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ON UNIFORMLY STRONGLY PRIME Γ-SEMIRINGS (II)

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Abstract

The Uniformly strongly prime k-radical of a Γ-semiring is a special class which we study via its operator semiring.

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1. Introduction

In 1987, D.M. Olson [7] introduced the notion of uniformly strongly prime radical in rings. In 1988, G.L. Booth and N.J. Groenwald [1] extended this notion of uniformly strongly prime radical to Γ-ring. In 1977, G.A.P. Heyman and C. Roos [6] introduced the notion of essential extension of rings.

In this paper we introduce the notions of uniformly right strongly prime ideal and uniformly left strongly prime ideal of a Γ-semiring and show that these two notions are equivalent. Also we study uniformly strongly prime k-radical of a Γ-semiring via its operators semirings as G.L. Booth and N.J. Groenwald did in case of Γ-ring. Some earlier works on the operator semiring of a Γ-semiring may be found in [4]. We obtain a relation between uniformly strongly prime k-radical of a Γ-semiring and with that of its matrix Γ-semiring via operator semiring. Lastly, we introduce the notion of super t-system in a Γ-semiring and obtain the relation between the uniformly strongly prime k-radical and super t-system in a Γ-semiring.

2. Uniformly strongly prime Γ-semirings

Definition 2.1 ([4]). Let S and Γ be two additive commutative semigroups. Then S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ (image to be denoted by $a\alpha b$, for $a, b \in S$ and $\alpha \in \Gamma$) satisfying the following conditions:

(i) $a\alpha(b+c) = a\alpha b + a\alpha c$

(ii) $(a+b)\alpha c = a\alpha c + b\alpha c$

(iii) $a(\alpha+\beta)c = a\alpha c + a\beta c$

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$

for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Every semiring S is a Γ-semiring with $\Gamma = S$ where $a\alpha b$ denotes the product of elements $a, \alpha, b \in S$.

If S contains an element 0 such that $0+x = x = x+0$ and $0\alpha x = x\alpha 0 = 0$ for all $x \in S$, for all $\alpha \in \Gamma$, then 0 is called the zero of S.

Throughout this paper we assume that a Γ-semiring always contains a zero element.

Definition 2.2 ([4]). Let S be a Γ-semiring and L be the left operator semiring and R be the right operator semiring. If there exists an element

$$\sum_{i=1}^{m}[e_i, \delta_i] \in L \left(\text{respectively} \sum_{j=1}^{n}[\nu_j, f_j] \in R\right)$$

such that

$$\sum_{i=1}^{m} e_i\delta_i a = a \left(\text{respectively} \sum_{j=1}^{n} a\nu_j f_j = a\right) \text{ for all } \ a \in S$$

then S is said to have the left unity

$$\sum_{i=1}^{m}[e_i, \delta_i] \left(\text{respectively the right unity } \sum_{j=1}^{n}[\nu_j, f_j]\right).$$

Definition 2.3 ([4])**.** A nonempty subset I of a Γ-semiring S is called an ideal of S if $I+I \subseteq I$, $I\ \Gamma S \subseteq I$, $S\,\Gamma\, I \subseteq I$, where for subsets U, V of S and Δ of Γ,

$$U\Delta V = \left\{ \sum_{i=1}^{n} u_i\gamma_i v_i : u_i \in U, v_i \in V, \gamma_i \in \Delta \text{ and } n \text{ is a positive integer} \right\}$$

Definition 2.4. A Γ-semiring S is called uniformly right strongly prime if S and Γ contain finite subsets F and Δ respectively such that for any non zero $x(\neq 0) \in S$, $x\Delta F\Delta y = \{0\}$ implies that $y = 0$ for all $y \in S$. The pair $(F,\ \Delta)$ is called a uniform right insulator for S.

Analogously we can define uniformly left strongly prime Γ-semiring.

