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DISTRIBUTIVE ORDERED SETS AND RELATIVE PSEUDOCOMPLEMENTS

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Abstract

Brouwerian ordered sets generalize Brouwerian lattices. The aim of this paper is to characterize α-complete Brouwerian ordered sets in a manner similar to that used previously for pseudocomplemented, Stone, Boolean and distributive ordered sets. The sublattice $G(P)$ in the Dedekind-Mac Neille completion $DM(P)$ of an ordered set P generated by P is said to be the characteristic lattice of P. We can define a stronger notion of Brouwerianicity by demanding that both P and $G(P)$ be Brouwerian. It turns out that the two concepts are the same for finite ordered sets. Further, the so-called antiblocking property of distributive lattices is generalized to distributive ordered sets.

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The aim of this paper is to characterize Brouwerian ordered sets defined by Halaš in [4] in a manner similar to that used previously for pseudocomplemented, Stone, Boolean and distributive ordered sets, cf. [6] and [9].

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This characterization is generalized to α-complete Brouwerian ordered sets. The sublattice $G(P)$ in the Dedekind-Mac Neille completion $DM(P)$ of an ordered set P generated by P is said to be the characteristic lattice of P. Its basic properties were investigated in [7]. We can define a stronger notion of Brouwerianicity by demanding that both P and $G(P)$ be Brouwerian. It turns out that the two concepts are the same for finite ordered sets. It remains an open problem whether they coincide in general. Further, the so-called antiblocking property of distributive lattices is generalized to distributive ordered sets. Clearly, finite Brouwerian ordered sets generalize finite distributive lattices and therefore can be considered for applications where distributive ordered sets are too general.

1. Preliminaries

First we recall some important definitions and facts from [9].

The sets of all lower bounds and of all upper bounds of the union of subsets $A_1, \dots, A_n$ in an ordered set $(P, \le)$ will be denoted by $L_P(A_1, \dots, A_n)$ and $U_P(A_1, \dots, A_n)$ respectively, shortly $L(A_1, \dots, A_n)$ and $U(A_1, \dots, A_n)$. As usual, $\downarrow p := \{q \in P \mid q \le p\}$.

Lemma 1. *Let $P \subseteq Q$ be a subset of an ordered set $(Q, \le)$ endowed with the induced order such that $b = \bigvee(\downarrow b \cap P)$ for each $b \in Q$. Then $L_Q(P) = L_Q(Q) \cup L_P(P)$ and the following conditions are equivalent:*

(i) $L_P(P) = L_Q(Q) \cap P$;

(ii) $L_P(P) \subseteq L_Q(Q)$;

(iii) $L_Q(P) = L_Q(Q)$.

Proof. For each $b \in L_Q(P)$ we have $b = \bigvee(\downarrow b \cap P) = \bigvee(\downarrow b \cap L_P(P))$. This yields $L_Q(P) = L_Q(Q) \cup L_P(P)$ since $\downarrow b \cap L_P(P)$ is either empty or formed by the least element in P. Obviously (i) $\implies$ (ii). Now, from (ii) we obtain $L_Q(P) = L_Q(Q) \cup L_P(P) = L_Q(Q)$. This proves (ii) $\implies$ (iii). Implication (iii) $\implies$ (i) is easily verified with $L_P(P) = L_Q(P) \cap P = L_Q(Q) \cap P$. ∎

Definition. We say that a subset $P \subseteq Q$ is *dense* in an ordered set $(Q, \leq)$ if $b = \bigvee(\downarrow b \cap P)$ for each $b \in Q$. If in addition any of conditions (i)–(iii) from Lemma 1 is satisfied, then P is *strictly dense* in Q. It is *doubly dense* if it is both dense and dually dense.

Note that the definition of a dense subset admits that P and Q have different bottom elements, the bottom element of P being the unique atom in Q. Exactly this is avoided in our definition of a strictly dense subset. This fact is important when investigating pseudocomplemented ordered sets, the definition is used here only for the reason of consistency. It can be easily checked that a doubly dense subset is both strictly dense and dually strictly dense.

Lemma 8 from [9] can be generalized to

Lemma 2. *Let P be a dense subset of an ordered set $(Q, \leq)$ and $A, B \subseteq P$. Then*

$$L_P(A) \subseteq L_P(B) \Leftrightarrow L_Q(A) \subseteq L_Q(B),$$

and if moreover P is strictly dense in Q, then

$$L_P(A) \subseteq L_P(P) \Leftrightarrow L_Q(A) \subseteq L_Q(Q).$$

Proof. First we prove the former equivalence.

