 4.2. *Let V be a non-trivial solid variety of type $\tau = (n_i)_{i\in I}$ with $n_i > 0$ for all $i \in I$ such that $n := max\{n_i \mid i \in I\}$ exists. If s, $s' \in H_n$ with $s \neq s'$, then ${\sigma_s}^j \not\sim_{V-iso} {\sigma'_s}^j$ where $j \in I$ with $n_j = n$.*

Proof. Assume that $s \neq s'$. From $s \neq s'$ there follows that there is a $k \in \{1, \ldots, n\}$ with $s(k) \neq s'(k)$. For $i \in \{1, \ldots, n\} \setminus \{j\}$ let ${\sigma_s}^j(f_i) = f_i(x_1, \ldots, x_{n_i})$. Let $k \in \{1, \ldots, n\}$ and let $\mathcal{A}_k$ be a projection algebra of type τ with ${f_j}^{\mathcal{A}_k} = {e_k}^{n,A}$. Then $\mathcal{A}_k \in V$ since V is solid and $\mathcal{A}_k \not\cong \mathcal{A}_l$ for all $l \in \{1, \ldots, n\}$, $k \neq l$. Now we consider the derived algebras ${\sigma_s}^j(\mathcal{A}_k)$ and ${\sigma_{s'}}^j(\mathcal{A}_k)$ with fundamental operations ${\sigma_s}^j(f_i)^{\mathcal{A}_k}$ and ${\sigma_{s'}}^j(f_i)^{\mathcal{A}_k}$ for all $i \in I$, respectively. We have ${\sigma_s}^j(f_i) = f_i(x_1, \ldots, x_{n_i}) = {\sigma_{s'}}^j(f_i)$ for all $i \in I \setminus \{j\}$ and ${\sigma_s}^j(f_i)^{\mathcal{A}_k}$, ${\sigma_{s'}}^j(f_i)^{\mathcal{A}_k}$ are projections. Since ${f_j}^{\mathcal{A}_k} = {e_k}^{n,A}$ by definitions of ${\sigma_s}^j$ and ${\sigma_{s'}}^j$ we have ${\sigma_s}^j(f_j)^{\mathcal{A}_k} = (f_j(x_{s(1)}, \ldots, x_{s(n)}))^{\mathcal{A}_k} = {e_{s(k)}}^{n,A}$ and ${\sigma_{s'}}^j(f_j)^{\mathcal{A}_k} = {e_{s'(k)}}^{n,A}$. Since ${\sigma_s}^j(\mathcal{A}_k)$ and ${\sigma_{s'}}^j(\mathcal{A}_k)$ are different projection algebras over the same universes, we have ${\sigma_s}^j(\mathcal{A}_k) \not\cong {\sigma_{s'}}^j(\mathcal{A}_k)$ and then ${\sigma_s}^j \not\sim_{V-iso} {\sigma_{s'}}^j$. This proves the proposition. ∎

Because of $n^n - n \geq n^*$ we can sharpen Theorem 3.7 in the case of $isd_p(V)$ and obtain:

Theorem 4.3. *Let V be a non-trivial solid variety of type $\tau = (n_i)_{i\in I}$ with $n_i > 0$ for all $i \in I$ such that $n := max\{n_i \mid i \in I\}$ exists. Then $isd_p(V) \geq \prod_{i\in I} n_i + n^n - n$.*

