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ZERO-TERM RANK PRESERVERS OF INTEGER MATRICES

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Abstract

The zero-term rank of a matrix is the minimum number of lines (row or columns) needed to cover all the zero entries of the given matrix. We characterize the linear operators that preserve the zero-term rank of the $m \times n$ integer matrices. That is, a linear operator T preserves the zero-term rank if and only if it has the form $T(A) = P(A \circ B)Q$, where P, Q are permutation matrices and $A \circ B$ is the Schur product with B whose entries are all nonzero integers.

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1. Introduction and preliminaries

One of the most active and continuing subjects in matrix theory during the past century is the study of those linear operators on matrices that leave certain properties or relations of subsets invariant. The earliest papers in this subject are those of Frobenius and Kantor in 1897 (see [8]). Since much effort has been devoted to this type of problem, there have been several excellent survey papers, e.g., Marcus [6], and Li and Tsing [8].

Integer matrices also have been the subject of research by many authors. For many matrix functions such as rank, determinant and term rank, their preserver operators have been studied by many authors. The concept of term rank and zero-term rank are important in the combinatorial matrix theory. Term rank were studied in the combinatorial matrix theory [4] and permanent theory [7]. Moreover, term rank preserver were studied in [1]. We can consider the concept of zero-term rank of a matrix as the coordinate concept of term rank of a matrix. Johnson and Maybee used this zero-term rank to characterize the inverse zero pattern in [5]. Recently Beasley, Song and Lee [2] obtained characterizations of zero-term rank preservers of matrices over antinegative semiring. Also Beasley, Jun and Song [3] obtained characterizations of zero-term rank preservers of matrices over real fields. But there are few papers on zero-term rank of the integer matrices.

In this article, we obtain characterizations of the linear operators that preserve zero-term rank of integer matrices.

Let $M_{m,n}(\mathbb{Z})$ denote the set of all $m \times n$ matrices with entries in $\mathbb{Z}$, the ring of integers. Let $\mathbb{B}$ ={0,1} be the Boolean algebra. For $A \in M_{m,n}(\mathbb{Z})$, let $\overline{A}$ denote the $m \times n$ matrix with entries in $\mathbb{B}$ such that $\overline{a_{ij}} = 0$ if and only if $a_{ij} = 0$. Let E_{ij} be the matrix in $M_{m,n}(\mathbb{Z})$ which has a 1 in (i,j)entry and is zero elsewhere. We call E_{ij} a *cell.* A *weighted cell* is any nonzero integer multiple of a cell, i.e. αE_{ij} is a weighted cell for any nonzero integer α. Let J denote the $m \times n$ matrix all of whose entries are 1. A matrix A is said to *dominate* a matrix B if $a_{ij} = 0$ implies that $b_{ij} = 0$ and we write $A \geq B$.

The *zero-term rank* [5] of a matrix A, $z(A)$, is the minimum number of lines (row or columns) needed to cover all the zero entries of A. Of course, the *term rank* [1] of A, $t(A)$, is defined similarly for all the nonzero entries of A. Evidently the zero-term rank(or term rank) of a matrix is the zero-term rank(term-rank, respectively) of $\overline{A}$.

A linear operator $T : M_{m,n}(\mathbb{Z}) \to M_{m,n}(\mathbb{Z})$ *preserves zero-term rank* k if $z\,(T(A)) = k$ whenever $z(A) = k$. So a linear operator T *preserves zero-term rank* on $M_{m,n}(\mathbb{Z})$ if it preserves zero-term rank k for every $k \leq min\{m,n\}$.

Which linear operators over $M_{m,n}(\mathbb{Z})$ preserve zero-term rank? The operations of (1) permuting rows, (2) permuting columns and (3) (if $m = n$) transposing the matrices in $M_{m,n}(\mathbb{Z})$ are all linear, zero-term rank preserving operators on $M_{m,n}(\mathbb{Z})$.

If we take a fixed $m \times n$ matrix B in $M_{m,n}(\mathbb{Z})$, all of whose entries are nonzero integers, then its *Schur product* $A \circ B = [a_{ij}b_{ij}]$ with A has the same zero-term rank as does A. The operator $A \longmapsto A \circ B$ is linear. Similarly $A \longmapsto A \circ B$ is linear zero-term rank preserving operator. That these operators and their compositions are the only zero-term rank preservers is one of the consequence of Theorem 2.4 below.

A linear operator $T : M_{m,n}(\mathbb{Z}) \to M_{m,n}(\mathbb{Z})$ is called a (P, Q, B)*-operator* if there exist permutation matrices P and Q, and a matrix B, all of whose entries are nonzero, such that $T(A) = P(A \circ B)Q$ for all $A \in M_{m,n}(\mathbb{Z})$ or if $m = n$, $T(A) = P(A \circ B)^t Q$ for all $A \in M_{m,n}(\mathbb{Z})$.