$\Rightarrow$: $q \in L_Q(A)$ implies that $\downarrow q \cap P \subseteq L_P(A) \subseteq L_P(B) \subseteq L_Q(B)$, and hence $q = \bigvee(\downarrow q \cap P) \in L_Q(B)$.

$\Leftarrow$: $L_P(A) = P \cap L_Q(A) \subseteq P \cap L_Q(B) = L_P(B)$.

The latter equivalence is obtained by setting $B := P$ and applying condition (iii) from Lemma 1. ■

Definition. A subset C is a *cut* in an ordered set P if $C = LU(C)$.

The set of all cuts in P ordered by inclusion is called the *Dedekind-Mac Neille completion* of P and denoted $DM(P)$.

The lattice $G(P)$ generated in $DM(P)$ by P is called the *characteristic lattice* of P.

Observation 3. *Every ordered set P is doubly dense in $DM(P)$ and $G(P)$.*

For the sake of simplicity, an ordered set P is identified with the subset $\{\downarrow a \mid a \in P\}$ in $DM(P)$.

Lemma 4 (cf. [1])**.** *If P is a doubly dense subset of a complete lattice L, then L is isomorphic to $DM(P)$ over P.*

Definition. P is said to be a *generating* subset of a lattice L if L is generated by P as a lattice.

Lemma 5 ([7])**.** *If P is a doubly dense generating subset of a lattice L, then L is isomorphic to $G(P)$ over P.*

Definition. We say that an ordered set P is *distributive* if for each $a, b, c \in P$ we have $L(\{a\}, U(\{b, c\})) = LU(L(\{a, b\}), L(\{a, c\}))$. We say that an element b of an ordered set P is the *pseudocomplement* of an element a if $L(\{a, c\}) = L(P) \Leftrightarrow c \leq b$. We say that P is *pseudocomplemented* if each element $a \in P$ has a pseudocomplement a^* in P.

See [5] and [6] for details.

2. Conditions for distributivity

It is a commonplace that a lattice is distributive if and only if

$$c \wedge a \leq c \wedge b \ \&\ c \vee a \leq c \vee b \implies a \leq b,$$

cf. [12], p. 114. Since $c \wedge a \leq c \wedge b \iff c \wedge a \leq b$ and $c \vee a \leq c \vee b \iff a \leq c \vee b$, the above condition can be rewritten in the form

$$c \wedge a \leq b \ \&\ a \leq c \vee b \implies a \leq b.$$

Andreas Polyméris' query was how the last condition, used in investigations in artificial intelligence and called the *antiblocking property* by him, can be reformulated for ordered sets. It turns out that the situation is much similar to the lattice case.

Proposition 6. *Let $(P, \leq)$ be an ordered set. The following conditions are equivalent:*

(i) $(P, \leq)$ *is distributive;*

(ii) $L(\{v, z\}) \subseteq L(\{w\})$ & $U(\{w, z\}) \subseteq U(\{v\}) \implies v \leq w$ *for each* $v, w, z \in P$*;*

(iii) $L(\{v, z\}) \subseteq L(\{w, z\})$ & $U(\{w, z\}) \subseteq U(\{v, z\}) \implies v \leq w$ *for each* $v, w, z \in P$*.*

Proof. (i) $\iff$ (ii) was proved by Erné in [2], Corollary 2.6.

(ii) $\implies$ (iii) as $L(\{v, z\}) \subseteq L(\{w, z\})$ & $U(\{w, z\}) \subseteq U(\{v, z\})$ implies that $L(\{v, z\}) \subseteq L(\{w, z\}) \subseteq L(\{w\})$ & $U(\{w, z\}) \subseteq U(\{v, z\}) \subseteq U(\{v\})$, which by assumption yields $v \leq w$.