Theorem 2.5. *A Γ-semiring S is uniformly right strongly prime if and only if there exist finite subsets F of S and Δ of Γ such that for any two nonzero elements x and y of S there exist $f \in F$ and $\alpha, \beta \in \Delta$ such that $x\alpha f\beta y \neq 0$.*

Proof. Let S be a uniformly right strongly prime Γ-semiring and $(F,\ \Delta)$ be a uniform right insulator for S. Let x, y be any two nonzero elements of S. Suppose that $x\Delta F\Delta y = \{0\}$. Then $y = 0$, a contradiction. So there exist $f \in S$ and $\alpha, \beta \in \Delta$ such that $x\alpha f\beta y \neq 0$. ■

The converse follows by reversing the above argument.

Corollary 2.6. A Γ-semiring S is uniformly right strongly prime if and only if S is uniformly left strongly prime.

So we ignore the word right from uniformly right strongly prime.

Definition 2.7. A nonzero ideal I of a Γ-semiring S is called an essential ideal of S if for any nonzero ideal J of S, $I \bigcap J \neq (0)$.

Example 2.8. Let $S = \{r\omega : r \in \mathbf{Z}\}$ and $\Gamma = \{r\omega^2 : r \in \mathbf{Z}\}$, where ω be a cube root of unity and $\mathbf{Z}$ be the set of all integers. Then S is a Γ-semiring with usual addition and multiplication. Let $I = \{2r\omega : r \in \mathbf{Z}\}$. Then I is a nonzero ideal of S. Let J be any nonzero ideal of S. Then $I \bigcap J \neq (0)$. Hence I is an essential ideal of S.

Definition 2.9. A Γ-semiring T is said to be an essential extension of a Γ-semiring S if S is an essential ideal of T.

Definition 2.10. Let A be a nonempty subset of a Γ-semiring S. Right annihilator of A in S, denoted by $ann_R(A)$, is defined by $ann_R(A) = \{s \in S : A\Gamma s = \{0\}\}$.

Analogously we can define left annihilator $ann_L(A)$ of A in S. Annihilator of a nonempty subset A is denoted by $ann(A)$ which is a left as well as a right annihilator of A.

Remark 2.11. If S is a Γ-semiring then $ann_R(A)$ is a right ideal of S and $ann_L(A)$ is a left ideal of S. If A is an ideal of a Γ-semiring S then both annihilators are ideals of S.

Lemma 2.12. *Let S be a Γ-semiring and T be its essential extension. If S is a uniformly strongly prime Γ-semiring then for each nonzero x of T, $x\Delta F = \{0\}$ implies that $x \in ann_R(S)$ and $F\Delta x = \{0\}$ implies that $x \in ann_L(S)$, where $(F,\ \Delta)$ is a uniform insulator for S.*

Proof. Let $x\Delta F = \{0\}$. Then $s\alpha x\Delta F\Delta s\alpha x = \{0\}$ for all $s \in S$ and for all $\alpha \in \Gamma$. Since S is an ideal of T and $s \in S$, $s\alpha x \in S$. Again since S is uniformly strongly prime and $(F,\ \Delta)$ is a uniform insulator for S, $s\alpha x = 0$ for all $s \in S$ and for all $\alpha \in \Gamma$ i.e. $S\Gamma x = \{0\}$ i.e. $x \in ann_R(S)$ (By Definition 2.10). ■

Similarly we can prove that $F\Delta x = \{0\}$ implies that $x \in ann_L(S)$.

Lemma 2.13. *Let S be a uniformly strongly prime Γ-semiring and T be its essential extension. Then both annihilators of S in T are zero.*

Proof. Let $(F,\ \Delta)$ be a uniform insulator for S. If possible let $ann_R(S) \neq (0)$. Then $ann_R(S)$ is a nonzero ideal of T. Since S is an essential ideal of T, $ann_R(S) \bigcap S \neq (0)$. Let $x(\neq 0) \in ann_R(S) \bigcap S$. Then $S\Gamma x = \{0\}$. Now $x\Delta F\Delta x \subseteq x\Gamma S\Gamma x = \{0\}$. Since S is a uniformly strongly prime Γ-semiring, $x = 0$, a contradiction. Therefore $ann_R(S) = (0)$. ■

Similarly we can prove that $ann_L(S) = (0)$.

