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IMPLICATION ALGEBRAS

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Abstract

We introduce the concepts of pre-implication algebra and implication algebra based on orthosemilattices which generalize the concepts of implication algebra, orthoimplication algebra defined by J.C. Abbott [2] and orthomodular implication algebra introduced by the author with his collaborators. For our algebras we get new axiom systems compatible with that of an implication algebra. This unified approach enables us to compare the mentioned algebras and apply a unified treatment of congruence properties.

Keywords: implication algebra, pre-implication algebra, orthoimplication algebra, orthosemilattice, congruence kernel.

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1. Introduction

J.C. Abbott [1] introduced a concept of implication algebra in the sake to formalize the logical connective implication in the classical propositional logic. An *implication algebra* is a groupoid $\mathcal{A} = (A; \cdot)$ satisfiying the axioms

(I1) $(x \cdot y) \cdot x = x$ (*contraction*)

(I2) $(x \cdot y) \cdot y = (y \cdot x) \cdot x$ (*quasi-commutativity*)

(I3) $x \cdot (y \cdot z) = y \cdot (x \cdot z)$ (*exchange*).

The binary operation clearly formalizes the connective implication ($x \Rightarrow y$ is denoted by $x \cdot y$) and, as pointed out in [1], every property of the classical implication can be derived by using of (I1), (I2) and (I3). Among other results, J.C. Abbott [1] showed that in every implication algebra a binary relation $\leq$ can be introduced by the setting $x \leq y$ if $x \cdot y = 1$ where 1 is an algebraic constant of $\mathcal{A}$ given by the derived identity $x \cdot x = y \cdot y$. Futher, $(A; \leq)$ is an ordered set with a greatest element 1 which becomes a $\vee-$ semilattice with respect to the induced order $\leq$. Moreover, $x \vee y = (x \cdot y) \cdot y$ and for every $p \in A$, the interval $[p, 1]$ is a Boolean algebra where the complement a^p in $[p, 1]$ of an element $a \in [p, 1]$ is $a^p = a \cdot p$.

Conversely, having a $\vee-$ semilattice with a greatest element 1 where for any element p the interval $[p, 1]$ is a Boolean algebra, one can define $x \cdot y = (x \vee y)^y$ (the complement of $x \vee y$ in the interval $[y, 1]$) and the resulting groupoid is an implication algebra in the sense of [1].

Several attempts were made to generalize the concept of the implication algebra for nonclassical logics. G.M. Hardegree [6, 7] did it for logics suitable for quantum mechanics and based on orthomodular lattices. His so-called quasi-implication algebra is presented by the axioms

$$(x \cdot y) \cdot x = x \qquad \text{(which is (I1))}$$

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

$$((x \cdot y) \cdot (y \cdot x)) \cdot x = ((y \cdot x) \cdot (x \cdot y)) \cdot y.$$

This quasi-implication algebra induces a $\vee$-semilattice with 1 where every interval $[p, 1]$ is an orthomodular lattice but the converse construction is possible only if it has also a least element. Another attempt was done by J.C. Abbott [2]. He obtained a very simple axiom system (compatible with that of implication algebra) which works under the so-called compatibility condition. A similar implication algebra working without the compatibility condition was treated by the author and his collaborators in [5]. When ortholattices are considered instead of orthomodular ones, we derive an orthoimplication algebra, see [4]. In fact, the Hardegree quasi-implication algebra is based on the so-called "Sasaki implication" and that of Abbott [2] on "Dishkant implication." The comparision of both approaches can be found in the paper by Norman D. Megill and Mladen Pavičić [10]. An outstanding feature of the Hardegree's system is that it is complete but this has not been proven for the Abbott's system. A detailed description of these concepts can be also found in [10].

The aim of this paper is to unify our approach from [4, 5] toward that of [1] which enables us to study congruence properties common for all of these implication algebras.

In the sake of brevity, we will write often xy instead of $x \cdot y$.