In order to verify (iii) $\implies$ (i), suppose that $(P, \leq)$ is not distributive. Then there exist elements $a, b, c \in P$ such that

$$LU(L(\{a, b\}), L(\{a, c\})) \subset L(\{a\}, U(\{b, c\}))$$

and equivalently

$$UL(\{a\}, U(\{b, c\})) \subset U(L(\{a, b\}), L(\{a, c\})).$$

Let us denote

$$A := U(L(\{a, b\}), L(\{a, c\})) \setminus UL(\{a\}, U(\{b, c\}))$$

and

$$B := L(\{a\}, U(\{b, c\})) \setminus LU(L(\{a, b\}), L(\{a, c\})).$$

Obviously $A \neq \emptyset$ and $A \cap U(\{b, c\}) = \emptyset$. It is easy to see that $B \setminus L(\{w\}) \neq \emptyset$ for each $w \in A$. Indeed, $LU(L(\{a, b\}), L(\{a, c\})) \subseteq L(\{w\})$ and $L(\{a\}, U(\{b, c\})) \nsubseteq L(\{w\})$, and therefore $B \setminus L(\{w\}) = L(\{a\}, U(\{b, c\})) \setminus L(\{w\}) \neq \emptyset$. We distinguish two cases.

If $A \cap U(\{b\}) \neq \emptyset$, then we choose $w \in A \cap U(\{b\})$ and $v \in B \setminus L(\{w\})$. Clearly $v \nleq w$. Further we put $z := c$. Then

$$\begin{aligned} L(\{v,z\}) &= L(\{v,c\}) \\ &\subseteq L(\{a,c\}) && \text{as } L(\{v\}) \subseteq L(\{a\}) \\ &\subseteq L(\{w,c\}) = L(\{w,z\}) && \text{as } L(\{a,c\}) \subseteq L(\{w\}) \end{aligned}$$

and

$$\begin{aligned} U(\{w,z\}) &= U(\{w,c\}) \\ &\subseteq U(\{b,c\}) && \text{as } U(\{w\}) \subseteq U(\{b\}) \\ &\subseteq U(\{v,c\}) = U(\{v,z\}) && \text{as } U(\{b,c\}) \subseteq U(\{v\}). \end{aligned}$$

Thus (iii) is not satisfied.

Otherwise we choose $w \in A$ and $v \in B \setminus L(\{w\})$ arbitrarily. Clearly $v \not\leq w$. We put $z := b$. Then

$$\begin{aligned} L(\{v,z\}) &= L(\{v,b\}) \\ &\subseteq L(\{a,b\}) && \text{as } L(\{v\}) \subseteq L(\{a\}) \\ &\subseteq L(\{w,b\}) = L(\{w,z\}) && \text{as } L(\{a,b\}) \subseteq L(\{w\}) \end{aligned}$$

and

$$\begin{aligned} U(\{w,z\}) &= U(\{w,b\}) \\ &\subseteq U(L(\{a,b\}), L(\{a,c\}), \{b\}) && \text{as } L(\{a,b\}) \cup L(\{a,c\}) \subseteq L(\{w\}) \\ &\subseteq UL(\{a\}, U(\{b,c\}), \{b\}) && \text{as } A \cap U(\{b\}) = \emptyset \\ &\subseteq U(\{v,b\}) = U(\{v,z\}) && \text{as } UL(\{a\}, U(\{b,c\})) \subseteq U(\{v\}). \end{aligned}$$

Thus (iii) is not satisfied. ■

Notice that condition (ii) can be rewritten as follows.

For each elements $v, w, z \in P$ such that $v \nleq w$ either there is an element $v' \in L(v, z)$ such that $v' \nleq w$ or there is an element $w' \in U(w, z)$ such that $v \nleq w'$.

For lattices, a closely related assertion holds.

Observation 7. *A lattice is distributive if and only if*

$$c \wedge a = c \wedge b \;\&\; c \vee a = c \vee b \implies a = b.$$

The analogous assertion for ordered sets doesn't hold. It follows from the preceding that we have

$$L(\{v, z\}) = L(\{w, z\}) \;\&\; U(\{v, z\}) = U(\{w, z\}) \implies v = w$$

in every distributive ordered set. However, the converse is not true. Indeed, the ordered set visualized in Figure 1 is not distributive.

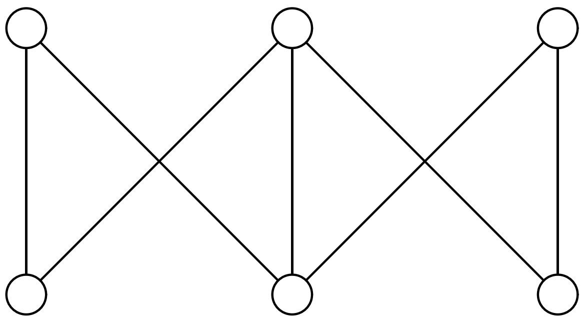


Figure 1.