Theorem 2.14. *Any essential extension of a uniformly strongly prime Γ-semiring S is a uniformly strongly prime Γ-semiring.*

Proof. Let $(F,\ \Delta)$ be a uniform insulator for S and T be an essential extension of S. Let x be a nonzero element of T. Then $x\Delta F$ and $F\Delta x$ both are nonzero. Suppose $x\Delta F = \{0\}$ then by Lemma 2.12, $x \in ann_R(S)$. Also by Lemma 2.13, $ann_R(S) = (0)$, which implies that $x = 0$, a contradiction. Therefore $x\Delta F \neq \{0\}$. Similarly $F\Delta x \neq \{0\}$. Let y, z be two nonzero elements of T. Then there exist $f_1,\ f_2 \in F$ and $\alpha_1,\ \alpha_2 \in \Delta$ such that $y\alpha_1 f_1 \neq 0$ and $f_2\alpha_2 z \neq 0$. Since S is an ideal of T, so $y\alpha_1 f_1,\ f_2\alpha_2 z \in S$. Again since S is uniformly strongly prime and $(F,\ \Delta)$ is a uniform insulator for S, there exist $\alpha, \beta \in \Delta$ and $f \in F$ such that $y\alpha_1 f_1 \alpha f \beta f_2 \alpha_2 z \neq 0$. Let $F^{'} = \{f_1\alpha f\beta f_2 : y\alpha_1 f_1\alpha f\beta f_2\alpha_2 z \neq 0;\ f_1, f, f_2 \in F;\ \alpha, \beta, \alpha_1, \alpha_2 \in \Delta;\ y, z \in T\}$. Then $F^{'} \subseteq S \subseteq T$ is a finite subset, since F and Δ are finite subsets. Hence by Theorem 2.5, T is uniformly strongly prime Γ-semiring with insulator $(F',\ \Delta)$. ■

Definition 2.15. Let S be a Γ-semiring and T be a nonempty subset of S. Then T is said to be a Γ-subsemiring of S if for $t_1, t_2 \in T$ and $\alpha \in \Gamma$, $t_1 + t_2,\ t_1\alpha t_2 \in T$.

Remark 2.16. Every ideal of a Γ-semiring S is a Γ-subsemiring of S.

Lemma 2.17. *If S is a uniformly strongly prime Γ-semiring and I is an ideal of S, then I is also a uniformly strongly prime Γ-subsemiring.*

Proof. Let S be a uniformly strongly prime Γ-semiring and $(F,\ \Delta)$ be a uniform insulator for S. If $I = (0)$ then obviously I is a uniformly strongly prime Γ-subsemiring. Suppose $I \neq (0)$ and r be a fixed nonzero element of I. Let $F^{'} = \{f_1\alpha r\beta f_2 : f_1, f_2 \in F;\ \alpha, \beta \in \Delta\}$. Since I is an ideal of S and $F,\ \Delta$ are finite subsets, $F^{'}$ is a finite subset of I. Let $x(\neq 0) \in I$ and $y \in I$. Now $x\Delta F^{'}\Delta y = \{0\}$ implies that $x\gamma f_1\alpha r\beta f_2\delta y = 0$ for all $f_1, f_2 \in F$ and for all $\alpha, \beta, \gamma, \delta \in \Delta$ i.e. $x\Delta F\Delta(r\beta f_2\delta y) = \{0\}$ for all $f_2 \in F$ and for all $\beta, \delta \in \Delta$. Since $r\beta f_2\delta y \in S$ for all $f_2 \in F$ and for all $\beta, \delta \in \Delta$ and S is a usp Γ-semiring with $x \neq 0$, $r\beta f_2\delta y = 0$ for all $f_2 \in F$ and for all $\beta, \delta \in \Delta$ i.e. $r\Delta F\Delta y = \{0\}$. By previous argument $y = 0$ as $r \neq 0$. Hence $(F^{'},\ \Delta)$ is a uniform insulator for I. Thus I is a uniformly strongly prime Γ-subsemiring. ■

3. Uniformly strongly prime k-radicals

Definition 3.1 ([3])**.** An ideal P of a Γ-semiring S is called a uniformly strongly prime ideal (usp ideal) if S and Γ contain finite subsets F and Δ respectively such that $x\Delta F\Delta y \subseteq P$ implies that $x \in P$ or $y \in P$ for all $x, y \in S$.

