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BIPARTITE PSEUDO MV-ALGEBRAS

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Abstract

A bipartite pseudo MV-algebra A is a pseudo MV-algebra such that $A = M \cup M^{\sim}$ for some proper ideal M of A. This class of pseudo MV-algebras, denoted **BP**, is investigated. The class of pseudo MV-algebras A such that $A = M \cup M^{\sim}$ for all maximal ideals M of A, denoted $\mathbf{BP}_0$, is also studied and characterized.

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1. Preliminaries

In the theory of MV-algebras, the classes **BP** and $\mathbf{BP}_0$ are defined and studied by A. Di Nola, F. Liguori and S. Sessa in [3] and investigated by R. Ambrosio and A. Lettieri in [1]. Here we define and investigate the classes **BP** and $\mathbf{BP}_0$ of pseudo MV-algebras and we give some characterizations of them. Pseudo MV-algebras were introduced by G. Georgescu and A. Iorgulescu in [5] and later by J. Rachůnek in [6] (here called generalized MV-algebras or, in short, GMV-algebras) and they are a non-commutative generalization of MV-algebras.

Let $A = (A, \oplus, ^{-}, ^{\sim}, 0, 1)$ be an algebra of type $(2, 1, 1, 0, 0)$. Set $x \cdot y = (y^{-} \oplus x^{-})^{\sim}$ for any $x, y \in A$. We consider that the operation $\cdot$ has priority to the operation $\oplus$, i.e., we will write $x \oplus y \cdot z$ instead of $x \oplus (y \cdot z)$. The algebra A is called a *pseudo MV-algebra* if for any $x, y, z \in A$ the following conditions are satisfied:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = 0 \oplus x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$;

(A4) $1^{\sim} = 0; 1^{-} = 0$;

(A5) $(x^{-} \oplus y^{-})^{\sim} = (x^{\sim} \oplus y^{\sim})^{-}$;

(A6) $x \oplus x^{\sim} \cdot y = y \oplus y^{\sim} \cdot x = x \cdot y^{-} \oplus y = y \cdot x^{-} \oplus x$;

(A7) $x \cdot (x^{-} \oplus y) = (x \oplus y^{\sim}) \cdot y$;

(A8) $(x^{-})^{\sim} = x$.

If the addition $\oplus$ is commutative, then both unary operations $^{-}$ and $^{\sim}$ coincide and then A is an MV-algebra.

Throughout this paper A will denote a pseudo MV-algebra. We will write $x^{\approx}$ instead of $(x^{\sim})^{\sim}$. For any $x \in A$ and $n = 0, 1, 2, \ldots$ we put

$$0x = 0 \text{ and } (n+1)\,x = nx \oplus x;$$

$$x^{0} = 1 \text{ and } x^{n+1} = x^{n} \cdot x.$$

Proposition 1.1 (Georgescu and Iorgulescu [5]). *The following properties hold for any $x, y \in A$:*

(a) $0^{-} = 1$;

(b) $1^{\approx} = 1$;

(c) $(x^{\sim})^{-} = x$;

(d) $(x^{-})^{\approx} = x^{\sim}$;

(e) $(x \oplus y)^{-} = y^{-} \cdot x^{-}; (x \oplus y)^{\sim} = y^{\sim} \cdot x^{\sim}$;

(f) $(x \cdot y)^{-} = y^{-} \oplus x^{-}; (x \cdot y)^{\sim} = y^{\sim} \oplus x^{\sim}$;

(g) $(x \oplus y)^{\approx} = x^{\approx} \oplus y^{\approx}$.

We define

$$x \leqslant y \Longleftrightarrow x^- \oplus y = 1.$$

As it is shown in [5], $(A, \leqslant)$ is a lattice in which the join $x \vee y$ and the meet $x \wedge y$ of any two elements x and y are given by:

$$\begin{aligned} x \vee y &= x \oplus x^\sim \cdot y = x \cdot y^- \oplus y; \\ x \wedge y &= x \cdot \big(x^- \oplus y\big) = (x \oplus y^\sim) \cdot y. \end{aligned}$$

For every pseudo MV-algebra A we set $\mathcal{L}(A) = (A, \vee, \wedge, 0, 1)$.

Proposition 1.2 (Georgescu and Iorgulescu [5])**.** *Let $x, y \in A$. Then the following properties hold:*

(a) $x \leqslant y \Longleftrightarrow y^- \leqslant x^-$;

(b) $x \leqslant y \Longleftrightarrow y^\sim \leqslant x^\sim$.