In [2], the linear operators which preserve zero-term rank of matrices over antinegative semiring were shown to be (P, Q, B)-operators.

We now state the result for later reference.

Theorem 1.1 [2]**.** *Let $\mathbb{S}$ be an antinegative semiring. Suppose T is a linear operator on $M_{m,n}(\mathbb{S})$. Then the following statements are equivalent:*

(i) *T is a (P, Q, B)-operator;*

(ii) *T preserves zero-term rank;*

(iii) *T preserves zero-term rank 1 and $T(J) \geq J$.*

In the followings, we assume that T is a linear operator on $M_{m,n}(\mathbb{Z})$ with $m, n > 1$.

2. ZERO-TERM RANK PRESERVERS ON $M_{m,n}(\mathbb{Z})$

We give a lemma upon which the main theorem will rely.

Lemma 2.1. *If $T : M_{m,n}(\mathbb{Z}) \to M_{m,n}(\mathbb{Z})$ preserves zero-term ranks 0 and 1, then T maps each cell to a nonzero multiple of some cell which induces a bijection on the set of indices $\{1, 2, \cdots, m\} \times \{1, 2, \cdots, n\}$.*

Proof. Since T preserves zero-term rank 0, we have

$$T(J) = V = (v_{ij}) \tag{1}$$

for some $V \in M_{m,n}(\mathbb{Z})$ with $v_{ij} \neq 0$ for all (i,j). If $T(E_{ij}) = 0$, then $T(J - E_{ij}) = T(J)$. But the zero-term rank of $T(J) = V$ is zero while the zero-term rank of $T(J - E_{ij})$ is 1 since T preserves zero-term rank 1. This contradiction implies that

$$T(E_{ij}) \neq 0 \tag{2}$$

for all (i,j). Since the zero-term rank of $T(J - E_{ij})$ is 1, there is some pair (r,s) such that the (r,s) entry of $T(J - E_{ij})$ is zero. Let

$$T(E_{ij}) = U = (u_{hk}). \tag{3}$$

Then $V = T(J) = T(J - E_{ij}) + T(E_{ij})$ and hence $v_{rs} = u_{rs}$. If some nonzero integers u_{hk} and v_{hk} are distinct, then the zero-term rank of $u_{hk}J - v_{hk}E_{ij}$ is zero while the (h,k) entry of $T(u_{hk}J - v_{hk}E_{ij}) = u_{hk}T(J) - v_{hk}T(E_{ij})$ is $u_{hk}v_{hk} - v_{hk}u_{hk} = 0$. This is a contradiction to the fact that T preserves zero-term rank 0. Thus if u_{hk} is not zero, then

$$u_{hk} = v_{hk}. \tag{4}$$

If we put $T(J - E_{ij}) = W = (w_{ij})$, we must have $U + W = V$. Hence if u_{rs} is not zero, then we have $u_{rs} + w_{rs} = v_{rs}$ and hence $v_{rs} + w_{rs} = v_{rs}$ by (4). Thus $w_{rs} = 0$. Since the zero-term rank of $T(J - E_{ij})$ is 1, all the zero entries of W lie in a single row or column. Without loss of generality, we may assume that all zero entries of W and hence all nonzero entries of U lie in row r.

Suppose that $T(E_{ij}) = U = (u_{hk})$ and $T(E_{cd}) = G = (g_{ef})$ with $(i,j) \neq (c,d)$. If the (r,s) entries of both U and G are not zero, then $v_{rs} = u_{rs} = g_{rs}$ by (4) and hence $T(E_{ij} + E_{cd})$ has (r,s) entry $2v_{rs}$, which is not zero. Hence $T(2J - E_{ij} - E_{cd})$ has zero in the (r,s) entry and then $z(T(2J - E_{ij} - E_{cd})) \geq 1$ while $z(2J - E_{ij} - E_{cd}) = 0$, which is a contradiction to the fact that T preserves zero-term rank 0. Since $T(E_{ij}) \neq 0$ for all (i,j) by (2), $T(E_{ij})$ must be a nonzero multiple of one cell by pigeon hole principle. Since $T(J) = V$ has zero-term rank 0, the mapping T must induce a bijection on the set of indices $\{1, 2, \cdots, m\} \times \{1, 2, \cdots, n\}$. ∎

If $T : M_{m,n}(\mathbb{Z}) \to M_{m,n}(\mathbb{Z})$ is a linear operator, define $\overline{T} : M_{m,n}(\mathbb{B}) \to M_{m,n}(\mathbb{B})$ by $\overline{T}(\overline{A}) = \sum_{i=1}^{m} \sum_{j=1}^{n} \overline{T(a_{ij}E_{ij})}$ for any $A \in M_{m,n}(\mathbb{Z})$.

