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FOLDNESS OF COMMUTATIVE IDEALS IN BCK-ALGEBRAS

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Abstract

This paper deals with some properties of n-fold commutative ideals and n-fold weak commutative ideals in BCK-algebras. Afterwards, we construct some algorithms for studying foldness theory of commutative ideals in BCK-algebras.

Keywords: BCK-algebra, fuzzy point, n-fold commutative ideals, n-fold weak commutative ideals.

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1. Introduction

The concept of fuzzy subset was introduced in the middle of the sixties by Zadeh [16]. He defined a fuzzy subset of a set X as a function $A : X \longrightarrow [0, 1]$. Based on this definition, Xi [15] introduced in 1991 the notion of fuzzy ideals in BCK-algebras. This work enlightened on the usefulness of ideals theory in general development of BCI/BCK/BL/MV-algebras. From logical point of view, various ideals correspond to various sets of provable formula, see [2, 3, 4, 10, 11, 12] and the references therein.

The tricky point when studying fuzzy mathematics lies in how to carry out the ordinary concept to the fuzzy case. In other words, how to pick out the rational generalization from the large number of available approaches. The particularity of fuzzy ideals compared to ordinary ideals is that one can not say which one of the BCK-algebra elements belongs (or not) to the fuzzy ideals under consideration.

In this paper we study the foldness theory of commutative ideals in BCK-algebras. This theory can be considered as a natural generalization of commutative ideals. Indeed, given any BCK-algebra X, we use the concept of fuzzy point to characterize n-fold commutative ideals in X.

The remainder of this paper is as follows: In Section 2, we recall some important properties of BCK-algebras and their ideals. In Sections 3 and 4 we give some characterizations of n-fold commutative ideals and n-fold weak commutative ideals. Finally, we construct some algorithms for studying n-fold commutative, n-fold weak commutative ideals and their fuzzification in BCK-algebras.

2. Background

For some background information see ([1, 2, 5, 10]). An algebra $(X, *, 0)$ of type $(2, 0)$ is called BCK-algebra iff $\forall\ x, y, z \in X$ the following conditions hold:

BCK-1. $((x * y) * (x * z)) * (z * y) = 0;$

BCK-2. $(x * (x * y)) * y = 0;$

BCK-3. $x * x = 0;$

BCK-4. $0 * x = 0;$

BCK-5. $x * y = 0 \textit{ and } y * x = 0 \Longrightarrow x = y.$

A binary relation $\leq$ can be defined on X by

BCK-6. $x \leq y \Longleftrightarrow x * y = 0,$

then $(X, \leq)$ is a partially ordered set with the least element 0.

The following properties also hold in any BCK-algebra ([1, 10, 14, 15]):

1. $x * 0 = x$;
2. $x * y = 0$ and $y * z = 0 \Longrightarrow x * z = 0$;
3. $x * y = 0 \Longrightarrow (x * z) * (y * z) = 0$ and $(z * y) * (z * x) = 0$;
4. $(x * y) * z = (x * z) * y$;
5. $(x * y) * x = 0$;
6. $x * (x * (x * y)) = x * y$.

Let $(X, *, 0)$ be a BCK-algebra.

A fuzzy subset of a BCK-algebra X is a function

$$\mu : X \longrightarrow [0, 1].$$

Let ξ be the family of all fuzzy sets in X. For $x \in X$ and $\lambda \in (0, 1]$, $x_\lambda \in \xi$ is a fuzzy point iff

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } \ x = y, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\tilde{X} = \{x_\lambda : x \in X, \lambda \in (0, 1]\}$ the set of all fuzzy points on X and we define a binary operation on $\tilde{X}$ as follows:

$$x_\lambda * y_\mu = (x * y)_{\min(\lambda,\mu)}.$$

It is easy to verify that $\forall$ x_λ, y_μ, $z_\alpha \in \tilde{X}$, the following conditions hold:

BCK-1'. $((x_\lambda * y_\mu) * (x_\lambda * z_\alpha)) * (z_\alpha * y_\mu) = 0_{\min(\lambda,\mu,\alpha)}$;

BCK-2'. $[x_\lambda * (x_\lambda * y_\mu)] * y_\mu = 0_{\min(\lambda,\mu)}$;

BCK-3'. $x_\lambda * x_\mu = 0_{\min(\lambda,\mu)}$;

BCK-4'. $0_\mu * x_\lambda = 0_{\min(\lambda,\mu)}$.