Definition 3.2. Let S be a Γ-semiring. The uniformly strongly prime k-radical (usp k-radical) of a Γ-semiring S, denoted by $\tau(S)$ is defined by

$$\tau(S) = \bigcap_{P \in \Lambda_S} P,$$

where Λ_S denote the set of all usp k-ideals of the Γ-semiring S.

Definition 3.3. Let S be a Γ-semiring and L (respectively R) be its left (respectively right) operator semiring, the usp k-radical of L (respectively R), denoted by $\tau(L)$ (respectively $\tau(R)$) is defined by

$$\tau(L) = \bigcap_{A \in \Lambda_L} A,$$

where Λ_L denote the set of all usp k-ideals of the left operator semiring L (respectively $\tau(R) = \bigcap_{B \in \Lambda_R} B$, where Λ_R denote the set of all usp k-ideals of the right operator semiring R).

Theorem 3.4. *Let S be a Γ-semiring with left and right unities and L be its left operator semiring then $\tau(L)^+ = \tau(S)$ and $\tau(L) = \tau(S)^{+'}$.*

Proof. Let Λ_L and Λ_S denote the set of all usp k-ideals of L and S respectively, then

$$\tau(L) = \bigcap_{A \in \Lambda_L} A \quad \text{and} \quad \tau(S) = \bigcap_{P \in \Lambda_S} P\,.$$

Hence

$$(\tau(L))^+ = \left(\bigcap_{A \in \Lambda_L} A \right)^+ = \bigcap_{A \in \Lambda_L} A^+\,.$$

Since for every $A \in \Lambda_L,\ A^+ \in \Lambda_S$ (Cf. Theorem 2.23 of [3]),

$$\bigcap_{A \in \Lambda_S} A \subseteq \bigcap_{A \in \Lambda_L} A^+\,.$$

Hence

$$\tau(S) \subseteq (\tau(L))^+\,. \tag{1}$$

Again

$$\tau(S) = \bigcap_{P \in \Lambda_S} P = \bigcap_{P \in \Lambda_S} \left(P^{+'}\right)^{+} = \left(\bigcap_{P \in \Lambda_S} P^{+'}\right)^{+}.$$

Since for each $P \in \Lambda_S,\ P^{+'} \in \Lambda_L$(Cf. Theorem 2.23 of [3]),

$$\bigcap_{P \in \Lambda_L} P \subseteq \bigcap_{P \in \Lambda_S} P^{+'}.$$

Hence

$$\left(\bigcap_{P \in \Lambda_L} P\right)^{+} \subseteq \left(\bigcap_{P \in \Lambda_S} P^{+'}\right)^{+},$$

which implies that

$$(2) \qquad (\tau(L))^{+} \subseteq \tau(S).$$

Thus from (1) and (2) we get $\tau(S) = (\tau(L))^{+}$. ■

Similarly we can prove that $\tau(L) = \tau(S)^{+'}$.

Proposition 3.5. *Let S be a Γ-semiring with left and right unities and R be its right operator semiring then $\tau(R)^* = \tau(S)$ and $\tau(R) = \tau(S)^{*'}$.*

Corollary 3.6. *Let S be a Γ-semiring with left and right unities L and R be its left and right operator semirings then $\tau(L)^+ = \tau(S) = \tau(R)^*$.*

Theorem 3.7. *Let S be a semiring with identity 1 and $\tau^{'}(S),\ \tau(S)$ denote respectively the usp k-radical of the semiring S and the usp k-radical of the Γ-semiring S, where $\Gamma = S$, then $\tau^{'}(S) = \tau(S)$.*

Proof. By Theorem 2.26 of [3] every usp k-ideal of a semiring S is a usp k-ideal of the Γ-semiring S, where $\Gamma = S$. This follows that $\tau^{'}(S) = \tau(S)$.■

4. USP k-radicals in Matrix Γ-semiring

Definition 4.1 ([5])**.** Let S be a Γ-semiring and m, n be positive integers. We denote by S_{mn} and Γ_{nm} respectively the sets of $m \times n$ matrices with entries from S and $n \times m$ matrices with entries from Γ. Let $A, B \in S_{mn}$

and $\Delta \in \Gamma_{nm}$. Then $A\Delta B \in S_{mn}$ and $A + B \in S_{mn}$. Clearly, S_{mn} forms a Γ_{nm}-semiring with these operations. We call it the matrix Γ-semiring S or the matrix Γ_{nm}-semiring S_{mn} or simply the Γ_{nm}-semiring S_{mn}.