Following [4], we can consider the set $\mathrm{Inf}(A) = \{x \in A : x^2 = 0\}$. We have the following proposition.

Proposition 1.3 (Dymek and Walendziak [4])**.** *For every $x \in A$, the following conditions are equivalent:*

(a) $x \in \mathrm{Inf}(A)$;

(b) $2x^- = 1$;

(c) $2x^\sim = 1$.

By Proposition 1.3, $\mathrm{Inf}(A) = \{x \in A : 2x^- = 1\} = \{x \in A : 2x^\sim = 1\}$. We also have the following simple proposition.

Proposition 1.4. *The following conditions are equivalent for every $x \in A$ and $n \in \mathbb{N}$:*

(a) $x^n = 0$;

(b) $nx^- = 1$;

(c) $nx^\sim = 1$.

Proof. (a) $\Rightarrow$ (b): Let $x^n = 0$. Then, by Proposition 1.1, $nx^- = (x^n)^- = 0^- = 1$.

(b) $\Rightarrow$ (c): Suppose that $nx^- = 1$. Hence, by Proposition 1.1, $1 = 1^{\approx} = (nx^-)^{\approx} = n\,(x^-)^{\approx} = nx^{\sim}$.

(c) $\Rightarrow$ (a): Suppose that $nx^{\sim} = 1$. Applying Proposition 1.1, we obtain $0 = 1^- = (nx^{\sim})^- = \left[(x^{\sim})^-\right]^n = x^n$. ■

Let $\mathrm{N}(A) = \{x \in A : x^n = 0 \text{ for some } n \in \mathbb{N}\}$. Elements of $\mathrm{N}(A)$ are called the *nilpotent* elements of A. From Proposition 1.4 we see that $\mathrm{N}(A) = \{x \in A : nx^- = 1 \text{ for some } n \in \mathbb{N}\} = \{x \in A : nx^{\sim} = 1 \text{ for some } n \in \mathbb{N}\}$. It is obvious that $\mathrm{Inf}(A) \subseteq \mathrm{N}(A)$.

Definition 1.5. A subset I of A is called an *ideal* of A if it satisfies the following conditions:

(I1) $0 \in I$;

(I2) *If* $x, y \in I$, then $x \oplus y \in I$;

(I3) *If* $x \in I$, $y \in A$ and $y \leqslant x$, then $y \in I$.

Under this definition, $\{0\}$ and A are the simplest examples of ideals.

Proposition 1.6 (Walendziak [8])**.** *Let I be a nonvoid subset of A. Then I is an ideal of A if and only if I satisfies conditions* (I2) *and*

(I3') *If $x \in I$, $y \in A$, then $x \wedge y \in I$.*

Denote by $\mathrm{Id}(A)$ the set of ideals of A and note that $\mathrm{Id}(A)$ ordered by set inclusion is a complete lattice.

Remark 1.7. Let $I \in \mathrm{Id}(A)$.

(a) *If* $x, y \in I$, then $x \cdot y, x \wedge y, x \vee y \in I$.

(b) I is an ideal of the lattice $\mathcal{L}(A)$.

For every subset $W \subseteq A$, the smallest ideal of A which contains W, i.e., the intersection of all ideals $I \supseteq W$, is said to be the ideal *generated* by W, and will be denoted $(W]$. For every $z \in A$, the ideal $(z] = (\{z\}]$ is called the *principal ideal generated by z* (see [5]), and we have

$$(z] = \{x \in A : x \leqslant nz \text{ for some } n \in \mathbb{N}\}.$$

Definition 1.8. Let I be a proper ideal of A (i.e., $I \neq A$).

(a) I is called *prime* if, for all $I_1, I_2 \in \mathrm{Id}(A)$, $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$.

(b) I is called *regular* if $I = \bigcap X$ implies that $I \in X$ for every subset X of $\mathrm{Id}(A)$.

(c) I is called *maximal* if whenever J is an ideal such that $I \subseteq J \subseteq A$, then either $J = I$ or $J = A$.

By definition, each regular ideal is prime.

Proposition 1.9 (Walendziak [8]). *If $I \in \mathrm{Id}(A)$ is maximal, then I is prime.*

Definition 1.10. A *cover* of a proper ideal I of A is a unique least ideal I^* which properly contains I.

Definition 1.11. A pseudo MV-algebra A is called *normal-valued* if for any regular ideal I of A and any $x \in I^*$, $x \oplus I = I \oplus x$.