Remark 2.1. The condition BCK-5. is not true in $(\tilde{X},*)$. So the partial order $\leq$ in $(X,*)$ can not be extended to $(\tilde{X},*)$.

We can also establish the following conditions $\forall\ x_\lambda,\ y_\mu,\ z_\alpha \in \tilde{X}$:

1'. $x_\lambda * 0_\mu = x_{\min(\lambda,\mu)}$;

2'. $x_\lambda * y_\mu = 0_{\min(\lambda,\mu)}$ and $y_\mu * z_\alpha = 0_{\min(\mu,\alpha)} \Longrightarrow x_\lambda * z_\alpha = 0_{\min(\lambda,\mu)}$;

3'. $x_\lambda * y_\mu = 0_{\min(\lambda,\mu)} \Longrightarrow (x_\lambda * z_\alpha)*(y_\mu * z_\alpha) = 0_{\min(\lambda,\mu,\alpha)}$ and $(z_\alpha * y_\mu) * (z_\alpha * x_\lambda) = 0_{\min(\lambda,\mu,\alpha)}$;

4'. $(x_\lambda * y_\mu) * z_\alpha = (x_\lambda * z_\alpha) * y_\mu$;

5'. $(x_\lambda * y_\mu) * x_\lambda = 0_{\min(\lambda,\mu)}$;

6'. $x_\lambda * (x_\lambda * (x_\lambda * y_\mu)) = x_\lambda * y_\mu$.

We recall that if A is a fuzzy subset of a BCK-algebra X, then we have the following:

$$\tilde{A} = \{x_\lambda \in \tilde{X} \ :\ A(x) \geq \lambda, \lambda \in (0,1]\}. \tag{1}$$

$$\forall\ \lambda \in (0,1],\ \tilde{X}_\lambda = \{x_\lambda : x \in X\}, \quad \text{and} \quad \tilde{A}_\lambda = \{x_\lambda \in \tilde{X}_\lambda : A(x) \geq \lambda\}. \tag{2}$$

We have also $\tilde{X}_\lambda \subseteq \tilde{X},\ \tilde{A} \subseteq \tilde{X}, \tilde{A}_\lambda \subseteq \tilde{A}, \tilde{A}_\lambda \subseteq \tilde{X}_\lambda$ and one can easily check that $(\tilde{X}_\lambda, *, 0_\lambda)$ is a BCK-algebra.

Definition 2.1 [3]**.** A nonempty subset I of a BCK-algebra X is called an ideal if it satisfies

1. $0 \in I$;
2. $x * y \in I$ and $y \in I \Longrightarrow x \in I$.

Definition 2.2 [3]**.** A fuzzy subset A of a BCK-algebra X is a fuzzy ideal iff

1. $\forall\, x \in X,\ A(0) \geq A(x)$;

2. $\forall\, x, y \in X,\ A(x) \geq \min(A(x * y), A(y))$.

Definition 2.3. $\tilde{A}$ is a weak ideal of $\tilde{X}$ iff

1) $\forall\, \nu \in Im(A),\ 0_\nu \in \tilde{A}$;

2) $\forall\, x_\lambda, y_\mu \in \tilde{X}$, such that $x_\lambda * y_\mu \in \tilde{A}$ and $y_\mu \in \tilde{A}$, we have $x_{\min(\lambda,\mu)} \in \tilde{A}$.

Remark 2.2. Any weak ideal $\tilde{A}$ has the following property

$$(x_\lambda * y_\mu = 0_{\min(\lambda,\mu)} \text{ and } y_\mu \in \tilde{A}) \Rightarrow x_{\min(\lambda,\mu)} \in \tilde{A}.$$

Proof. Let $x_\lambda, y_\mu \in \tilde{X}$ such that $x_\lambda * y_\mu = 0_{\min(\lambda,\mu)}$ and $y_\mu \in \tilde{A}$.

$$y_\mu \in \tilde{A} \Longrightarrow A(y) \geq \mu.$$

Let $A(y) = \alpha$, using Definition 2.3 - 1) we obtain $0_\alpha \in \tilde{A}$.

So $A(0) \geq \alpha$. But $\alpha = A(y) \geq \mu \geq \min(\lambda, \mu)$. Therefore $0_{\min(\lambda,\mu)} \in \tilde{A}$. Finally, according to Definition 2.3 - 2), we have $x_{\min(\lambda,\mu)} \in \tilde{A}$. ■

A characterization of a weak ideal is given by the following theorem.