We denote the right operator semiring of the matrix Γ_{nm}-semiring S_{mn} by $[\Gamma_{nm}, S_{mn}]$ and the left one by $[S_{mn}, \Gamma_{nm}]$. If $x \in S$, the notation xE_{ij} will be used to denote a matrix in S_{mn} with x in the (i, j)-th entry and zeros elsewhere. The notation αE_{ij}, where $\alpha \in \Gamma$ will have a similar meaning. If $P \subseteq S$, P_{mn} will denote the set of all $m \times n$ matrices with entries from P. If $\Delta \subseteq \Gamma$, Δ_{nm} is similarly defined.

Theorem 4.2. *Let S be a Γ-semiring with left and right unities. Then S is a usp Γ-semiring if and only if S_{mn} is a usp Γ_{nm}-semiring for all positive integers m and n.*

Proof. S is a usp Γ-semiring if and only if L is a usp semiring if and only if L_m is a usp semiring if and only if S_{mn} is a usp Γ_{nm}-semiring (By Theorem 2.18 of [3] and by Lemma 3.13 of [2]). ■

Lemma 4.3. *Let S be a Γ-semiring and m, n be positive integers; then a nonempty subset P of S_{mn} is a usp k-ideal of S_{mn} if and only if $P = Q_{mn}$, for some usp k-ideal Q of S.*

Proof. Let L be the left operator semiring of S and L_m be the left operator semiring of the Γ_{nm}-semiring S_{mn}. Let P be a usp k-ideal of S_{mn}. Then by Theorem 2.23 of [3] (applied to Γ_{nm}-semiring S_{mn}) $P^{+'}$ is a usp k-ideal of the left operator semiring L_m of the Γ_{nm}-semiring S_{mn}. So $P^{+'} = Q_m$ for some usp k-ideal Q of L. This implies that $(P^{+'})^{+} = (Q_m)^{+}$ i.e. $P = (Q^{+})_{mn}$(By Theorem 2.10 and Proposition 3.4 of [5]). Again by Theorem 2.23 of [3], Q^{+} is an usp k-ideal of S. So the direct implication follows.

Conversely, suppose Q is a usp k-ideal of S. Then $Q^{+'}$ is a usp k-ideal of L (By Theorem 2.23 of [3]). So $(Q^{+'})_m$ is a usp k-ideal of L_m i.e. $(Q_{mn})^{+'}$ is a usp k-ideal of L_m by previous arguments. Hence $Q_{mn} = ((Q_{mn})^{+'})^{+}$ is an usp k-ideal of the Γ_{nm}-semiring S_{mn}. ■

Lemma 4.4. *Let S be a Γ-semiring and I be an ideal of S. Then $(S/I)_{mn}$ is isomorphic to S_{mn}/I_{mn} for all positive integers m, n.*

Proof. Let θ be a mapping of the Γ_{nm}-semiring $(S/I)_{mn}$ to the Γ_{nm}-semiring S_{mn}/I_{mn} defined by $\theta((x_{ij}/I)_{mn}) = (x_{ij})_{mn}/I_{mn}$. Let $(x_{ij}/I)_{mn}$,

$(y_{ij}/I)_{mn} \in (S/I)_{mn}$. Now $(x_{ij}/I)_{mn} = (y_{ij}/I)_{mn}$ if and only if $x_{ij}/I = y_{ij}/I$, for all i, j if and only if $x_{ij} + a_{ij} = y_{ij} + b_{ij}$, for some $a_{ij}, b_{ij} \in I$ and for all i, j if and only if $(x_{ij} + a_{ij})_{mn} = (y_{ij} + b_{ij})_{mn}$ if and only if $(x_{ij})_{mn} + (a_{ij})_{mn} = (y_{ij})_{mn} + (b_{ij})_{mn}$ for some matrices $(a_{ij})_{mn}$, $(b_{ij})_{mn} \in I_{mn}$ if and only if $(x_{ij})_{mn}/I_{mn} = (y_{ij})_{mn}/I_{mn}$ if and only if $\theta((x_{ij}/I)_{mn}) = \theta((y_{ij}/I)_{mn})$. Therefore θ is well defined as well as injective.