An element $x \neq 0$ of a pseudo MV-algebra A is called *infinitesimal* (see [7]) if x satisfies condition

$$nx \leqslant x^- \text{ for each } n \in \mathbb{N}.$$

Proposition 1.12. *Let A be a pseudo MV-algebra and $x \in A$. Then the following conditions are equivalent:*

(a) *x is infinitesimal;*

(b) *$nx \leqslant x^\sim$ for each $n \in \mathbb{N}$;*

(c) *$x \leqslant (x^-)^n$ for each $n \in \mathbb{N}$;*

(d) *$x \leqslant (x^\sim)^n$ for each $n \in \mathbb{N}$.*

Proof. (a) $\Leftrightarrow$ (b): See Rachůnek [7].

(b) $\Rightarrow$ (c): Let $nx \leqslant x^\sim$ for each $n \in \mathbb{N}$. Then, by Propositions 1.2(a) and 1.1(e), $x = (x^\sim)^- \leqslant (nx)^- = (x^-)^n$ for each $n \in \mathbb{N}$.

(c) $\Rightarrow$ (b): Let $x \leqslant (x^-)^n$ for each $n \in \mathbb{N}$. Then, by Propositions 1.1(e) and 1.2(b), $nx = \left[(nx)^-\right]^\sim = \left[(x^-)^n\right]^\sim \leqslant x^\sim$ for each $n \in \mathbb{N}$.

(a) $\Leftrightarrow$ (d): Analogous. ∎

Let us denote by $\mathrm{Infinit}(A)$ the set of all infinitesimal elements in A and by $\mathrm{Rad}(A)$ the intersection of all maximal ideals of A.

Proposition 1.13 (Rachůnek [7]). *Let A be a pseudo MV-algebra. Then:*

(a) $\mathrm{Rad}(A) \subseteq \mathrm{Infinit}(A)$.

(b) *If A is normal-valued, then* $\mathrm{Rad}(A) = \mathrm{Infinit}(A)$.

Proposition 1.14 (Dymek and Walendziak [4]). *Let A be a pseudo MV-algebra. Then* $\mathrm{Infinit}(A) \subseteq \mathrm{Inf}(A)$.

Proposition 1.15 (Dymek and Walendziak [4]). *Let A be a normal-valued pseudo MV-algebra. Then* $\mathrm{Inf}(A)$ *is an ideal of A if and only if* $\mathrm{Inf}(A) = \mathrm{Rad}(A)$.

2. Implicative ideals

Definition 2.1. An ideal I of A is called *implicative* if for any $x, y, z \in A$ it satisfies the following condition:

(Im) $(x \cdot y \cdot z \in I$ and $z^\sim \cdot y \in I) \Longrightarrow x \cdot y \in I$.

Proposition 2.2 (Walendziak [8]). *The implication* (Im) *is equivalent to*

(Im′) *For all $x, y, z \in A$, if $x \cdot y \cdot z^- \in I$ and $z \cdot y \in I$, then $x \cdot y \in I$.*

Proposition 2.3 (Walendziak [8]). *Let $I \in \mathrm{Id}(A)$. Then the following conditions are equivalent:*

(a) *I is implicative;*

(b) $\mathrm{N}(A) \subseteq I$;

(c) $\mathrm{Inf}(A) \subseteq I$.

Now we give an example of an ideal of a pseudo MV-algebra which is not implicative.

Example 2.4. Let A be the set of all increasing bijective functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$x \leqslant f(x) \leqslant x+1 \text{ for all } x \in \mathbb{R}.$$

Define the operations $\oplus, ^-, ^\sim$ and constans 0 and 1 as follows:

$$\begin{aligned}
(f \oplus g)(x) &= \min\{f(g(x)), x+1\}, \\
f^-(x) &= f^{-1}(x)+1, \\
f^\sim(x) &= f^{-1}(x+1), \\
0(x) &= x, \\
1(x) &= x+1.
\end{aligned}$$

Then $(A, \oplus, ^-, ^\sim, 0, 1)$ is a pseudo MV-algebra. Note that

$$\operatorname{Inf}(A) = \{f \in A : 2f^- = 1\} = \{f \in A : f(x) \leqslant f^{-1}(x)+1 \text{ for all } x \in \mathbb{R}\}$$

and the function $g(x) = x + \frac{1}{2}$ belongs to $\operatorname{Inf}(A)$. Observe that $\operatorname{Inf}(A)$ is not an ideal of A because $g \oplus g \notin \operatorname{Inf}(A)$. Now, define a function f as follows:

$$f(x) = \begin{cases} x+1 & \text{if } x \leqslant 0, \\ 1+\dfrac{x}{2} & \text{if } 0 < x < 2, \\ x & \text{if } x \geqslant 2. \end{cases}$$