Theorem 2.1 [10]**.** Suppose that A is a fuzzy subset of a BCK-algebra X, then the following conditions are equivalent:

1. A is a fuzzy ideal;

2. $\forall\, x_\lambda, y_\mu \in \tilde{A}$, $(z_\alpha * y_\mu) * x_\lambda = 0_{\min(\lambda,\mu,\alpha)} \Longrightarrow z_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$;

3. $\forall\, t \in (0, 1]$, the t-level subset $A^t = \{x \in X : A(x) \geq t\}$ is an ideal when $A^t \neq \emptyset$;

4. $\tilde{A}$ is a weak ideal.

3. Fuzzy n-fold commutative weak ideals

Throughout this paper, X always means a BCK-algebra and $\tilde{X}$ the set of fuzzy points on X.

Let us denote $(...((x*y)*y)*...)*y$ by $x*y^n$ and $(...((x_\lambda*y_\mu)*y_\mu)*...)*y_\mu$ by $x_\lambda * y_\mu^n$ (where y and y_μ occurs respectively n times) with $x, y \in X$, $x_\lambda, y_\mu \in \tilde{X}$.

We recall the following:

Definition 3.1. An nonempty subset I of a BCK-algebra X is called a commutative ideal of X if it satisfies

1. $0 \in I$;
2. $\forall\ x, y, z \in X,\ ((x*y)*z \in I \text{ and } z \in I) \Longrightarrow x*(y*(y*x)) \in I.$

An ideal I of a BCK-algebra X is commutative iff

$$\forall\ x, y \in X,\ x*y \in I \Longrightarrow x*(y*(y*x)) \in I.$$

Lemma 3.1 [10]**.** *For any fuzzy ideal A of X, if $x \leq y$, then $A(y) \leq A(x)$.*

Definition 3.2. A BCK-algebra X is n-fold commutative if for any $x, y \in X,\ x*y = x*(y*(y*x^n))$.

Theorem 3.1 [13]**.** *A BCK-algebra X is n-fold commutative iff for any $x, y \in X,\ x*(x*y) \leq y*(y*x^n)$.*

Definition 3.3. A nonempty subset I of a BCK-algebra X is an n-fold commutative ideal of X if it satisfies

1. $0 \in I$;
2. $\forall\ x, y, z \in X$,

$$((x*y)*z \in I \text{ and } z \in I) \Longrightarrow x*(y*(y*x^n)) \in I.$$

Lemma 3.2 [13]**.** *An ideal I of a BCK-algebra X is an n-fold commutative ideal iff*

$$\forall\ x, y \in X,\ \ x * y \in I \Longrightarrow x * (y * (y * x^n)) \in I.$$

Now, we give some characterizations of fuzzy n-fold commutative ideals in BCK-algebras.

Definition 3.4. A fuzzy subset A of X is called a fuzzy n-fold commutative ideal of X if it satisfies

1. $\forall\ x \in X,\ \ A(0) \geq A(x);$
2. $\forall\ x, y, z \in X,\ \ A(x * (y * (y * x^n))) \geq \min(A((x * y) * z), A(z)).$

Definition 3.5 [10]**.** $\tilde{A}$ is a commutative weak ideal of $\tilde{X}$ iff

1. $\forall\ \nu \in Im(A),\ \ 0_\nu \in \tilde{A};$
2. $\forall\ x_\lambda, y_\mu, z_\alpha \in \tilde{X}$ such that $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$, we have

$$x_{\min(\lambda,\ \alpha)} * (y_\mu * (y_\mu * x_{\min(\lambda,\ \alpha)})) \in \tilde{A}.$$

Definition 3.6. $\tilde{A}$ is an n-fold commutative weak ideal of $\tilde{X}$ iff

1. $\forall\ \nu \in Im(A),\ \ 0_\nu \in \tilde{A};$
2. $\forall\ x_\lambda, y_\mu, z_\alpha \in \tilde{X}$, if $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$, then

$$x_{\min(\lambda,\alpha)} * (y_\mu * (y_\mu * x^n_{\min(\lambda,\alpha)})) \in \tilde{A}.$$

Example 3.1. Let $X = \{0, 1, 2, 3, 4\}$ with $*$ defined by the following table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	4	4	4	0

By simple computations one can prove that $(X, *, 0)$ is a BCK-algebra.