Nevertheless, there are no elements $v \neq w$ and z with $L(\{v, z\}) = L(\{w, z\})$ & $U(\{v, z\}) = U(\{w, z\})$.

3. Brouwerian ordered sets

Definition. Let $(P, \leq)$ be an ordered set and $a \in P$, $B \subseteq P$. If the set $\{s \in P \mid L(\{a, s\}) \subseteq L(B)\}$ has a greatest element, we call it the *relative pseudocomplement* of a with respect to B. If $B = \{b\}$, then we denote it $a * b$.

Halaš defined relative pseudocomplements for one-element subsets B only in [4]. The notation is adopted from [3].
We can slightly reformulate the definition.

Observation 8. *An element r is the relative pseudocomplement of a with respect to B if and only if*

$$L(\{a, s\}) \subseteq L(B) \iff s \leq r.$$

Lemma 9. *Let $(Q, \leq)$ be an ordered set, $B \subseteq P \subseteq Q$ such that P is dense in Q and r be the relative pseudocomplement of a with respect to B in P with the induced order. Then r is the relative pseudocomplement of a with respect to B in Q.*

Proof. From the assumption it follows that $L_P(\{a, r\}) \subseteq L_P(B)$. Using Lemma 2 we immediately obtain $L_Q(\{a, r\}) \subseteq L_Q(B)$. If conversely $L_Q(\{a, c\}) \subseteq L_Q(B)$ for an element $c \in Q$, then $c = \bigvee(\downarrow c \cap P)$. Now, for each $p \in \downarrow c \cap P$, we have $L_Q(\{a, p\}) \subseteq L_Q(\{a, c\}) \subseteq L_Q(B)$, which by Lemma 2 yields $L_P(\{a, p\}) \subseteq L_P(B)$, and consequently $p \leq r$. Therefore $c \leq r$. ■

Definition. Let α be a cardinal number with $1 < \alpha$.

An ordered set $(P, \leq)$ is said to be *α-Brouwerian* if the relative pseudocomplement of a with respect to B exists for each $a \in P$ and each $B \subseteq P$ with $|B| < \alpha$.

An ordered set $(P, \leq)$ is said to be *α-lower-bounded* if there exists a subset $B \subseteq P$ with $|B| < \alpha$ and $L(B) = L(P)$.

An ordered set $(P, \leq)$ is said to be *α-complete* if each subset $B \subseteq P$ with $|B| < \alpha$ has an infimum in P.

We say that a subset P is *α-relative-pseudocomplement-closed* in an α-Brouwerian ordered set $(Q, \leq)$ if $a \in P$, $B \subseteq P$ and $|B| < \alpha$ together imply that the relative pseudocomplement of a with respect to B in Q is an element of P.

We say that a subset P is *relative-pseudocomplement-closed* in a 2-Brouwerian ordered set $(Q, \leq)$ if $a, b \in P$ implies that $a * b \in P$.

According to Halaš, we will speak about Brouwerian ordered sets instead of 2-Brouwerian ordered sets. Similarly to semimodularity, see [8], we obtain a whole spectrum of reasonable generalizations.

Observation 10. *For every ordered set P, the following conditions are equivalent:*

(i) *P is 3-complete;*

(ii) *P is $\aleph_0$-complete;*

(iii) *P is a topped meet-semilattice.*

Proposition 11. *Every α-Brouwerian α-lower-bounded ordered set is pseudocomplemented.*

Proof. Let $(P, \leq)$ be an α-Brouwerian α-lower-bounded ordered set. Then there exists a subset $B \subseteq P$ with $|B| < \alpha$ and $L(B) = L(P)$. Take $a \in P$. Since $(P, \leq)$ is α-Brouwerian, the set $\{s \in P \mid L(\{a, s\}) \subseteq L(B)\}$ has a greatest element r. Since $\{s \in P \mid L(\{a, s\}) \subseteq L(B)\} = \{s \in P \mid L(\{a, s\}) \subseteq L(P)\}$, r is the pseudocomplement of a. ∎

Observation 12. *Let α, β be cardinal numbers with $1 < \beta \leq \alpha$. Then every α-Brouwerian ordered set is β-Brouwerian because $|B| < \beta$ implies that $|B| < \alpha$. In particular, every α-Brouwerian ordered set is Brouwerian.*

Observation 13. *Every non-empty Brouwerian ordered set possesses the greatest element $\top = a * a$.*

Using Proposition 6, the proof of the following assertion formulated in [4] is very short.