Obviously $f \in A$. Let I be the ideal generated by f^-, i.e.,

$$I = \{h \in A : h \leqslant nf^- \text{ for some } n \in \mathbb{N}\}.$$

Observe that $f^-(1) = 1$ and thus $nf^-(1) = 1$ for every $n \in \mathbb{N}$. Hence $g(1) = 1.5 > nf^-(1)$ for all n, i.e., $g \notin I$. Therefore $\operatorname{Inf}(A) \nsubseteq I$ and so, by Proposition 2.3, I is not an implicative ideal of A.

Proposition 2.5 (Walendziak [8])**.** *If* $\operatorname{Inf}(A)$ *is an ideal, then* $\operatorname{Inf}(A)$ *is implicative.*

Proposition 2.6. *If* $\mathrm{Inf}(A)$ *is an ideal of* A*, then* $\mathrm{Inf}(A) = \mathrm{N}(A)$.

Proof. Assume that $\mathrm{Inf}(A)$ is an ideal of A. Then, by Proposition 2.5, it is implicative. So, by Proposition 2.3, $\mathrm{N}(A) \subseteq \mathrm{Inf}(A)$ and since $\mathrm{Inf}(A) \subseteq \mathrm{N}(A)$, we obtain $\mathrm{Inf}(A) = \mathrm{N}(A)$. ■

For a nonvoid subset B of a pseudo MV-algebra A we put:

$$B^{-} = \{x^{-} : x \in B\} \text{ and } B^{\sim} = \{x^{\sim} : x \in B\}.$$

Proposition 2.7. *Let* I *be a proper ideal of* A *such that* $I^{-} = I^{\sim}$ *and let* A_I *be a subalgebra of* A *generated by* I*. Then* $A_I = I \cup I^{-} = I \cup I^{\sim}$.

Proof. First, it is clear that $I \cup I^{-} = I \cup I^{\sim}$. Now, we prove that $I \cup I^{-}$ is a subalgebra of A. Since $0 \in I$, we have $1 = 0^{-} \in I^{-} \subseteq I \cup I^{-}$. Thus $0, 1 \in I \cup I^{-}$.

Take arbitrary $x \in I \cup I^{-}$. Then $x \in I$ or $x \in I^{-}$. If $x \in I$, then $x^{-} \in I^{-}$ and therefore $x^{-} \in I \cup I^{-}$. If $x \in I^{-}$, then $x \in I^{\sim}$. This entails $x = y^{\sim}$ for some $y \in I$ and hence $x^{-} = y \in I$. Therefore $x^{-} \in I \cup I^{-}$ for any $x \in I \cup I^{-}$. Similarly, if $x \in I \cup I^{-}$, then $x^{\sim} \in I \cup I^{\sim} = I \cup I^{-}$.

Now, we show that $x \oplus y, x \cdot y \in I \cup I^{-}$ for every $x, y \in I \cup I^{-}$. We consider four cases.

Case 1. $x, y \in I$.
Since I is an ideal, $x \oplus y, x \cdot y \in I \subseteq I \cup I^{-}$.

Case 2. $x \in I, y \in I^{-}$.
Then, $x \cdot y \leqslant x$ and $x \in I$ entail $x \cdot y \in I \subseteq I \cup I^{-}$. Since $y \in I^{-}$, we have $y = z^{-}$, where $z \in I$ and hence, by Proposition 1.1(f), $x \oplus y = x \oplus z^{-} = (x^{\sim})^{-} \oplus z^{-} = (z \cdot x^{\sim})^{-} \in I^{-}$ because $z \cdot x^{\sim} \in I$. Thus $x \oplus y, x \cdot y \in I \cup I^{-}$.

Case 3. $x \in I^{-}, y \in I$.
Analogous.

Case 4. $x, y \in I^{-}$.
We have $x \oplus y = z^{-} \oplus t^{-} = (t \cdot z)^{-} \in I^{-}$ for some $t, z \in I$. Similarly, $x \cdot y = z^{-} \cdot t^{-} = (t \oplus z)^{-} \in I^{-}$. Therefore $x \oplus y, x \cdot y \in I \cup I^{-}$.