Proof. For $n = 1$ the inequalitiy is clearly valid. Assume that $n > 1$. Then there is an element $j \in I$ such that $n_j = n$ and there are exactly $\prod_{i\in I} n_i$ different projection hypersubstitutions of type τ. By Proposition 4.1 we have $\sigma \not\sim_{V-iso} \sigma'$ if $\sigma \neq \sigma'$ are different projection hypersubstitutions and therefore $P(V)/\sim_{V-iso}$ contains at least $\prod_{i\in I} n_i$ pairwise different blocks. By Proposition 4.2, $P(V)/\sim_{V-iso}$ contains n^n pairwise different blocks generated by hypersubstitutions of the form ${\sigma_s}^j$. Now we verify that no projection hypersubstitution collapses with a hypersubstitution of the form ${\sigma_s}^j$ where s is non-constant. Suppose that there are a projection hypersubstitution σ and a non-constant mapping $s \in H_n$ such that $\sigma \sim_{V-iso} {\sigma_s}^j$. From the definitions of σ and ${\sigma_s}^j$ we have $\sigma(f_j) = x_{j_l}$, $j_l \in \{1, \ldots, n\}$ and ${\sigma_s}^j(f_j) = f_j(x_{s(1)}, \ldots, x_{s(n)})$. Since s is not constant, there is an integer $k \in \{1, \ldots, n\}$ with $x_{s(k)} \neq x_{j_l}$. This implies $f_j(x_{s(1)}, \ldots, x_{s(n)}) \approx x_{j_l} \notin IdV$ since V is non-trivial and solid. Therefore, there is an algebra $\mathcal{A} \in V$ with

$\mathcal{A} \not\models f_j(x_{s(1)}, \ldots, x_{s(n)}) \approx x_{j_l}$. From $\sigma \sim_{V-iso} \sigma_s{}^j$ we obtain an isomorphism h from $\sigma(\mathcal{A})$ onto $\sigma_s{}^j(\mathcal{A})$ and then $h(a_{j_l}) = h(\sigma(f_j)^{\mathcal{A}}(a_1, \ldots, a_n)) = \sigma_s{}^j(f_j)^{\mathcal{A}}(h(a_1), \ldots, h(a_n)) = (f_j(x_{s(1)}, \ldots, x_{s(n)}))^{\mathcal{A}}(h(a_1), \ldots, h(a_n))$ for all $a_1, \ldots, a_n \in A$. It follows $(f_j(x_{s(1)}, \ldots, x_{s(n)}))^{\mathcal{A}}(a_1, \ldots, a_n) = a_{j_l} = e_{j_l}{}^{n,A}(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in A$, i.e. $(f_j(x_{s(1)}, \ldots, x_{s(n)}))^{\mathcal{A}} = e_{j_l}{}^{n,A}$. This means that $\mathcal{A} \models f_j(x_{s(1)}, \ldots, x_{s(n)}) \approx x_{j_l}$, a contradiction.

Since there are exactly n hypersubstitutions mapping f_j to a term of the form $f_j(x_c, \ldots, x_c)$ and f_i to $f_i(x_1, \ldots, x_{n_i})$, $i \neq j$ where $c \in \{1, \ldots, n\}$ we get $isd_p(V) \geq \prod_{i \in I} n_i + n^n - n$. ■

5. Varieties of bands

We are particularly interested in the following varieties of bands:

$TR = Mod\{x_1 \approx x_2\}$,

$LZ = Mod\{x_1x_2 \approx x_1\}$,

$RZ = Mod\{x_1x_2 \approx x_2\}$,

$SL = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1{}^2 \approx x_1, x_1x_2 \approx x_2x_1\}$,

$RB = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3 \approx x_1x_3, x_1{}^2 \approx x_1\}$,

$NB = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1{}^2 \approx x_1, x_1x_2x_3x_4 \approx x_1x_3x_2x_4\}$,

$RegB = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1{}^2 \approx x_1, x_1x_2x_1x_3x_1 \approx x_1x_2x_3x_1\}$,

$LN = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1{}^2 \approx x_1, x_1x_2x_3 \approx x_1x_3x_2\}$,

$RN = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1{}^2 \approx x_1, x_1x_2x_3 \approx x_2x_1x_3\}$,

$LReg = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1{}^2 \approx x_1, x_1x_2 \approx x_1x_2x_1\}$,

$RReg = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1{}^2 \approx x_1, x_1x_2 \approx x_2x_1x_2\}$,

$LQN = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1{}^2 \approx x_1, x_1x_2x_3 \approx x_1x_2x_1x_3\}$,

$RQN = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1{}^2 \approx x_1, x_1x_2x_3 \approx x_1x_3x_2x_3\}$.