Lemma 14. *Every Brouwerian ordered set is distributive.*

Proof. From $L(\{v,z\}) \subseteq L(\{w\})$ & $U(\{w,z\}) \subseteq U(\{v\})$ we obtain that $z \leq v * w$ & $U(\{w,z\}) \subseteq U(\{v\})$ by Observation 8. Hence $v * w \in U(\{w,z\}) \subseteq U(\{v\})$, and therefore $L(\{v\}) = L(\{v, v*w\}) \subseteq L(\{w\})$. Finally $v \leq w$. ∎

It is obvious that being α-Brouwerian is an instance of being Brouwerian even for lattices. On the other hand, every Brouwerian lattice is α-Brouwerian for each $\alpha \leq \aleph_0$ and every complete Brouwerian lattice is α-Brouwerian for any cardinal number α.

Proposition 15. *For each cardinal number α with $1 < \alpha$ and every non-empty ordered set P, the following conditions are equivalent:*

(i) *P is α-Brouwerian;*

(ii) *P is α-complete and Brouwerian.*

Proof. (i) $\Longrightarrow$ (ii) Let $(P, \leq)$ be a non-empty α-Brouwerian ordered set, $B \subseteq P$ and $|B| < \alpha$. P is Brouwerian by Observation 12. The relative pseudocomplement r of the top element $\top$, see Observation 13, with respect to B is the infimum of B. Indeed, $r = \bigvee\{s \mid L(\{s\}) = L(\{\top, s\}) \subseteq L(B)\} = \bigvee\{s \mid s \in L(B)\} = \bigwedge B$. To sum up, P is α-complete.

(ii) $\Longrightarrow$ (i) Let $(P, \leq)$ be an α-complete and Brouwerian ordered set. Let $a \in P$ and $B \subseteq P$ such that $|B| < \alpha$. By assumption, there exist the infimum of B and the relative pseudocomplement $a * \bigwedge B$. Clearly $L(B) = L(\{\bigwedge B\})$ and consequently

$$L(\{a,s\}) \subseteq L(B) \iff L(\{a,s\}) \subseteq L\Big(\Big\{\bigwedge B\Big\}\Big)$$
$$\iff s \leq a * \bigwedge B$$

by Observation 8. By the same argument, $a * \bigwedge B$ is the relative pseudocomplement of a with respect to B. ∎

Corollary 16. *For every ordered set P, the following conditions are equivalent:*

(i) *P is* 3*-Brouwerian;*

(ii) *P is* $\aleph_0$*-Brouwerian;*

(iii) *P is a Brouwerian meet-semilattice.*

Proof. The assertion is obviously true if P is empty. If P is non-empty, it follows from the preceding proposition and Observation 10. ■

Notice that finite Brouwerian ordered sets generalize finite distributive lattices, so that they may be useful in such applications where finite distributive ordered sets are too general. In contrast to lattices, finite distributive ordered sets need not be Brouwerian.

Observation 17. *The ordered set* $\mathbf{2}^4 \setminus \{(0,0,1,1)\}$ *is a distributive non-Brouwerian ordered set.*

Recall that finite Brouwerian lattices are precisely finite distributive lattices.

Observation 18. *For every finite ordered set P and each cardinal number α with $2 < \alpha$, the following conditions are equivalent:*

(i) *P is α-Brouwerian;*

(ii) *P is* 3*-Brouwerian;*

(iii) *P is a Brouwerian/distributive lattice.*

Lemma 19. *In every Brouwerian ordered set, $a \leq c$ & $b \leq s$ implies that $c * b \leq a * s$.*

Proof. $L(\{a, c * b\}) \subseteq L(\{c, c * b\}) \subseteq L(\{b\}) \subseteq L(\{s\})$. By Observation 8, $c * b \leq a * s$. ■

Proposition 20. *An α-Brouwerian dense subset of an α-Brouwerian ordered set is α-relative-pseudocomplement-closed in it.*

Proof. The proof follows immediately from Lemma 9. ■

Proposition 21. *An α-relative-pseudocomplement-closed dense subset of an α-Brouwerian ordered set is α-Brouwerian.*