Finally, we get that $I \cup I^{-}$ is a subalgebra (containing I) of an algebra A and from this reason, $A_I \subseteq I \cup I^{-}$. It is obvious that $I \cup I^{-} \subseteq A_I$. ■

Remark 2.8. The assumption $I^{-} = I^{\sim}$ in Proposition 2.7 is necessary. Indeed, consider the pseudo MV-algebra A from Example 2.4. Take an ideal

$$I = \{h \in A : h \leqslant nf^{-} \text{ for some } n \in \mathbb{N}\}$$

generated by f^{-}, where

$$f(x) = \begin{cases} x+1 & \text{if } x \leqslant 0, \\ 1+\dfrac{x}{2} & \text{if } 0 < x < 2, \\ x & \text{if } x \geqslant 2. \end{cases}$$

Thus $f \in I^{\sim}$. Since $f(1) = 1.5 > nf^{-}(1) = 1$ and $f^{\sim}(1) = 2 > nf^{-}(1)$, we have $f \notin I$ and $f^{\sim} \notin I$. Hence $f^{-} \notin I^{-}$ and $f \notin I^{-}$. Consequently we obtain $I^{-} \neq I^{\sim}$ and $f \notin I \cup I^{-}$, but $f \in A_I$.

Proposition 2.9 (Dymek and Walendziak [4])**.** *Let I be a prime ideal of A. Then the following conditions are equivalent:*

(a) *I is implicative*;

(b) $A = I \cup I^{\sim} (= I \cup I^{-})$.

Proposition 2.10 (Dymek and Walendziak [4])**.** *Let I be a proper ideal of A. If $A = I \cup I^{\sim} (= I \cup I^{-})$, then I is a maximal ideal of A generating A.*

Let us denote by IRad(A) the intersection of all implicative ideals of A. It is clear that IRad(A) is an implicative ideal of A, in fact, it is the smallest implicative ideal of A. By Propositions 1.13, 1.14 and 2.3, we have a ladder of inclusions:

(1) $$\mathrm{Rad}(A) \subseteq \mathrm{Infinit}(A) \subseteq \mathrm{Inf}(A) \subseteq \mathrm{N}(A) \subseteq \mathrm{IRad}(A).$$

Theorem 2.11. $(\mathrm{N}(A)] = \mathrm{IRad}(A)$.

Proof. Since $\mathrm{N}(A) \subseteq (\mathrm{N}(A)]$, it follows that $(\mathrm{N}(A)]$ is implicative. It is the smallest implicative ideal containing $\mathrm{N}(A)$ and hence the thesis. ■

Remark 2.12. We have also $(\mathrm{Inf}(A)] = \mathrm{IRad}(A)$ because $(\mathrm{Inf}(A)]$ is the smallest implicative ideal of A containing $\mathrm{Inf}(A)$.

Corollary 2.13. Inf(A) *is an ideal of* A *iff* $\mathrm{Inf}(A) = \mathrm{N}(A) = \mathrm{IRad}(A)$.

Theorem 2.14. IRad(A) *is a prime ideal of* A *iff* $A = \mathrm{IRad}\,(A) \cup (\mathrm{IRad}\,(A))^{\sim}$.

Proof. Let IRad(A) be a prime ideal of A. Since IRad(A) is implicative, we have, by Proposition 2.9, that $A = \mathrm{IRad}(A) \cup (\mathrm{IRad}\,(A))^{\sim}$.

If $A = \mathrm{IRad}(A) \cup (\mathrm{IRad}\,(A))^{\sim}$, then it is easy to see that IRad(A) is a maximal ideal of A. Hence, by Proposition 1.9, it is a prime ideal of A. ∎

Corollary 2.15. IRad(A) *is a prime ideal of* A *iff* $A = \mathrm{IRad}\,(A) \cup (\mathrm{IRad}\,(A))^{-}$.

3. Bipartite pseudo MV-algebras

Now, we define the class **BP** of *bipartite* pseudo MV-algebras as follows: $A \in \mathbf{BP}$ iff $A = M \cup M^{\sim}$ for some proper ideal M of A. By Proposition 2.10, we have that if $A \in \mathbf{BP}$, then there is a maximal ideal of A generating A.