In [8] the author determined the dimension of every subvariety of the variety $RegB$. This means that $id_p(V)$ for these varieties is known. Now we determine $id_p(V)$ for every variety of bands. Since our proofs for subvarieties of $RegB$ are quite different from the proofs in [8] we will give here the full proof. In [6] Proposition 4.1 was proved that for each variety of bands $\sim_V = \sim_{V-iso}$. Therefore $d_p(V) = isd_p(V)$ for each variety of bands. Moreover it was proved that

$$
\begin{aligned}
d_p(V) = 1 \quad &\text{iff} \quad V \in \{TR, LZ, RZ, SL\},\\
d_p(V) = 2 \quad &\text{iff} \quad V \in \{LN, RN, LReg, RReg\},\\
d_p(V) = 3 \quad &\text{iff} \quad V \text{ is not dual solid and } V \notin \{LZ, RZ, LN, RN,\\
&\qquad\quad LReg, RReg, LQN, RQN\},\\
d_p(V) = 4 \quad &\text{iff} \quad V \text{ is dual solid and } V \notin \{TR, SL, NB, RegB\}\\
&\qquad\quad \text{or } V \in \{LQN, RQN\},\\
d_p(V) = 6 \quad &\text{iff} \quad V \in \{NB, RegB\}.
\end{aligned}
$$

We note that a variety of type $\tau = (2)$ is called *dual* solid if $\hat{\sigma}_{x_2x_1}[s] \approx \hat{\sigma}_{x_2x_1}[t] \in IdV$ for every identity $s \approx t$ satisfied in V. (σ_t denotes the hypersubstitution mapping the binary operation symbol f to the binary term t.) Now we are interested in $id_p(V)$ for every variety of bands. Since $1 \leq id_p(V) \leq d_p(V)$ for $V \in \{TR, LZ, RZ, SL\}$ we get $id_p(V) = 1$.

Now we have:

Theorem 5.1. *Let V be a variety of bands. Then*

(i) $id_p(V) = 1$ *iff* $V \in \{TR, LZ, RZ, SL\}$,

(ii) $id_p(V) = 2$ *iff* $V \in \{LN, RN, LReg, RReg\}$,

(iii) $id_p(V) = 3$ *iff* V *is not dual solid and* $V \notin \{LZ, RZ, LN, RN, LReg, RReg, LQN, RQN\}$, *or* V *is dual solid and* $V \notin \{TR, SL, NB, RegB\}$

(iv) $id_p(V) = 4$ *iff* $V \in \{LQN, RQN\}$,

(v) $id_p(V) = 5$ *iff* $V \in \{NB, RegB\}$.

Proof. If $V \in \{TR, LZ, RZ, SL\}$, then $id_p(V) = 1$.

For $V \in \{LN, RN, LReg, RReg\}$ the quotient set $P(V)/\!\sim_V \mid P(V)$ consists of precisely two classes. In each case it is easy to see that the derived varieties are different. As an example we consider $P(LN)/\!\sim_{LN} \mid P(LN) = \{[\sigma_{x_1}]_{\sim_{LN} \mid P(LN)}, [\sigma_{x_1x_2}]_{\sim_{LN} \mid P(LN)}\}$ (see proof of Theorem 4.2 in [6]). Clearly $\sigma_{x_1} \not\approx_{LN} \mid P(LN)\sigma_{x_1x_2}$. This proves (ii).