Proof. Let P be an α-relative-pseudocomplement-closed dense subset of an α-Brouwerian ordered set $(Q, \leq)$. Suppose that $a, r \in P$, $B \subseteq P$, $|B| < \alpha$ and r is the relative pseudocomplement of a with respect to B in Q. For $p \in P$ we obtain $L_P(\{a,p\}) \subseteq L_P(B) \iff L_Q(\{a,p\}) \subseteq L_Q(B) \iff p \leq r$ according to Lemma 2 and Observation 8. ∎

Lemma 22. *Let $(P, \leq)$ be a Brouwerian ordered set, $a, r \in P$ and $B \subseteq P$. If r is the relative pseudocomplement of a with respect to B, then $r = \bigwedge\{a * s \mid s \in B\}$.*

Proof. For each $s \in B$ we obtain $L(\{a,r\}) \subseteq L(B) \subseteq L(\{s\})$, which in turn yields $r \leq a * s$. Hence $r \in L(\{a * s \mid s \in B\})$. Conversely, for any $c \in L(\{a * s \mid s \in B\})$ we have $(\forall s \in B) L(\{a,c\}) \subseteq L(\{s\})$, and consequently $L(\{a,c\}) \subseteq L(B)$. Therefore $c \leq r$. To sum up, r is the infimum of $\{a * s \mid s \in B\}$ in P. ∎

Lemma 23. *If P is a Brouwerian ordered set and B, C cuts in P, then $L(\{a * s \mid a \in C \;\&\; s \in U(B)\})$ is the relative pseudocomplement of C with respect to $\{B\}$ in $DM(P)$.*

Proof. Let $(P, \leq)$ be a Brouwerian ordered set and B, C cuts in P. Put $R := L(\{a*s \mid a \in C \;\&\; s \in U(B)\})$. It suffices to verify that $R \cap C \subseteq B$, and whenever $Q \cap C \subseteq B$ for a cut Q, then $Q \subseteq R$. For each $a \in R \cap C$ and each $s \in U(B)$ we have $a \leq a * s$ and therefore $L(\{a\}) = L(\{a, a * s\}) \subseteq L(\{s\})$, which yields $a \in B$. Hence $R \cap C \subseteq B$. Conversely, let Q be a cut such that $Q \cap C \subseteq B$. Then for each $a \in C$, $q \in Q$ and $s \in U(B)$ we obtain $L(\{q,a\}) \subseteq Q \cap C \subseteq B \subseteq L(\{s\})$, and therefore $q \leq a * s$. This in turn yields $Q \subseteq R$. ∎

Proposition 24. *If P is an α-Brouwerian ordered set, then the Dedekind-Mac Neille completion $DM(P)$ of P is Brouwerian, and P is α-relative-pseudocomplement-closed in $DM(P)$.*

Proof. Let $(P, \leq)$ be an α-Brouwerian ordered set. The first part of the assertion follows from the preceding lemma and Observation 12. In view of Proposition 15, $DM(P)$ is α-Brouwerian. Now, for $c \in P$, $B \subseteq P$ where

$|B| < \alpha$, $\mathbf{B} = \{\downarrow b \mid b \in B\}$ and $C = \downarrow c$, the relative pseudocomplement of C with respect to $\mathbf{B}$ is $\bigwedge\{C * S \mid S \in \mathbf{B}\} = \bigwedge\{L(\{a * s \mid a \in C \;\&\; s \in U(S)\} \mid S \in \mathbf{B}\} = \bigwedge\{L(\{a * s \mid a \in \downarrow c \;\&\; s \in U(\downarrow b) \mid b \in B\}\} = \bigwedge\{L(\{a * s \mid a \in L(\{c\}) \;\&\; s \in U(\{b\})\}) \mid b \in B\} = \bigwedge\{\downarrow(c * b) \mid b \in B\} = \downarrow \bigwedge\{c * b \mid b \in B\}$ according to Lemma 19, which is the relative pseudocomplement of c with respect to B in P in virtue of Lemma 22. ∎

Observation 25. *In our setting, the empty ordered set is a Brouwerian lattice, and every one-element set is a Brouwerian complete lattice.*

Theorem 26. *For every ordered set P and each cardinal number α with $1 < \alpha$, the following conditions are equivalent:*

(i) *P is α-Brouwerian;*

(ii) *$DM(P)$ is Brouwerian and P is α-relative-pseudocomplement-closed in it;*

(iii) *P is an α-relative-pseudocomplement-closed doubly dense subset of a complete Brouwerian lattice;*

(iv) *P is an α-relative-pseudocomplement-closed dense subset of an α-Brouwerian lattice.*

Proof. The assertion of the theorem holds for the empty ordered set in view of Observation 25. Suppose that P is a non-empty ordered set.