First, recall that a pseudo MV-algebra A is said to be *symmetric* if $x^{-} = x^{\sim}$ for any $x \in A$. It is shown in [2] that the variety of symmetric pseudo MV-algebras contains as a proper subvariety the variety of all MV-algebras. We have the following proposition.

Proposition 3.1. *Let* A *be a symmetric pseudo* MV*-algebra. Then* $A \in \mathbf{BP}$ *if and only if* A *is generated by some maximal ideal.*

Proof. Let A be a symmetric pseudo MV-algebra. If $A \in \mathbf{BP}$, then, by Proposition 2.10, there is a maximal ideal of A generating A.

Conversely, assume that A is generated by some maximal ideal M. Since A is symmetric, we have $M^{-} = M^{\sim}$. Hence, by Proposition 2.7, $A = M \cup M^{\sim}$. Therefore $A \in \mathbf{BP}$. ∎

Proposition 3.2 (Dymek and Walendziak [4, Th. 3.5])**.** $A \notin \mathbf{BP}$ *iff* $(\mathrm{Inf}\,(A)] = A$.

Remark 3.3. Observe that for the pseudo MV-algebra A from Example 2.4, $(\mathrm{Inf}\,(A)] = A$. Thus, by Proposition 3.2, $A \notin \mathbf{BP}$.

Proposition 3.4. *If* Inf(A) *is a proper ideal of* A*, then* $A \in \mathbf{BP}$.

Proof. Assume that Inf(A) is a proper ideal of A. It is clear that there exists a maximal ideal M of A such that $\mathrm{Inf}(A) \subseteq M$. Then, by Proposition 2.3, M is implicative. From Proposition 2.9 we conclude that $A = M \cup M^{\sim}$. Thus $A \in \mathbf{BP}$. ∎

Proposition 3.5. *$A \in \mathbf{BP}$ iff there exists an ideal I of A which is prime and implicative.*

Proof. Follows from Proposition 2.9. ■

Theorem 3.6. *The class* $\mathbf{BP}$ *is closed under subalgebras.*

Proof. Let $A \in \mathbf{BP}$. Then there exists a proper ideal M of A such that $A = M \cup M^{\sim}$. Let B be a subalgebra of A. Then $I = M \cap B$ is a proper ideal of B. Since $(B \cap M)^{\sim} = B \cap M^{\sim}$, we have

$$\begin{aligned} B &= B \cap A = B \cap (M \cup M^{\sim}) = (B \cap M) \cup (B \cap M^{\sim}) \\ &= (B \cap M) \cup (B \cap M)^{\sim} = I \cup I^{\sim}. \end{aligned}$$

Therefore $B \in \mathbf{BP}$. ■

Let A_t be a pseudo MV-algebra for $t \in T$ and let $A = \prod_{t \in T} A_t$ be the direct product of A_t. We can consider the canonical projection $\mathrm{pr}_t : A \to A_t$ which is, of course, a homomorphism of pseudo MV-algebras. If $t \in T$ and I_t is a proper ideal of A_t, then it is easily seen that $\mathrm{pr}_t^{-1}(I_t)$ is a proper ideal of A and that $\mathrm{pr}_t^{-1}\left(I_t^{-}\right) = \left[\mathrm{pr}_t^{-1}(I_t)\right]^{-}$ and $\mathrm{pr}_t^{-1}(I_t^{\sim}) = \left[\mathrm{pr}_t^{-1}(I_t)\right]^{\sim}$.

Theorem 3.7. *Let A and A_t for $t \in T$ be pseudo MV-algebras such that $A = \prod_{t \in T} A_t$. If $A_{t_0} \in \mathbf{BP}$ for some $t_0 \in T$, then $A \in \mathbf{BP}$.*

Proof. Since $A_{t_0} \in \mathbf{BP}$, we have $A_{t_0} = M_{t_0} \cup M_{t_0}^{\sim}$ for some proper ideal M_{t_0} of A_{t_0}. From the above discussion, $\mathrm{pr}_{t_0}^{-1}(M_{t_0})$ is a proper ideal of A and

$$\begin{aligned} A &= \mathrm{pr}_{t_0}^{-1}(A_{t_0}) = \mathrm{pr}_{t_0}^{-1}\left(M_{t_0} \cup M_{t_0}^{\sim}\right) = \mathrm{pr}_{t_0}^{-1}(M_{t_0}) \cup \mathrm{pr}_{t_0}^{-1}\left(M_{t_0}^{\sim}\right) \\ &= \mathrm{pr}_{t_0}^{-1}(M_{t_0}) \cup \left[\mathrm{pr}_{t_0}^{-1}(M_{t_0})\right]^{\sim}. \end{aligned}$$