2. Orthoimplication algebras

Definition 1. A groupoid $\mathcal{A} = (A; \cdot)$ satisfying the identities (I1) (contraction) and (I2) (quasi-commutativity) will be called a *pre-implication algebra.*

Hence, an implication algebra is a pre-implication algebra satisfying the exchange identity.

Theorem 1. *Let $\mathcal{A} = (A; \cdot)$ be a pre-implication algebra. Then $\mathcal{A}$ satisfies the identity $x{\cdot}x = y{\cdot}y$, i.e., there is an algebraic constant 1 such that $x{\cdot}x = 1$ and $\mathcal{A}$ satisfies the identities* $\quad 1 \cdot x = x, \quad x \cdot 1 = 1 \quad$ *and* $\quad x \cdot (x \cdot y) = x \cdot y.$

Proof. Applying (I1) twice, we get

$$x(xy) = ((xy)x)(xy) = xy. \tag{A}$$

Applying (I1), (I2) and this identity, we have

$$xx = ((xy)x)x = (x(xy))(xy) = (xy)(xy).$$

Hence, using this and (I2),

$$xx = (xy)(xy) = ((xy)y)((xy)y) = ((yx)x)((yx)x = (yx)(yx) = yy$$

thus there is a constant, say 1, such that $x \cdot x = 1$ is an identity of $\mathcal{A}$.
Using (I1) and (A), we get $1 \cdot x = (xx)x = x \quad$ and $\quad x \cdot 1 = x(xx) = xx = 1.$ ∎

Lemma 2. *Let $\mathcal{A} = (A; \cdot)$ be a pre-implication algebra. Introduce a binary relation $\leq$ on A as follows:*

$$x \leq y \quad \textit{if and only if} \quad x \cdot y = 1.$$

Then $\leq$ is reflexive, antisymmetrical and $x \leq 1$ for every $x \in A$.

Proof. By Theorem 1, $x \cdot x = 1$ for each $x \in A$ thus $\leq$ is reflexive. Suppose $x \leq y$ and $y \leq x$. Then

$$x \cdot y = 1,\ y \cdot x = 1 \text{ and } x = 1 \cdot x = (yx)x = (xy)y = 1 \cdot y = y,$$

i.e., it is antisymmetric. Finally, $x \cdot 1 = 1$ gives $x \leq 1$ for each $x \in A$. ■

In what follows, we will change several names of implication algebras already introduced. We are motivated by the facts that the original names do not correspond to the assigned lattice properties.

For example, the name orthoimplication algebra was used by J.C. Abbott [2] for the implication algebra derived from an orthomodular lattice. We are going to show that our orthoimplication algebra defined below corresponds really to an orthosemilattice and hence that of Abbott will be called an orthomodular implication algebra in the sequel.

Definition 2. By an *orthoimplication algebra* is meant a pre-implication algebra satisfying the identity

$$(((xy)y)z)(xz) = 1 \quad (\textit{antitony identity}).$$

Lemma 3. *Let $\mathcal{A} = (A, \cdot)$ be an orthoimplication algebra and $\leq$ is defined as in Lemma 2. Then $\leq$ is an order on A and for every $a, b, c \in A$,*

$$a \leq b \textit{ implies } bc \leq ac \quad (\textit{antitony of } \leq).$$

Proof. By Lemma 2, we need only to show transitivity of $\leq$. Suppose $x \leq y$ and $y \leq z$. Then $xy = 1$, $yz = 1$ and

$$xz = 1(xz) = (yz)(xz) = ((1y)z)(xz) = (((xy)y)z)(xz) = 1$$

due to the antitony identity. Thus $x \leq z$ and hence $\leq$ is an order on A.

Suppose $a, b, c \in A$ and $a \leq b$. Then $ab = 1$ and hence $(ab)b = 1 \cdot b = b$. This yields

$$(bc)(ac) = (((ab)b)c)(ac) = 1$$

by the antitony identity thus we have $bc \leq ac$. ■

Call $\leq$ the *induced order* of $\mathcal{A} = (A, \cdot)$.