(i)⇒(ii) was proved in Proposition 24.

(ii)⇒(iii) as P is doubly dense in $DM(P)$.

(iii)⇒(iv) follows a fortiori in view of Proposition 15.

(iv)⇒(i) follows by Proposition 21. ∎

Corollary 27. *For every ordered set P, the following conditions are equivalent:*

(i) *P is Brouwerian;*

(ii) *$DM(P)$ is Brouwerian and P is relative-pseudocomplement-closed in it;*

(iii) *P is a relative-pseudocomplement-closed doubly dense subset of a complete Brouwerian lattice;*

(iv) *P is a relative-pseudocomplement-closed dense subset of a Brouwerian lattice.*

Observation 28. *For every ordered set P and each cardinal number α with $1 < \alpha$, the following conditions are equivalent:*

(i) *P is an α-relative-pseudocomplement-closed dense subset of an α-Brouwerian lattice.*

(ii) *P is an α-relative-pseudocomplement-closed strictly dense subset of an α-Brouwerian lattice.*

Hence we can replace the assumption that P be dense by strictly dense in condition (iv). The fact that $DM(P)$ is Brouwerian whenever P is Brouwerian was proved by Erné in [2], Corollary 3.3.

We can consider a stronger notion of Brouwerianicity.

Theorem 29. *For every ordered set P and each cardinal number α with $1 < \alpha$, the following conditions are equivalent:*

(i) *both P and $G(P)$ are α-Brouwerian;*

(ii) *$G(P)$ is α-Brouwerian and P is α-relative-pseudocomplement-closed in it;*

(iii) *P is an α-relative-pseudocomplement-closed doubly dense generating subset of an α-Brouwerian lattice.*

Proof. The assertion of the theorem holds for the empty ordered set in view of Observation 25. Suppose that P is a non-empty ordered set.

(i)$\Rightarrow$(ii) follows by Proposition 20 as P is doubly dense in $G(P)$.

(ii)$\Rightarrow$(iii) as P is a doubly dense generating subset of $G(P)$.

(iii)$\Rightarrow$(i) follows by Proposition 21 and Lemma 5. ∎

Corollary 30. *For every ordered set P, the following conditions are equivalent:*

(i) *both P and $G(P)$ are Brouwerian;*

(ii) *$G(P)$ is Brouwerian and P is relative-pseudocomplement-closed in it;*

(iii) *P is a relative-pseudocomplement-closed doubly dense generating subset of a Brouwerian lattice.*

Theorem 31. *For every finite ordered set P, the following conditions are equivalent:*

(i) *P is Brouwerian;*

(ii) *$G(P)$ is Brouwerian/distributive and P is relative-pseudocomplement-closed in it;*

(iii) *P is a relative-pseudocomplement-closed doubly dense generating subset of a Brouwerian/distributive lattice;*

(iv) *P is a relative-pseudocomplement-closed dense subset of a Brouwerian/distributive lattice.*

Proof. Since $DM(P)$ is finite and $G(P) = DM(P)$, the assertion of the theorem follows immediately from Theorems 26 and 29. ∎

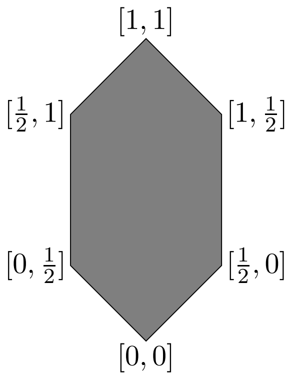
A pseudocomplemented distributive ordered set the characteristic lattice of which is not pseudocomplemented was constructed in [9].