Clearly θ is onto and semigroup homomorphism under addition.
Now

$$\theta\Big((x_{ij}/I)_{mn}(\alpha_{ij})_{nm}(y_{ij}/I)_{mn}\Big)$$

$$= \theta\left(\left(\sum_{k=1}^{n}\sum_{l=1}^{m}(x_{ik}\alpha_{kl}y_{lj})/I\right)_{mn}\right)$$

$$= \left(\sum_{k=1}^{n}\sum_{l=1}^{m}(x_{ik}\alpha_{kl}y_{lj})\right)_{mn}/I_{mn}$$

$$= ((x_{ij})_{mn}(\alpha_{ij})_{nm}(y_{ij})_{mn})/I_{mn}$$

$$= ((x_{ij})_{mn}/I)(\alpha_{ij})_{nm}((y_{ij})_{mn}/I)$$

$$= \theta((x_{ij}/I)_{mn})\iota((\alpha_{ij})_{nm})\theta((y_{ij}/I)_{mn}).$$

This shows that (θ, ι) is an isomorphism of $(S/I)_{mn}$ onto S_{mn}/I_{mn} , where ι is the identity mapping from Γ_{nm} onto Γ_{nm}. ■

Proposition 4.5. *Let A, B be two ideals of a Γ-semiring S. Then $(A \cap B)_{mn} = A_{mn} \cap B_{mn}$, where A_{mn}, B_{mn} are two ideals of Γ_{nm}-semiring S_{mn}.*

Theorem 4.6. *Let S be a Γ-semiring. Then $\tau(S_{mn}) = (\tau(S))_{mn}$ for all positive integers m, n.*

Proof. By Definition 3.2,

$$\tau(S_{mn}) = \bigcap_{P \in \Lambda_{S_{mn}}} P,$$

where $\Lambda_{S_{mn}}$ denote the set of all usp k-ideals of the Γ-semiring S_{mn} and

$$\tau(S) = \bigcap_{Q \in \Lambda_S} Q,$$

where Λ_S denote the set of all usp k-ideals of the Γ-semiring S.
Therefore

$$(\tau(S))_{mn} = \left(\bigcap_{Q \in \Lambda_S} Q \right)_{mn} = \bigcap_{Q \in \Lambda_S} Q_{mn}$$

(By Proposition 4.5). Hence by Lemma 4.3 we get the result. ∎

5. Special classes

Definition 5.1 . A class $\wp$ of Γ-semirings is called hereditary if I is an ideal of a Γ-semiring S and $S \in \wp$ implies that $I \in \wp$.

Definition 5.2 . A class $\wp$ of Γ-semirings is called closed under essential extension if I is an essential ideal of a Γ-semiring S and $I \in \wp$ implies that $S \in \wp$.

Definition 5.3 . The class $\wp$ of Γ-semirings is called a special class if

(i) $\wp$ consists of prime Γ-semirings,

(ii) $\wp$ is hereditary and

(iii) $\wp$ is closed under essential extension.

Theorem 5.4 . A class $\pounds$ of uniformly strongly prime Γ-semirings is a special class.