Corollary 4. *Every orthoimplication algebra satisfies the identity* $x(yx) = 1$.

Proof. By Lemma 2, $y \leq 1$ and, due to Lemma 3, $x = 1 \cdot x \leq y \cdot x$ whence $x(yx) = 1$. ■

Lemma 5. *Let* $\mathcal{A} = (A, \cdot)$ *be an orthoimplication algebra. Then*

(a) $a \leq (ab)b, \quad b \leq (ab)b$;

(b) *if* $a, b \leq c$ *then* $(ab)b \leq c$.

Proof. (a) Applying quasi-commutativity and Corollary 4, we derive

$$a((ab)b) = a((ba)a) = 1 \quad \text{thus} \quad a \leq (ab)b,$$

analogously $b((ab)b) = 1$ yields $b \leq (ab)b$.

Suppose $a, b \leq c$. Then, by Lemma 3, $cb \leq ab$ and hence $(ab)b \leq (cb)b = (bc)c = 1 \cdot c = c$. ■

Corollary 6. *Let* $\mathcal{A} = (A, \cdot)$ *be an orthoimplication algebra. Then* $a \vee b = (ab)b$ *is the supremum of* a, b *w.r.t. the induced order* $\leq$.

Remark. This justifies the name *antitony identity.* Since $x \leq x \vee y$, the identity properly says $(x \vee y)z \leq xz$ which means a right antitony of $\leq$.

Theorem 7. *Let* $\mathcal{A} = (A, \cdot)$ *be an orthoalgebra. Then* $\mathcal{A}$ *is a* $\vee-$*semilattice with a greatest element* 1 *with respect to the induced order. For each* $p \in A$, *the interval* $[p, 1]$ *is an ortholattice where the orthocomplement* a^p *of* $a \in [p, 1]$ *in this interval is* $a^p = a \cdot p$.

Proof. By Lemma 2 and Corollary 6, $\mathcal{A}$ becomes a $\vee-$semilattice with a greatest element 1 where $a \vee b = (ab)b$. Let $p \in A$ and $a \in [p, 1]$. By Lemma 2 and Lemma 3, $a \leq 1$ and hence $a^p = ap \geq 1p = p$ thus $a^p \in [p, 1]$ and, due to Lemma 3, the mapping $a \mapsto a^p$ is antitone. Moreover, $a^{pp} = (ap)p = a \vee p = a$ thus it is an antitone involution, i.e., a bijection of $[p, 1]$ onto itself with $p^p = 1$ and $1^p = p$ and hence for $a, b \in [p, 1]$

$$(a^p \vee b^p)^p \quad \text{is the infimum of} \quad a, b$$

with respect to $\leq$, i.e., $a \wedge_p b = (a^p \vee b^p)^p$ is the operation meet in $[p,1]$ and $([p,1], \vee, \wedge_p)$ is a lattice. Futher,

$$a \vee a^p = a \vee (ap) = ((ap)a)a = aa = 1$$

thus $a \wedge_p a^p = (a^p \vee a)^p = 1^p = p$, i.e., a^p is an orthocomplement of a in $[p,1]$. ∎

We are going to introduce a concept of orthosemilattice which is a generalization of the concept of generalized orthomodular lattice introduced in [9], see also [8].

Definition 3. A semilattice $(A, \vee)$ with a greatest element 1 is called an *orthosemilattice* if for each $p \in A$ the interval $[p,1]$ is an ortholattice with respect to the induced order.

Let $(A, \vee)$ be an orthosemilattice, $p \in A$ and $a \in [p,1]$. From now on, we will denote by a^p the orthocomplement of a in $[p,1]$.

It means that in every orthosemilattice $(A, \vee)$ a set of mappings $a \mapsto a^p (p \in A)$ is given, each defined on an interval $[p,1]$ such that it is an orthocomplementation on $[p,1]$.