For the variety NB, by the proof of Theorem 4.2 in [6] we have $P(NB)/\!\sim_{NB} \mid P(NB) = Hyp(2)/\!\sim_{NB} = \{[\sigma_{x_1}]_{\sim_{NB}}, [\sigma_{x_2}]_{\sim_{NB}}, [\sigma_{x_1x_2}]_{\sim_{NB}}, [\sigma_{x_2x_1}]_{\sim_{NB}}, [\sigma_{x_1x_2x_1}]_{\sim_{NB}}, [\sigma_{x_2x_1x_2}]_{\sim_{NB}}\}$ and $|P(NB)/\!\sim_{NB}| = 6$. From the results of the previous section, we get $\sigma_{x_1} \not\approx_{NB} \sigma_{x_2}$, $\sigma_{x_1} \not\approx_{NB} \sigma_{x_1x_2}$, $\sigma_{x_2} \not\approx_{NB} \sigma_{x_1x_2}$, $\sigma_{x_1x_2} \approx_{NB} \sigma_{x_2x_1}$ by Lemma 3.4. Now we show that $\sigma_{x_1x_2x_1} \not\approx_{NB} \sigma_{x_1}$, $\sigma_{x_1x_2x_1} \not\approx_{NB} \sigma_{x_2}$, $\sigma_{x_1x_2x_1} \not\approx_{NB} \sigma_{x_1x_2}$, $\sigma_{x_1x_2x_1} \not\approx_{NB} \sigma_{x_2x_1x_2}$. If $\sigma_{x_1x_2x_1} \approx_{NB} \sigma_{x_1}$, then $Id\sigma_{x_1x_2x_1}(NB) = Id\sigma_{x_1}(NB)$. Since $x_1x_2 \approx x_1 \in Id\sigma_{x_1}(NB)$, so $x_1x_2 \approx x_1 \in Id\sigma_{x_1x_2x_1}(NB)$ there follows $x_1x_2x_1 \approx x_1 \in IdNB$, a contradiction since $x_1x_2x_1 \approx x_1 \notin IdNB$. Therefore $\sigma_{x_1x_2x_1} \not\approx_{NB} \sigma_{x_1}$. Similarly we show that $\sigma_{x_1x_2x_1} \not\approx_{NB} \sigma_{x_2}$. If $\sigma_{x_1x_2x_1} \approx_{NB} \sigma_{x_1x_2}$, then $Id\sigma_{x_1x_2x_1}(NB) = IdNB$. Clearly $x_1x_2x_1 \approx x_1x_2 \in Id\sigma_{x_1x_2x_1}(NB)$ since $\hat{\sigma}_{x_1x_2x_1}[x_1x_2x_1] \approx x_1(x_2x_1x_2)x_1 \approx x_1x_2x_1x_2x_1 \approx x_1x_2x_1$ and $\hat{\sigma}_{x_1x_2x_1}[x_1x_2] \approx x_1x_2x_1$ there follows $x_1x_2x_1 \approx x_1x_2 \in IdNB$, a contradiction. Therefore $\sigma_{x_1x_2x_1} \not\approx_{NB} \sigma_{x_1x_2}$.

If $\sigma_{x_1x_2x_1} \approx_{NB} \sigma_{x_2x_1x_2}$, then $Id\sigma_{x_1x_2x_1}(NB) = Id\sigma_{x_2x_1x_2}(NB)$. Since $x_1x_2x_1 \approx x_1x_2 \in Id\sigma_{x_1x_2x_1}(NB)$, so $x_1x_2x_1 \approx x_1x_2 \in Id\sigma_{x_2x_1x_2}(NB)$ and there follows $\hat{\sigma}_{x_2x_1x_2}[x_1x_2x_1] = x_1(x_2x_1x_2)x_1 \approx x_1x_2x_1 \approx x_2x_1x_2 = \hat{\sigma}_{x_2x_1x_2}[x_1x_2] \in IdNB$, a contradiction since $x_1x_2x_1 \approx x_2x_1x_2 \notin IdNB$. Therefore $\sigma_{x_1x_2x_1} \not\approx_{NB} \sigma_{x_2x_1x_2}$. In a similar way we conclude for $\sigma_{x_2x_1x_2}$. Remark that $\sigma_{x_1x_2}(NB) = \sigma_{x_2x_1}(NB)$ by Lemma 3.4. This means that $P(NB)/\!\approx_{NB} = \{[\sigma_{x_1}]_{\approx_{NB}}, [\sigma_{x_2}]_{\approx_{NB}}, [\sigma_{x_1x_2}]_{\approx_{NB}}, [\sigma_{x_1x_2x_1}]_{\approx_{NB}}, [\sigma_{x_2x_1x_2}]_{\approx_{NB}}\}$, i.e. $id_p(NB) = 5$. In a similar way we prove that $id_p(RegB) = 5$. This shows (v).