Hence $A \in \mathbf{BP}$. ■

Corollary 3.8. *The class* $\mathbf{BP}$ *is closed under direct products.*

Further, we define the class $\mathbf{BP}_0$ of pseudo MV-algebras as follows: $A \in \mathbf{BP}_0$ iff $A = M \cup M^{\sim}$ for all maximal ideals M of A. Note that if $A \in \mathbf{BP}_0$, then A is generated by all its maximal ideals. Remark that if A is a symmetric pseudo MV-algebra, then $A \in \mathbf{BP}_0$ if and only if A is generated by all its maximal ideals. Clearly, $\mathbf{BP}_0 \subseteq \mathbf{BP}$.

Theorem 3.9. $A \in \mathbf{BP}_0$ *iff* $\mathrm{Inf}(A) = \mathrm{Rad}(A)$.

Proof. Let $A \in \mathbf{BP}_0$. Then $A = M \cup M^{\sim}$ for every maximal ideal M of A. By Propositions 2.9 and 2.3, $\mathrm{Inf}(A) \subseteq M$ for every maximal ideal M of A and hence $\mathrm{Inf}(A) \subseteq \mathrm{Rad}(A)$. Thus, by (1), $\mathrm{Inf}(A) = \mathrm{Rad}(A)$.

Now, assume that $\mathrm{Inf}(A) = \mathrm{Rad}(A)$. Then $\mathrm{Inf}(A) \subseteq M$ for every maximal ideal M of A. By Propositions 2.3 and 2.9 we obtain that $A = M \cup M^{\sim}$ for every maximal ideal M of A. Thus $A \in \mathbf{BP}_0$. ■

Corollary 3.10. *If* $A \in \mathbf{BP}_0$*, then* $\mathrm{Inf}(A) = \mathrm{N}(A)$.

Proof. From Theorem 3.9 we conclude that $\mathrm{Inf}(A)$ is an ideal of A. By Proposition 2.6, $\mathrm{Inf}(A) = \mathrm{N}(A)$. ■

Corollary 3.11. $A \in \mathbf{BP}_0$ *iff* $\mathrm{Rad}(A)$ *is an implicative ideal of* A.

Proof. Let $A \in \mathbf{BP}_0$. Then, by Theorem 3.9, $\mathrm{Inf}(A) \subseteq \mathrm{Rad}(A)$ and hence, by Proposition 2.3, $\mathrm{Rad}(A)$ is an implicative ideal of A.

Conversely, assume that $\mathrm{Rad}(A)$ is an implicative ideal of A. Then, by Proposition 2.3, $\mathrm{Inf}(A) \subseteq \mathrm{Rad}(A)$ and thus, by (1), $\mathrm{Inf}(A) = \mathrm{Rad}(A)$. Therefore, by Theorem 3.9, $A \in \mathbf{BP}_0$. ■

Theorem 3.12. *Let A be a pseudo MV-algebra. Then the following are equivalent:*

(a) $A \in \mathbf{BP}_0$;

(b) $\mathrm{Rad}(A) = \mathrm{Infinit}(A) = \mathrm{Inf}(A) = \mathrm{N}(A) = \mathrm{IRad}(A)$;

(c) *every maximal ideal of A is implicative.*

Proof. (a) $\Rightarrow$ (b): Let $A \in \mathbf{BP}_0$. Then, by (1) and Theorem 3.9, $\mathrm{Rad}(A) = \mathrm{Infinit}(A) = \mathrm{Inf}(A)$. Hence $\mathrm{Inf}(A)$ is an ideal of A and by Corollary 2.13, $\mathrm{Inf}(A) = \mathrm{N}(A) = \mathrm{IRad}(A)$. Therefore (b) is true.

(b) $\Rightarrow$ (c): Since $\mathrm{Inf}(A) = \mathrm{Rad}(A)$, $\mathrm{Inf}(A) \subseteq M$ for every maximal ideal M of A and by Proposition 2.3, every maximal ideal M of A is implicative.