We are going to prove the converse of Theorem 2. As noted in the introduction, such a converse was proved by J.C. Abbott for implication algebras and the so-called semi-Boolean algebras but an analogous result was not reached by Hardegree [6] in the general case.

Theorem 8. *Let $(A, \vee)$ be an orthosemilattice. Define the binary operation $x \cdot y = (x \vee y)^y$. Then $\mathcal{A} = (A, \cdot)$ is an orthoimplication algebra.*

Proof. Let $x, y, z \in A$. Then

$$\begin{aligned}
(xy)x &= ((x \vee y)^y \vee x)^x = ((x \vee y)^y \vee (x \vee y))^x = 1^x = x;\\
(xy)y &= ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y = y \vee x\\
&= (y \vee x)^{xx} = ((y \vee x)^x \vee x)^x = (yx)x;\\
(((xy)y)z)(xz) &= ((x \vee y)z)(xz) = ((x \vee y \vee z)^z \vee (x \vee z)^z)^{(x \vee y)^z}\\
&= ((x \vee z)^z)^{(x \vee z)^z} = 1.
\end{aligned}$$

∎

Remark. We have set up an uniform approach to implication algebras. Namely, if a pre-implication algebra satisfies:

(i) the *exchange identity* then it is an implication algebra which corresponds to a $\vee$-semilattice with 1 where every interval $[p, 1]$ is a Boolean algebra (see [1]);

(ii) the identity $x((yx)z) = xz$ then it is an orthomodular implication algebra which corresponds to a $\vee$-semilattice with 1 where every interval $[p, 1]$ is an orthomodular lattice (see [2]);

(iii) the *antitony identity* then it is an orthoimplication algebra which corresponds to a $\vee$-semilattice with 1 where every interval $[p, 1]$ is an ortholatice.

It is worth noticing that in the both Abbott's cases (i) and (ii), the axioms hold on a Boolean algebra or an orthomodular lattice, respectively. Contrary to this, the axioms of an orthoimplication algebra holds on an orthosemilattice as shown above but not on an ortholattice in general. In particular, the second axiom of the orthoimplication algebra fails e.g. on the hexagon O_6.

3. Congruence properties

The uniform approach enables us to investigate several congruence properties in the same way. Let us note that some of them were already treated in [1] or [4] for particular cases.

Since pre-implication algebras are defined by the identities (I1) and (I2), the class $\mathcal{V}$ of all pre-implication algebras forms a variety which includes a subvariety of orthoimplication algebras and it has a subvariety of orthomodular implication algebras and its subvariety is a variety of implication algebras.

Denote by $Con\mathcal{A}$ the congruence lattice of an algebra $\mathcal{A}$; i.e. $Con\mathcal{A}$ is the set of all congruences on $\mathcal{A}$ ordered by set inclusion where the operation meet coincides with set intersection.

Recall that an algebra $\mathcal{A}$ is called *congruence distributive* if the congruence lattice $Con\mathcal{A}$ is distributive; further, $\mathcal{A}$ is *congruence 3-permutable* if $\Theta \circ \Phi \circ \Theta = \Phi \circ \Theta \circ \Phi$ for every Θ, $\Phi \in Con\mathcal{A}$. If it is the case then, of course, $\Theta \vee \Phi = \Theta \circ \Phi \circ \Theta$ holds in $Con\mathcal{A}$. A variety $\mathcal{V}$ is *congruence distributive* or *congruence 3-permutable* if every member $\mathcal{A} \in \mathcal{V}$ has the corresponding property.

Theorem 9. *The variety $\mathcal{V}$ of pre-implication algebras is congruence distributive.*

Proof. It is well-known that $\mathcal{V}$ is congruence distributive if and only if there exist so-called Jónsson terms, i.e., ternary terms $t_0, \ldots, t_n$ such that $t_0(x,y,z) = x$, $t_n(x,y,z) = z$, $t_i(x,y,x) = x$ for each $i \in \{0, \ldots, n\}$ and $t_i(x,x,z) = t_{i+1}(x,x,z)$ for i even and $t_i(x,z,z) = t_{i+1}(x,z,z)$ for i odd.