Let $t_1, t_2 \in (0, 1]$ and let define a fuzzy subset $A : X \longrightarrow [0, 1]$ by

$$t_1 = A(0) = A(1) = A(2) = A(3) > A(4) = t_2.$$

One can easily check that for any $n > 2$,

$$\begin{aligned}\tilde{A} =& \{0_\lambda : \lambda \in (0, t_1]\} \cup \{1_\lambda : \lambda \in (0, t_1]\} \cup \{2_\lambda : \lambda \in (0, t_1]\} \\ & \cup \{3_\lambda : \lambda \in (0, t_1]\} \cup \{4_\lambda : \lambda \in (0, t_1]\}\end{aligned}$$

is an n-fold commutative weak ideal.

Remark 3.1. $\tilde{A}$ is a 1-fold commutative weak ideal of a BCK-algebra X iff $\tilde{A}$ is a commutative weak ideal of X.

Theorem 3.2. *If A is a fuzzy subset of X, then A is a fuzzy n-fold commutative ideal iff $\tilde{A}$ is an n-fold commutative weak ideal.*

Proof.

$\Longrightarrow$ – Let $\lambda \in Im(A)$, it is easy to prove that $0_\lambda \in \tilde{A}$;

– Let $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$,

$$A((x * y) * z) \geq \min(\lambda, \mu, \alpha) \textit{ and } A(z) \geq \alpha.$$

Since A is a fuzzy n-fold commutative ideal, we have

$$A(x * (y * (y * x^n))) \geq \min(A((x * y) * z),$$

$$A(z)) \geq \min(\min(\lambda, \mu, \alpha), \alpha) = \min(\lambda, \mu, \alpha).$$

Therefore,

$$(x * (y * (y * x^n)))_{\min(\lambda,\mu,\alpha)} = x_{\min(\lambda,\alpha)} * (y_\mu * (y_\mu * x^n_{\min(\lambda,\alpha)})) \in \tilde{A}.$$

$\Longleftarrow$ – Let $x \in X$, it is easy to prove that $A(0) \geq A(x)$.

– Let $x, y, z \in X$ and let $A((x * y) * z) = \beta$ and $A(z) = \alpha$, then

$$((x * y) * z)_{\min(\beta,\alpha)} = (x_\beta * y_\alpha) * z_\alpha \in \tilde{A} \text{ and } z_\alpha \in \tilde{A}.$$

Since $\tilde{A}$ is n-fold commutative weak ideal, we have

$$x_{\min(\beta,\alpha)} * (y_\alpha * (y_\alpha * x^n_{\min(\beta,\alpha)})) = (x * (y * (y * x^n)))_{\min(\beta,\alpha)} \in \tilde{A}.$$

Thus $A(x * (y * (y * x^n))) \geq \min(\beta, \alpha) = \min(A((x * y) * z), A(z))$.

■

Proposition 3.1. *In an n-fold commutative BCK-algebra, every weak ideal is an n-fold commutative weak ideal.*

Proof. The proof is straigthforward. ■

Corollary 3.1. *In an n-fold commutative BCK-algebra, every fuzzy ideal is a fuzzy n-fold commutative ideal.*

Proposition 3.2. *An n-fold commutative weak ideal is an ideal. But the converse does not hold in general.*

Proof. Let $x_\lambda, y_\mu \in \tilde{A}$, then

$$x_\lambda * y_\mu = (x_\lambda * 0_\mu) * y_\mu \in \tilde{A}.$$

Since $\tilde{A}$ is an n-fold commutative weak ideal, we have

$$x_{\min(\lambda,\mu)} = x_{\min(\lambda,\mu)} * (0_\mu * (0_\mu * x^n_{\min(\lambda,\mu)})) \in \tilde{A}.$$

For the converse, let $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ defined by the following table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Obviously, $(X, *, 0)$ is a BCK-algebra.