Theorem 32. *There exists a Brouwerian ordered set the characteristic lattice of which is not Brouwerian.*

Proof. Let $\langle 0, 1\rangle$ be the closed interval in $\mathbb{R}$. Put

$$D := \left\{ [a_1, a_2] \in \langle 0,1\rangle \times \langle 0,1\rangle \mid |a_1 - a_2| \leq \frac{1}{2} \right\}$$

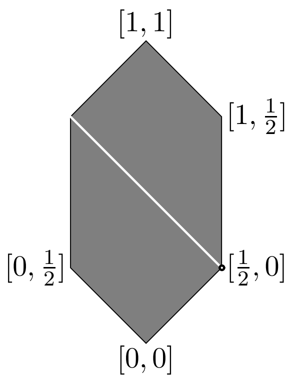
D



and

$$A := \left(D \setminus \left\{ [a_1, a_2] \in D \mid a_1 = \frac{1}{2} \right\} \right) \cup \left\{ \left[\frac{1}{2}, 0 \right] \right\}.$$

A



Both D and A are equipped with the induced order from the cartesian product.

Obviously, D is a subframe in $\langle 0,1\rangle \times \langle 0,1\rangle$, and therefore complete and Brouwerian. Consequently, $D \cong DM(D)$. Further, A is doubly dense in D and therefore $DM(A) \cong D$. We shall verify that A is Brouwerian. In view of Corollary 27, we have to show that A is relative-pseudocomplement closed in D. Let $[a_1, a_2], [b_1, b_2] \in A$. Assume that

$$[a_1, a_2] * [b_1, b_2] = \left[\frac{1}{2}, c\right]$$

and

$$c \neq 0. \tag{1}$$

This implies that

$$a_1 \wedge b_1 = a_1 \wedge \frac{1}{2} \tag{2}$$

$$a_2 \wedge b_2 = a_2 \wedge c \tag{3}$$

$$b_1 \leq \frac{1}{2} \tag{4}$$

$$b_2 \leq c \tag{5}$$

and

$$a_1 \not\leq b_1 \text{ or } a_2 \not\leq b_2. \tag{6}$$

Case $b_1 = \frac{1}{2}$.
Clearly, $b_2 = 0$ and hence $a_2 = 0$ by (1) and (3). Consequently, $a_1 \leq \frac{1}{2}$ which contradicts (6).

Case $b_1 < \frac{1}{2}$.
In virtue of (2), $a_1 \leq b_1 < \frac{1}{2}$. This together with (6) implies that $b_2 < a_2$. Further, $c = b_2$ by (3). To sum up,

$$[a_1, a_2] \wedge \left[\frac{1}{2} + \frac{1}{2}c, c\right] = [a_1, c] = [a_1, a_2] \wedge \left[\frac{1}{2}, c\right]$$

which implies that $c = 0$ contrary to (1).

Now, $[1, \frac{1}{2}] * ([0, \frac{1}{2}] \vee [\frac{1}{2}, 0]) = [1, \frac{1}{2}] * [\frac{1}{2}, \frac{1}{2}] = [\frac{1}{2}, 1]$. It is easy to see that $[\frac{1}{2}, 1]$ is meet-irreducible and join-irreducible in D, and therefore $[\frac{1}{2}, 1] \notin G(A)$. ■

In [10], weakly pseudocomplemented ordered sets were defined for which characteristic lattices are also weakly pseudocomplemented. If we analogously define weakly Brouwerian ordered sets, then we obtain infinitely distributive ordered sets studied in [11].

References

[1] B.A. Davey and H.A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge 1990.

[2] M. Erné, *Distributive laws for concept lattices*, Algebra Universalis **30** (1993), 538–580.

[3] G. Grätzer, *General Lattice Theory*, Akademie-Verlag, Berlin 1978.

[4] R. Halaš, *Pseudocomplemented ordered sets*, Archivum Math. (Brno) **29** (1993), 153–160.

[5] J. Larmerová and J. Rachůnek, *Translations of distributive and modular ordered sets*, Acta Univ. Palack. Olom., Math. **27** (1988), 13–23.

[6] J. Niederle, *Boolean and distributive ordered sets: characterization and representation by sets*, Order **12** (1995), 189–210.

[7] J. Niederle, *Identities in ordered sets*, Order **15** (1999), 271–278.

[8] J. Niederle, *Semimodularity and irreducible elements*, Acta Sci. Math. (Szeged) **64** (1998), 351–356.

[9] J. Niederle, *On pseudocomplemented and Stone ordered sets*, Order **18** (2001), 161–170.

[10] J. Niederle, *On pseudocomplemented and Stone ordered sets, addendum*, Order **20** (2003), 347–349.

[11] J. Niederle, *On infinitely distributive ordered sets*, Math. Slovaca **55** (2005), 495–502.

[12] G. Szász, *Einführung in die Verbandstheorie*, Akadémiai Kiadó, Budapest 1962.

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