Take $n = 3$, $t_0(x,y,z) = x$, $t_3(x,y,z) = z$ and

$$t_1(x,y,z) = (y(zx))x, \quad t_2(x,y,z) = (xy)z.$$

Then clearly $t_0(x,y,x) = t_3(x,y,x) = x$,

$$t_1(x,y,x) = (y(xx))x = (y1)x = 1x = x,$$

$$t_2(x,y,x) = (xy)x = x.$$

For i even, $t_0(x,x,z) = x = (x(zx))x = t_1(x,x,z)$,

$$t_2(x,x,z) = (xx)z = 1z = z = t_2(x,x,z).$$

For i odd, $t_1(x,z,z) = (z(zx))x = (zx)x = (xz)z = t_2(x,z,z)$. ■

Theorem 10. *The variety $\mathcal{V}$ of pre-implication algebras is congruence 3-permutable.*

Proof. It is well-known that this congruence condition is characterized by the existence of ternary terms t_1, t_2 satisfying the identities

$$t_1(x,z,z) = x, \;\; t_1(x,x,z) = t_2(x,z,z), \;\; t_2(x,x,z) = z.$$

One can consider $t_1(x,y,z) = (zy)x$ and $t_2(x,y,z) = (xy)z$.

It is an easy excercise to verify the desired identities. ■

Let $\Theta \in \mathcal{A}$ where $\mathcal{A}$ is a pre-implication algebra. The congruence class $[1]_\Theta$ will be called a *congruence kernel* of Θ.

Theorem 11. *Let $\mathcal{A}$ be a pre-implication algebra. Then every congruence on $\mathcal{A}$ is determined by its kernel, i.e., if $\Theta, \Phi \in Con\mathcal{A}$ and $[1]_\Theta = [1]_\Phi$ then $\Theta = \Phi$.*

Proof. Suppose $\Theta, \Phi \in Con\mathcal{A}$ with $[1]_\Theta = [1]_\Phi$. Let $\langle x, y\rangle \in \Theta$. Then $\langle xy, 1\rangle = \langle xy, yy\rangle \in \Theta$ thus $xy \in [1]_\Theta = [1]_\Phi$, i.e., $\langle xy, 1\rangle \in \Phi$ and hence $\langle (xy)y, y\rangle = \langle (xy)y, 1y\rangle \in \Phi$. Analogously it can be shown $\langle (yx)x, x\rangle \in \Phi$. Since $(xy)y = (yx)x$, we conclude $\langle x, y\rangle \in \Phi$ proving $\Theta \subseteq \Phi$. The converse inclusion can be shown analogously. ■

Theorem 11 gives rise to rhe natural question how to characterize congruence kernels in pre-implication algebras. We will do this in two different ways, i.e., by the so-called deductive system and by a closedness with respect to the corresponding terms.

Definition 4. Let $\mathcal{A} = (A, \cdot)$ be a pre-implication algebra. A subset $I \subseteq A$ with $1 \in I$ is called a *deductive system* of $\mathcal{A}$ if it satisfies the conditions

(d1) $x \in I$ and $yz \in I$ imply $(xy)z \in I$;

(d2) $xy \in I$ and $yx \in I$ imply $(xz)(yz) \in I$ and $(zx)(zy) \in I$.

Remark. Take $x = y$ in (d1), one obtains

(MP) $x \in I$ and $xz \in I$ imply $z \in I$

which is a rule analogous to Modus Ponens in the deductive logic. It justifies the name deductive system for such a subset.