Let us define a fuzzy subset $A : X \longrightarrow [0, 1]$ by

$$A(0) = 1,\ A(1) = \frac{1}{2},\ A(2) = A(3) = A(4) = \frac{1}{3}.$$

It is easy to check that

$$\begin{aligned}\tilde{A} =& \{0_\lambda : \lambda \in (0, 1]\} \cup \left\{1_\lambda : \lambda \in \left(0, \frac{1}{2}\right]\right\} \cup \left\{2_\lambda : \lambda \in \left(0, \frac{1}{3}\right]\right\} \\ &\cup \left\{3_\lambda : \lambda \in \left(0, \frac{1}{3}\right]\right\} \cup \left\{4_\lambda : \lambda \in \left(0, \frac{1}{3}\right]\right\}\end{aligned}$$

is a weak ideal, but not an n-fold commutative weak ideal because

$(2_1 * 3_1) * 0_1 = 0_1 \in \tilde{A}$ and $0_1 \in \tilde{A}$, but $2_1 * (3_1 * (3_1 * 2^n_1)) = 2_1 \notin \tilde{A}$. ■

Corollary 3.2. *A fuzzy n-fold commutative ideal is a fuzzy ideal. But the converse does not hold in general.*

The following theorem gives a characterization of an n-fold commutative weak ideal.

Theorem 3.3. *Suppose that $\tilde{A}$ is a weak ideal (namely A is a fuzzy ideal by Theorem 2.1), then the following conditions are equivalent:*

1. *A is a fuzzy n-fold commutative ideal;*
2. *$\forall\ x_\lambda, y_\mu \in \tilde{X}$ such that $x_\lambda * y_\lambda \in \tilde{A}$, we have*
$$x_{\min(\lambda,\mu)} * \Big(y_\mu * \Big(y_\mu * x^n_{\min(\lambda,\mu)}\Big)\Big) \in \tilde{A};$$
3. *$\forall\ t \in (0,1]$, the t-level subset $A^t = \{x \in X : A(x) \geq t\}$ is an n-fold commutative ideal when $A^t \neq \emptyset$;*
4. *$\forall\ x, y, z \in X$, $A(x * (y * (y * x^n))) \geq A(x * y)$;*
5. *$\tilde{A}$ is an n-fold commutative weak ideal.*

Proof.
1. $\Rightarrow$ 2. Let $x_\lambda * y_\mu \in \tilde{A}$. Since A is a fuzzy n-fold commutative, we have

$$A(x * (y * (y * x^n))) \geq \min(A((x * y) * (x * y)),$$

$$A(x * y)) \geq \min(A(0), A(x * y)) = A(x * y) \geq \min(\lambda, \mu).$$

Therefore,

$$(x * (y * (y * x^n)))_{\min(\lambda,\mu)} = x_{\min(\lambda,\mu)} * \Big(y_\mu * \Big(y_\mu * x^n_{\min(\lambda,\mu)}\Big)\Big) \in \tilde{A}.$$

2. $\Rightarrow$ 3. – Obviously, $\forall\ t \in (0,1]$, $0 \in A^t$.

– Let $(x * y) * z \in A^t$ and $z \in A^t$, then we have

$$((x * y) * z)_t = (x_t * y_t) * z_t \in \tilde{A} \text{ and } z_t \in \tilde{A}.$$

Since $\tilde{A}$ is a weak ideal, we have $x_t * y_t = (x * y)_t \in \tilde{A}$. Using the hypothesis, we obtain

$$x_t * (y_t * (y_t * x_t^n)) = (x * (y * (y * x^n)))_t \in \tilde{A}, \text{ hence } x * (y * (y * x^n)) \in A^t.$$

By vertue of Lemma 3.2, we obtain that $A^t = \{x \in X : A(x) \geq t\}$ is an n-fold commutative ideal.

3. $\Rightarrow$ 4. Let $x, y \in X$ and $t = A(x*y)$, then $x*y \in A^t$. Since A^t is an n-fold commutative ideal, we have

$$x*(y*(y*x^n)) \in A^t, \text{ hence } A(x*(y*(y*x^n))) \geq t = A(x*y).$$

4. $\Rightarrow$ 5. – Let $\lambda \in Im(A)$. Obviously, $0_\lambda \in \tilde{A}$.
– Let $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$. Since $\tilde{A}$ is a weak ideal, we obtain $(x*y)_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$. According to the hypothesis, we obtain

$$A(x*(y*(y*x^n))) \geq A(x*y) \geq \min(\lambda, \mu, \alpha),$$

hence

$$\begin{aligned}&\big(x*\big(y*\big(y*x^n\big)\big)\big)_{\min(\lambda,\mu,\alpha)}\\ &= x_{\min(\lambda,\mu)} * \Big(y_\mu * \Big(y_\mu * x^n_{\min(\lambda,\alpha)}\Big)\Big) \in \tilde{A}.\end{aligned}$$

5. $\Rightarrow$ 1. Follows from Theorem 3.2. ■

Theorem 3.4. *Let $\tilde{A}$ and $\tilde{B}$ be two weak ideals such that $\tilde{A} \subseteq \tilde{B}$, and $A(0) = B(0)$. If $\tilde{A}$ is an n-fold commutative weak ideal, then $\tilde{B}$ is also n-fold commutative weak ideal.*

Proof. To prove the theorem, we need the following result.