Proposition 2.2. *If $T : M_{m,n}(\mathbb{Z}) \to M_{m,n}(\mathbb{Z})$ preserves zero-term ranks* 0 *and* 1*, then we have the following:*

(1) *T preserves term rank* 1*;*

(2) *T maps a row into a row (or a column if $m = n$);*

(3) *For $m = n$, if T maps a row into a row(or a column), then all rows must be mapped to some rows(columns, respectively) under T;*

(4) *T preserves term rank k for $k \geq 2$.*

Proof. (1) Suppose that T does not preserve term rank 1. Then there exist some distinct cells E_{ij} and E_{il} on the same row(or column) such that

$$T(E_{ij} + E_{il}) = b_{ij}E_{rs} + b_{il}E_{pq}$$

with $p \neq r$ and $q \neq s$ by Lemma 2.1. So the zero-term rank of $J - E_{ij} - E_{il}$ is 1. But the zero-term rank of its image is 2 since

$$\begin{aligned} z(T(J - E_{ij} - E_{il})) &= z(\overline{T(J - E_{ij} - E_{il})}) \\ &= z(J - E_{rs} - E_{pq}) \\ &= 2\,. \end{aligned}$$

This shows that T does not preserve zero-term rank 1, which is a contradiction. Hence T preserves term rank 1.

(2) Suppose T does not map a row into a row(or a column if $m = n$). Then T does not preserve term rank 1. This contradicts (1).

(3) Let R_i and C_j denote the ith row and jth column respectively. If $T(R_1) \subseteq R_i$ but $T(R_2) \subseteq C_j$ then $R_1 + R_2$ has $2n$ cells but $R_i + C_j$ has $2n - 1$ cells. This contradicts the bijectivity of the corresponding map of T on the set of indices $\{1, 2, \cdots, m\} \times \{1, 2, \cdots, n\}$ by Lemma 2.1. Thus we have the result.

(4) Since T maps a row into a row(or a column if $m = n$) by (2). T does not increase term rank of a matrix. Suppose there exists a matrix X such that $t(X) = k$ and $t(T(X)) < k$. Since the corresponding map of T on indices is bijective by Lemma 2.1, we can take a 2×2 submatrix A of X such that $A = a_1 E_{ij} + a_2 E_{kl}$ but $T(A)$ has term rank 1, where a_1, a_2 are nonzero integers and $i \neq k, j \neq l$. Then T maps two rows into one row (or one column if $m = n$), which is a contradiction. Therefore T preserves term rank k. ■

Theorem 2.3. *If $T : M_{m,n}(\mathbb{Z}) \to M_{m,n}(\mathbb{Z})$ preserves zero-term ranks 0 and 1, then T is a (P, Q, B)-operator.*

Proof. From the Lemma 2.1, $\overline{T}$ is bijective on the set of cells in $M_{m,n}(\mathbb{B})$. Thus for any $A \in M_{m,n}(\mathbb{Z})$,

$$\overline{T(A)} = \overline{\sum_{i=1}^{m}\sum_{j=1}^{n} T(a_{ij}E_{ij})} = \sum_{i=1}^{m}\sum_{j=1}^{n} \overline{T(a_{ij}E_{ij})} = \overline{T}(\overline{A}).$$

Thus, since T preserves zero-term rank 1, we have that $\overline{T}$ does also. By Theorem 1.1, $\overline{T}$ is a (P, Q, B)-operator on $M_{m,n}(\mathbb{B})$, where $B = J$. Thus the mapping $\overline{A} \mapsto \overline{P^t T(A) Q^t}$ is the identity linear operator on $M_{m,n}(\mathbb{B})$. That is, $P^t T(E_{ij}) Q^t = b_{ij} E_{ij}$ for some pair (i, j)(or perhaps $P^t T(E_{ij}) Q^t = b_{ij} E_{ji}$ in case $m = n$). Then, $T(A) = P(A \circ B)Q$ for all $A \in M_{m,n}(\mathbb{Z})$ or $m = n$ and $T(A) = P(A \circ B)^t Q$ for all $A \in M_{m,n}(\mathbb{Z})$. ■

Theorem 2.4. *For a linear operator T on $M_{m,n}(\mathbb{Z})$, the following are equivalent:*

(i) *T preserves zero-term ranks 0 and 1;*

(ii) *T is a (P, Q, B)-operator;*

(iii) *T preserves zero-term rank.*

Proof. Proposition 2.2 shows that (i) implies (ii). Obviously (ii) implies (iii) and (iii) implies (i). ■

Thus we had characterizations of the linear operators that preserve zero-term rank of integer matrices.

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