Lemma 12. *Let $\mathcal{A} = (A, \cdot)$ be a pre-implication algebra. If $I \subseteq A$ is a congruence kernel then it is a deductive system of $\mathcal{A}$.*

Proof. Suppose $I = [1]_\Theta$ for some $\Theta \in Con\mathcal{A}$. If $x \in I$ and $yz \in I$ then $\langle x, 1\rangle \in \Theta$ and $\langle yz, 1\rangle \in \Theta$ thus $(xy)z \;\Theta\; (1y)z = yz \;\Theta\; 1$ giving $\langle (xy)z, 1\rangle \in \Theta$, i.e., $(xy)z \in I$. Hence, I satisfies (d1).

If $xy \in I$ and $yx \in I$ then, similarly as in the proof of Theorem 11, one can derive $\langle x, y\rangle \in \Theta$. Hence also $\langle xz, yz\rangle \in \Theta$ and $\langle zx, zy\rangle \in \Theta$ which yields $(xz)(yz) \in I$ and $(zx)(zy) \in I$ proving that I satisfies (d2). ■

Theorem 13. *Let $\mathcal{A} = (A; \cdot)$ be an orthoimplication algebra. Then $I \subseteq A$ is a congruence kernel if and only if I is a deductive system; then it is a kernel of Θ_I given by the setting*

$$\text{(S)} \qquad \langle x, y\rangle \in \Theta_I \quad \text{if} \quad xy \in I \quad \text{and} \quad yx \in I.$$

Proof. Define Θ_I on A by (S). If $x \in I$ then $1 \cdot x = x \in I$ and $x \cdot 1 = 1 \in I$ thus $\langle x, 1\rangle \in \Theta_I$. If $\langle x, 1\rangle \in \Theta_I$ then $x = 1 \cdot x \in I$, hence $I = [1]_{\Theta_I}$. All we need to show is that $\Theta_I \in Con\mathcal{A}$. Evidently, Θ_I is reflexive and symmetrical.

Suppose $\langle x, y\rangle \in \Theta_I$ and $\langle y, z\rangle \in \Theta_I$. Then $xy \in I, yz \in I$ and, by (d1), also $((xy)y)z \in I$. By the antitony identity, $(((xy)y)z) \cdot (xz) = 1 \in I$. Applying (MP) (see the foregoing Remark) we conclude $xz \in I$. Analogously it can be shown $zx \in I$, i.e. $\langle x, z\rangle \in \Theta_I$. Hence, Θ_I is transitive, i.e. an equivalence on A.

Suppose $\langle a, b\rangle \in \Theta_I$ and $\langle c, d\rangle \in \Theta_I$. By using of (d2) we easily derive $\langle ac, bd\rangle \in \Theta_I$ and $\langle bc, bd\rangle \in \Theta_I$. Due to transitivity of Θ_I, one has $\langle ac, bd\rangle \in\in \Theta_I$. Thus $\Theta_I \in Con\mathcal{A}$. We have shown that every deductive system I is a kernel of the congruence Θ_I. The converse implication follows by Lemma 12. ∎

Definition 5. Let $\mathcal{A} = (A; \cdot)$ be a pre-implication algebra and $t(x_1, \dots, x_n, y_1, \dots, y_m)$ be a term functions of $\mathcal{A}$ (in two sorts of variables). A subset $B \subseteq A$ is said to be *y-closed with respect to t* if $t(a_1, \dots, a_n, b_1, \dots, b_m) \in B$ for every $b_1, \dots, b_m \in B$ and $a_1, \dots, a_n \in A$.

Lemma 14. *Let $\mathcal{A} = (A; \cdot)$ be a pre-implication algebra and $t(x_1, \dots, x_n, y_1, \dots, y_m)$ be a term function of $\mathcal{A}$ such that $t(x_1, \dots, x_n, 1, \dots, 1) = 1$ is an identity of $\mathcal{A}$. Let $I \subseteq A$ be a congruence kernel. Then I is y-closed w.r.t. t.*