For the variety LQN we have $P(LQN)/\!\sim_{LQN} \mid P(LQN) = \{[\sigma_{x_1}]_{\sim_{LQN} \mid P(LQN)}, [\sigma_{x_2}]_{\sim_{LQN} \mid P(LQN)}, [\sigma_{x_1x_2}]_{\sim_{LQN} \mid P(LQN)}, [\sigma_{x_1x_2x_1}]_{\sim_{LQN} \mid P(LQN)}\}$. By the same way as above we show that $\sigma \not\approx_{LQN} \mid P(LQN)\sigma'$ where

$\sigma, \sigma' \in \{\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_1x_2}, \sigma_{x_1x_2x_1}\}$, $\sigma \neq \sigma'$. This shows that $P(LQN) /\!\approx_{LQN} \mid P(LQN) = \{[\sigma_{x_1}]_{\approx_{LQN} \mid P(LQN)}, [\sigma_{x_2}]_{\approx_{LQN} \mid P(LQN)}, [\sigma_{x_1x_2}]_{\approx_{LQN} \mid P(LQN)}, [\sigma_{x_1x_2x_1}]_{\approx_{LQN} \mid P(LQN)}\}$, i.e. $id_p(LQN) = 4$. Similarly we can prove that $id_p(RQN) = 4$. This shows (iv).

Let V be a dual solid variety different from TR, SL, RB, NB and $RegB$. Then $P(V)/\!\sim_V \mid P(V) = \{[\sigma_{x_1}]_{\sim_V \mid P(V)}, [\sigma_{x_2}]_{\sim_V \mid P(V)}, [\sigma_{x_1x_2}]_{\sim_V \mid P(V)}, [\sigma_{x_2x_1}]_{\sim_V \mid P(V)}\}$ (see proof of Theorem 4.2 in [6]). By the same idea as above we can show that $\sigma \not\approx_V \mid P(V)\sigma'$ where $\sigma, \sigma' \in \{\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_1x_2}\}$, $\sigma \neq \sigma'$. So $id_p(V) = 3$.

Finally, let V is a non-dual solid variety different from $LZ, RZ, LN, RN, LReg, RReg, LQN, RQN$, then $P(V)/\!\sim_V \mid P(V) = \{[\sigma_{x_1}]_{\sim_V \mid P(V)}, [\sigma_{x_2}]_{\sim_V \mid P(V)}, [\sigma_{x_1x_2}]_{\sim_V \mid P(V)}\}$. Similarly as above, we get that each representative of different blocks of $P(V)/\!\sim_V \mid P(V)$ cannot be $\approx_V \mid P(V)$-related. Therefore $id_p(V) = 3$ and this shows (iii).

Since all possible cases are considered we get the second direction of (i). ■

References

[1] St. Burris and H.P. Sankappanavar, *A course in Universal Algebra*, Springer-Verlag, New York, Heidelberg, Berlin 1981.

[2] K. Denecke and J. Koppitz, *Fluid, unsolid, and completely unsolid varieties*, Algebra Colloquium 7:4 (2000), 381–390.

[3] K. Denecke and R. Marszałek, *Binary Relations on Monoids of Hypersubstitutions*, Algebra Colloquium 4:1 (1997), 49–64.

[4] K. Denecke and S.L. Wismath, *Hyperidentities and clones*, Gordon and Breach Science Publishers, 2000.

[5] K. Denecke and S.L. Wismath, *Universal Algebra and Applications in Theoretical Computer Science*, Boca Raton, London, Washington, D.C.: Chapman & Hall/CRC 2002.

[6] K. Denecke, J. Koppitz and R. Srithus, *The Degree of Proper Hypersubstitutions*, preprint 2005.

[7] K. Denecke, J. Koppitz and R. Srithus, *N-fluid Varieties*, preprint 2005.

[8] E. Graczyńska, *M-solid Quasivarieties*, preprint 2006.

[9] E. Graczyńska and D. Schweigert, *The Dimension of a Variety*, preprint 2006.

[10] J. Koppitz and K. Denecke, *M-solid Varieties of Algebras*, Springer 2006.

[11] J. Płonka, *Proper and inner hypersubstitutions of varieties*, p. 106–116 in: *"Proceedings of the International Conference Sommer School on General Algebra and Ordered Sets"*, Olomouc 1994.

Received 26 October 2006
Revised 6 November 2006