Proof. Since uniformly strongly prime implies prime, so $\pounds$ consists of all prime Γ-semirings. By Lemma 2.17, we get $\pounds$ is hereditary class. Again by Theorem 2.14, $\pounds$ is closed under essential extension. Hence $\pounds$ is a special class. ∎

Proposition 5.5. *The following conditions are equivalent for any class ρ of prime Γ-semirings:*

(i) *if I is an ideal of S, $I \in \rho$ and $ann(I) = (0)$, then $S \in \rho$*

(ii) *ρ is closed under essential extension.*

Proof. (i)⇒(ii) Let I be an essential ideal of S and $I \in \rho$. Let $x \in I \cap ann(I)$ then $x \in I$ and $x \in ann(I)$ which implies that $x \in I$ and $I\,\Gamma\, x = \{0\} = x\,\Gamma\, I$. So $x\,\Gamma\, I\,\Gamma x = \{0\}$ which implies that $< x > \; \Gamma \; < x >= \{0\}$. Since (0) is a prime ideal of S so it follows that $x = 0$. Therefore $ann(I) = (0)$. Hence by (i), $S \in \rho$ which imply (ii).

(ii)⇒(i) Let ρ be closed under essential extension. Let $I \in \rho$ be an ideal of S with $ann(I) = (0)$. Let L be an ideal of S such that $I \cap L = (0)$. Now $I\,\Gamma\, L \subseteq I \cap L = (0)$ and $L\,\Gamma\, I \subseteq I \cap L = (0)$. So $I\,\Gamma\, L = (0) = L\,\Gamma\, I$ which implies that $L \subseteq ann(I) = (0)$, so $L = (0)$. Therefore I is an essential ideal of S. Also $I \in \rho$. Hence by (ii) $S \in \rho$. ■

A Γ-semiring S is called a us(1) prime if it has an insulator of the form $(\{x\}, \{\gamma\})$ where $x \in S$ and $\gamma \in \Gamma$.

As in Theorem 5.4 we can show that the class $\pounds_1$ of all us(1) prime Γ-semirings is a special class.

Definition 5.6. A pair of subsets $(T,\ I)$ of a Γ-semiring S is called a super t-system of S if

(i) I is an ideal of S

(ii) $T \cap I \subseteq (0)$ and

(iii) there exist finite subsets F of S and Δ of Γ such that for all $a, b \in S \backslash I$, $a \Delta F \Delta b \cap T \neq \phi$.

The pair $(F,\ \Delta)$ will be called an insulator of the super t-system. Therefore I is a uniformly strongly prime ideal of S if and only if $(S \setminus I,\ I)$ is a super t-system.

Theorem 5.7 *For a Γ-semiring S, $\tau(S) = \{a \in S$: whenever $(T,\ I)$ is a super t-system in S with $a \in T$ then $0 \in T\}$.*

Proof. Let $H = \{a \in S$: whenever $(T,\ I)$ is a super t-system in S with $a \in T$ then $0 \in T\}$. Let $a \in \tau(S)$. Suppose $(T,\ I)$ is a super t-system in S with $a \in T$ and $0 \notin T$. So $T \cap I = \phi$. Let $(F,\ \Delta)$ be an insulator of the super t-system. By Zorn's lemma there exists a maximal ideal P such that $I \subseteq P$ and $T \cap P = \phi$. We now prove that P is a uniformly strongly prime ideal of S with $(F,\ \Delta)$ be its uniform insulator. If possible let there exist $x, y \notin P$ such that $x\Delta F\Delta y \subseteq P$. Since $x, y \notin P$, then $x, y \notin I$ which implies that $x, y \in S \setminus I$ and $(x\Delta F\Delta y) \cap T \subseteq P \cap T = \phi$, a contradiction, since $(T,\ I)$ is a super t-system in S. So $x\Delta F\Delta y \subseteq P$ implies that $x \in P$ or $y \in P$ which implies that P is a uniformly strongly prime ideal of S. Hence $a \in \tau(S)$ implies that $a \in P$. Again $a \in T \Rightarrow a \notin P$ as $T \cap P = \phi$, which is a contradiction. So $0 \in T$. Then $a \in H \Rightarrow \tau(S) \subseteq H$.

Conversely, let $a \notin \tau(S)$. Then there exists a uniformly strongly prime k-ideal and hence an ideal say Q of S such that $a \notin Q$. Then $a \in S \setminus Q$, so $(S \setminus Q,\ Q)$ is a super t-system with $a \in S \setminus Q$ but $0 \notin S \setminus Q$, so $a \notin H$. Hence $H \subseteq \tau(S)$. This completes the proof. ■

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