Proof. Suppose $I = [1]_\Theta$ for some $\Theta \in Con\mathcal{A}$. Let $a_1, \dots, a_n \in A, b_1, \dots, b_m \in I$. Then $\langle b_i, 1\rangle \in \Theta$ for $i = 1, \dots, m$ and hence

$$t(a_1, \dots, a_n, b_1, \dots, b_m) \; \Theta \; t(a_1, \dots, a_n, 1, \dots, 1) = 1$$

proving $t(a_1, \dots, a_n, b_1, \dots, b_m) \in [1]_\Theta = I$. ∎

Notation. In what follows, we will fix the following term functions of pre-implication algebras:

$$t_1(x,y) = x \cdot y$$

$$t_2(x_1,x_2,y_1,y_2) = (x_1x_2)[y_2((y_1x_1)x_2)]$$

$$t_3(x_1,x_2,y) = (x_1x_2)(x_1(yx_2))$$

$$t_4(x_1,x_2,x_3,y) = [(x_1x_2)(x_1(yx_3))] \cdot [(x_1x_2)(x_1x_3)]$$

$$t_5(x_1,x_2,x_3,y) = [(x_1x_2)((yx_3)x_2)] \cdot [(x_1x_2)(x_3x_2)].$$

It is immediately clear that $t_i(x_1,\dots,x_n,1,\dots,1) = 1$ for $i = 1,\dots,5$ and, by Lemma 14, every congruence kernel of each pre-implication algebra is y-closed w.r.t. $t_1,\dots,t_5$. We are going to prove also the converse:

Theorem 15. *Let $\mathcal{A} = (A;\cdot)$ be an orthoimplication algebra. Then $I \subseteq A$ with $1 \in I$ is a congruence kernel if and only if I is y-closed with respect to $t_1,\dots,t_5$.*

Proof. Due to Lemma 14 and Theorem 13, we need only to show that if $I \subseteq A$ with $1 \in I$ is y-closed with respect to $t_1,\dots,t_5$ then I is a deductive system of $\mathcal{A}$.

Take $x_1 = x_2 = x$ in t_2, we obtain $t(x,y_1,y_2) = (y_2((y_1x))x$. Of course, if I is y-closed w.r.t. t_2 then I is also y-closed w.r.t. t. Suppose $a \in I$ and $ab \in I$. Then $b = 1 \cdot b = ((ab)(ab))b = t(b,a,ab) \in I$ thus I satisfies the condition (MP).

Suppose now $x \in I$. Then

$$\text{(B)} \qquad \begin{aligned} &(xy)y = (yx)x = t_1(yx,x) \in I \quad \text{and} \\ &(yz)[((xy)y)((xy)z)] = t_2(y,z,(xy)y) \in I. \end{aligned}$$

Suppose now $x \in I$ and $yz \in I$. By using of (MP), we derive from (B) also $((xy)y)((xy)z) \in I$ and, due to $(xy)y \in I$, we obtain $(xy)z \in I$. Hence, I satisfies (d1).

Now, let $xy \in I$ and $yx \in I$. Then

$$(zx)[z((yx)x)] = t_3(z, x, yx) \in I \quad \text{and}$$

$$[(zx)(z((xy)y] \cdot [(zx)(zy)] = t_4(z, x, y, xy) \in I.$$

Since $(xy)y = (yx)x$, we apply (MP) to conclude $(zx)(zy) \in I$.

Further, $(xz)(((yx)x)z) = t_2(x, z, yx, 1) \in I$ and $(yx)x = (xy)y$ thus also $(xz)(((xy)y)z) \in I$ and

$$[(xz)(((xy)y)z)]((xz)(yz)) = t_5(x, z, y, xy) \in I.$$

Due to (MP), we conclude $(xz)(yz) \in I$. Hence, I satisfies also (d2) and by Theorem 13, I is a congruence kernel. ■

Remark. In spite of Theorem 11, I is a congruence kernel in an orthoimplication algebra $\mathcal{A}$ if and only if it is the kernel of Θ_I i.e. if $I = [1]_\Theta$ z then

$$\langle x, y\rangle \in \Theta \quad \text{if and only if} \quad xy \in I \quad \text{and} \quad yx \in I.$$

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