Lemma 3.3 [13]**.** *If I and J are two ideals of X such that $I \subseteq J$ with I n-fold commutative, then J is also n-fold commutative.*

Using this lemma, we can prove Theorem 3.4 as follows:

To prove that $\tilde{B}$ is n-fold commutative, it suffices to show that $\forall\ t \in (0,1]$, B^t is n-fold commutative ideal when $B^t \neq \emptyset$.

Since $A(0) = B(0)$, it is clear that $A^t \neq \emptyset$ when $B^t \neq \emptyset$.

$$\tilde{A} \subseteq \tilde{B} \Longrightarrow A^t \subseteq B^t.$$

Since $\tilde{A}$ is n-fold commutative, A^t is also n-fold commutative. According to Lemma 3.3, B^t is also n-fold commutative. So, $\tilde{B}$ is also n-fold commutative.

■

Consequence 3.1. $\forall\ \lambda \in Im(A)$, if $\{0_\lambda\}$ is an n-fold commutative weak ideal, then $\tilde{A}$ is also an n-fold commutative weak ideal.

Corollary 3.3. *Let A and B be two fuzzy ideals of X such that $A \leq B$ and $B(0) = A(0)$. If A is a fuzzy n-fold commutative ideal, then B is also a fuzzy n-fold commutative ideal.*

4. FUZZY n-FOLD WEAK COMMUTATIVE WEAK IDEALS

In this section, we define and give some characterizations of fuzzy n-fold weak commutative weak ideals in BCK-algebras.

Let us recall the following results.

Definition 4.1. A nonempty subset I of X is called an n-fold weak commutative ideal of X if it satisfies

1. $0 \in I$;
2. $\forall\ x, y, z \in X,\ (x * (x * y^n)) * z \in I$ and $z \in I \Longrightarrow y * (y * x) \in I$.

Lemma 4.1 [13]. *An ideal I of a BCK-algebra X is an n-fold weak commutative ideal iff*

$$\forall\ x, y, z \in X,\ x * (x * y^n) \in I \Longrightarrow y * (y * x) \in I.$$

Definition 4.2. A fuzzy subset A of X is called a fuzzy n-fold weak commutative ideal of X if it satisfies

1. $\forall\ x \in X,\ A(0) \geq A(x)$;
2. $\forall\ x, y, z \in X,\ A(y * (y * x)) \geq \min(A((x * (x * y^n)) * z), A(z))$.

Definition 4.3. $\tilde{A}$ is a weak commutative weak ideal of $\tilde{X}$ iff

1. $\forall\ \nu \in Im(A),\ 0_\nu \in \tilde{A}$;
2. $\forall\ x_\lambda, y_\mu, z_\alpha \in \tilde{X}$,

$$((x_\lambda * (x_\lambda * y_\mu)) * z_\alpha \in \tilde{A} \text{ and } z_\alpha \in \tilde{A}) \Rightarrow (y_\mu * (y_\mu * x_{\min(\lambda,\alpha)})) \in \tilde{A}.$$

Definition 4.4. $\tilde{A}$ is an n-fold weak commutative weak ideal of $\tilde{X}$ iff

1. $\forall\, \nu \in Im(A),\ 0_\nu \in \tilde{A}$;
2. $\forall\, x_\lambda, y_\mu, z_\alpha \in \tilde{X}$,

$$((x_\lambda * (x_\lambda * y_\mu^n)) * z_\alpha \in \tilde{A} \text{ and } z_\alpha \in \tilde{A}) \Rightarrow (y_\mu * (y_\mu * x_{\min(\lambda,\alpha)})) \in \tilde{A}.$$

Example 4.1. Let $X = \{0, 1, 2, 3\}$ with $*$ defined by the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	3	0

By simple computations one can prove that $(X, *, 0)$ is a BCK-algebra.
Let $t_1, t_2 \in (0, 1]$ and let us define a fuzzy subset $A : X \longrightarrow [0, 1]$ by

$$t_1 = A(0) = A(1) = A(2) > A(3) = t_2.$$

It is easy to check that for any $n > 2$,

$$\tilde{A} = \{0_\lambda : \lambda \in (0, t_1]\} \cup \{1_\lambda : \lambda \in (0, t_1]\}$$

$$\cup \{2_\lambda : \lambda \in (0, t_1]\} \cup \{3_\lambda : \lambda \in (0, t_2]\}$$

is an n-fold weak commutative weak ideal.