(c) $\Rightarrow$ (a): Since every maximal ideal M of A is implicative, we obtain by Proposition 2.9, $A = M \cup M^{\sim}$ for every maximal ideal M of A. Thus $A \in \mathbf{BP}_0$. ■

Theorem 3.13. *Let A be a normal-valued pseudo MV-algebra. Then the following are equivalent:*

(a) $A \in \mathbf{BP}_0$;

(b) $\mathrm{Inf}(A)$ *is an ideal of* A;

(c) $\mathrm{Rad}(A) = \mathrm{Infinit}(A) = \mathrm{Inf}(A) = \mathrm{N}(A) = \mathrm{IRad}(A)$;

(d) *every maximal ideal of A is implicative.*

Proof. (a) $\Rightarrow$ (b): Follows from Theorem 3.9.

(b) $\Rightarrow$ (c): Follows from (1), Proposition 1.15 and Corollary 2.13.

(c) $\Rightarrow$ (d), (d) $\Rightarrow$ (a): Follow from Theorem 3.12. ∎

From [2, Proposition 4.9], for any pseudo MV-algebras A, B we have:

$$\mathrm{Rad}\,(A \times B) = \mathrm{Rad}\,(A) \times \mathrm{Rad}\,(B)\,. \tag{2}$$

Lemma 3.14. *Let A, B be any pseudo MV-algebras. Then* $\mathrm{Inf}(A \times B) = \mathrm{Inf}(A) \times \mathrm{Inf}(B)$.

Proof. Let $(x, y) \in \mathrm{Inf}(A \times B)$. Then $(x, y)^2 = \left(x^2, y^2\right) = (0, 0)$ and hence $x^2 = y^2 = 0$. Thus $x \in \mathrm{Inf}(A)$ and $y \in \mathrm{Inf}(B)$, i.e., $(x, y) \in \mathrm{Inf}(A) \times \mathrm{Inf}(B)$.

Now, let $x \in \mathrm{Inf}(A)\,, y \in \mathrm{Inf}(B)$. Then $x^2 = y^2 = 0$. Hence $(x, y)^2 = (0, 0)$, i.e., $(x, y) \in \mathrm{Inf}(A \times B)$. Therefore $\mathrm{Inf}(A \times B) = \mathrm{Inf}(A) \times \mathrm{Inf}(B)$. ∎

From (2), Lemma 3.14 and Theorem 3.9 we obtain the following theorem.

Theorem 3.15. *Let A, B be any pseudo MV-algebras. Then $A, B \in \mathbf{BP}_0$ iff $A \times B \in \mathbf{BP}_0$.*

We shall end the paper with two examples. The first one is an example of a pseudo MV-algebra which belongs to $\mathbf{BP}_0$, while the second one is an example of a pseudo MV-algebra which is in $\mathbf{BP}$ and is not in $\mathbf{BP}_0$.

Example 3.16 (Dymek and Walendziak [4]**).** Let $B = \{(1, y) : y \geqslant 0\} \cup \{(2, y) : y \leqslant 0\}$, $\mathbf{0} = (1, 0)$, $\mathbf{1} = (2, 0)$. For any $(a, b), (c, d) \in B$, we define operations $\oplus, ^-, ^\sim$ as follows:

$$
\begin{aligned}
(a,b)\oplus(c,d) &= \begin{cases} (1,b+d) & \textit{if } a=c=1, \\ (2,ad+b) & \textit{if } ac=2 \textit{ and } ad+b\leqslant 0, \\ (2,0) & \textit{in other cases.} \end{cases} \\
(a,b)^{-} &= \left(\frac{2}{a},-\frac{2b}{a}\right), \\
(a,b)^{\sim} &= \left(\frac{2}{a},-\frac{b}{a}\right).
\end{aligned}
$$

Then $B=(B,\oplus,^{-},^{\sim},\mathbf{0},\mathbf{1})$ is a pseudo MV-algebra. Let $M=\{(1,y): y\geqslant 0\}$. Then M is the unique maximal ideal of B and $B=M\cup M^{\sim}$ is generated by M. Thus $B\in\mathbf{BP}_0$ and so $B\in\mathbf{BP}$. Note that M is an implicative ideal of B and $\mathrm{Rad}(B)=\mathrm{Infinit}(B)=\mathrm{Inf}(B)=\mathrm{N}(B)=\mathrm{IRad}(B)=M$.

Example 3.17. Let A be the pseudo MV-algebra from Example 2.4 and B be the pseudo MV-algebra from Example 3.16. Since $B\in\mathbf{BP}$, we conclude, by Theorem 3.7, $A\times B\in\mathbf{BP}$. But, by Theorem 3.15, $A\times B\notin\mathbf{BP}_0$ because $A\notin\mathbf{BP}_0$.

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