Remark 4.1. $\tilde{A}$ is an 1-fold weak commutative weak ideal of a BCK-algebra X iff $\tilde{A}$ is a weak commutative weak ideal.

Theorem 4.1. *If A is a fuzzy subset of X, then A is a fuzzy n-fold weak commutative ideal iff $\tilde{A}$ is an n-fold weak commutative weak ideal.*

Proof.

$\Longrightarrow$ – Let $\lambda \in Im(A)$. Obviously $0_\lambda \in \tilde{A}$;

– Let $(x_\lambda * (x_\lambda * y_\mu^n)) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$, then

$$A((x * (x * y^n)) * z) \geq \min(\lambda, \mu, \alpha) \text{ and } A(z) \geq \alpha.$$

Since A is a fuzzy n-fold weak commutative ideal, we have

$$A(y * (y * x)) \geq \min(A((x * (x * y)) * z),$$

$$A(z)) \geq \min(\min(\lambda, \mu, \alpha), \alpha) = \min(\lambda, \mu, \alpha).$$

Therefore $(y * (y * x))_{\min(\lambda,\mu,\alpha)} = y_\mu * (y_\mu * x_{\min(\lambda,\alpha)}) \in \tilde{A}$.

$\Longleftarrow$ – Let $x \in X$, it is easy to prove that $A(0) \geq A(x)$;

– Let $x, y, z \in X$, $A((x * (x * y^n)) * z) = \beta$ and $A(z) = \alpha$.
Then,

$$((x * (x * y^n)) * z)_{\min(\beta,\alpha)} = (x_\beta * (x_\beta * y_\beta^n)) * z_\alpha \in \tilde{A} \text{ and } z_\alpha \in \tilde{A}.$$

Since $\tilde{A}$ is n-fold weak commutative weak ideal, we have

$$y_\beta * (y_\beta * x_{\min(\beta,\alpha)}) = (y * (y * x))_{\min(\beta,\alpha)} \in \tilde{A}.$$

Hence, $A(y * (y * x)) \geq \min(\beta, \alpha) = \min(A((x * (x * y^n)) * z), A(z))$.

■

Proposition 4.1. *In an n-fold commutative BCK-algebra, the concepts of weak ideals, n-fold commutative weak ideals and n-fold weak commutative weak ideals are the same.*

Proof. The proof is straightforward. ■

Corollary 4.1. *In an n-fold commutative BCK-algebra, the concepts of fuzzy ideals, fuzzy n-fold commutative ideals and fuzzy n-fold weak commutative ideals are the same.*

Proposition 4.2. *An n-fold weak commutative weak ideal is a weak ideal.*

Proof. By setting $y_\mu = x_\lambda$ in Definition 4.4 and using the fact that $x*x = 0$ and $x * 0 = x$, one obtains that

$$\forall\ x_\lambda, z_\alpha \in \tilde{X} \text{ such that } x_\lambda * z_\alpha \in \tilde{A} \text{ and } z_\alpha \in \tilde{A},\ x_{\min(\lambda,\alpha)} \in \tilde{A}.$$

■

Corollary 4.2. *A fuzzy n-fold weak commutative ideal is a fuzzy ideal.*

The following theorem summarizes a characterization of an n-fold weak commutative weak ideal.

Theorem 4.2. *Suppose that $\tilde{A}$ is a weak ideal (namely A is a fuzzy ideal by Theorem* 2.1), *then the following conditions are equivalent:*

1) *A is fuzzy n-fold weak commutative ideal;*
2) *$\forall\ x_\lambda, y_\mu \in \tilde{X}$ such that $x_\lambda * (x_\lambda * y^n_{\min(\lambda,\mu)}) \in \tilde{A}$, we have*
$$y_\mu * (y_\mu * x_{\min(\lambda,\mu)}) \in \tilde{A};$$
3) *$\forall\ t \in (0,1]$, the t-level subset $A^t = \{x \in X : A(x) \geq t\}$ is an n-fold weak commutative ideal when $A^t \neq \emptyset$;*
4) *$\forall\ x, y \in X,\ A(y * (y * x)) \geq A(x * (x * y^n))$;*
5) *$\tilde{A}$ is an n-fold weak commutative weak ideal.*

Proof.
1) $\Rightarrow$ 2) Let $x_\lambda * (x_\lambda * y^n_{\min(\lambda,\mu)}) \in \tilde{A}$. Since A is a fuzzy n-fold weak commutative ideal, we have

$$A(y * (y * x)) \geq \min(A((x * (x * y^n)) * 0),$$

$$A(0)) = A((x * (x * y^n))) \geq \min(\lambda, \mu).$$

Therefore, $(y * (y * x))_{\min(\lambda,\mu)} = y_\mu * (y_\mu * x_{\min(\lambda,\mu)}) \in \tilde{A}$.

2) $\Rightarrow$ 3) – Obviously, $\forall\, t \in (0,1]$, $0 \in A^t$.

– Let $x * (x * y^n) \in A^t$, we have

$$(x * (x * y^n))_t = x_t * (x_t * y_t^n) \in \tilde{A}.$$

By virtue of the hypothesis, one obtains $y_t*(y_t*x_t) \in \tilde{A}$, therefore $y * (y * x) \in A^t$. Using Lemma 4.1, we can conclude that $A^t = \{x \in X : A(x) \geq t\}$ is an n-fold weak commutative ideal.

3) $\Rightarrow$ 4) Let $x, y \in X$ and $t = A(x * (x * y^n))$, then $x * (x * y^n) \in A^t$. Since A^t is an n-fold weak commutative ideal, we have

$$y * (y * x) \in A^t, \text{ therefore } A(y * (y * x)) \geq t = A(x * (x * y^n)).$$

4) $\Rightarrow$ 5) – Let $\lambda \in Im(A)$, it is clear that $0_\lambda \in \tilde{A}$.

– Let $(x_\lambda * (x_\lambda * y_\mu^n)) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$. Since $\tilde{A}$ is a weak ideal, $(x * (x * y^n))_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$. Using the hypothesis, we obtain

$$A(y * (y * x)) \geq A(x * (x * y^n)) \geq \min(\lambda, \mu, \alpha).$$

From this, one can deduce that

$$(y * (y * x))_{\min(\lambda,\mu,\alpha)} = y_\mu * (y_\mu * x_{\min(\lambda,\alpha)}) \in \tilde{A}.$$

5) $\Rightarrow$ 1) Follows from Theorem 4.1. ∎

Theorem 4.3. *Theorem* 3.4*, its corollary (Corollary* 3.3*) and consequence (Consequence* 3.1*) are valid if "n-fold commutative" is replaced by "n-fold weak commutative".*

Proof.
The proof is similar to that of Theorem 3.4 and is therefore omitted. ∎

Appendix A

Algorithms

Algorithm for BCK-algebras

```
Input(X : set, *: binary operation)
Output("X is a BCK-algebra or not")
Begin
 If X = ∅ then
   go to (1.);
 EndIf
 If 0 ∉ X then
   go to (1.);
 EndIf
 Stop:=false;
 i := 1;
 While i ≤ |X| and not(Stop) do
  If x_i * x_i ≠ 0 then
    Stop:=true;
  EndIf
  If 0 * x_i ≠ 0 then
    Stop:=true;
  EndIf
  j := 1
  While j ≤ |X| and not(Stop) do
   If (x_i * (x_i * y_j)) * y_j ≠ 0 then
     Stop:=true;
   EndIf
   If (x_i * y_j = 0) and (y_j * x_i = 0) then
    If x_i ≠ y_j then
      Stop:=true;
    EndIf
   EndIf
   k := 1;
   While k ≤ |X| and not(Stop) do
    If ((x_i * y_j) * (x_i * z_k)) * (z_k * y_j) ≠ 0 then
      Stop:=true;
    EndIf
   EndWhile
  EndWhile
 EndWhile
 If Stop then
  (1.) Output("X is not a BCK-algebra")
 Else
  Output("X is a BCK-algebra")
 EndIf
End
```

Algorithm for ideals of BCK-algebras

```
Input(X: BCK-algebra, I: subset of X);
Output("I is an ideal of X or not");
Begin
 If I = ∅ then
   go to (1.);
 EndIf
 If 0 ∉ I then
   go to (1.);
 EndIf
 Stop:=false;
 i := 1;
 While i ≤ |X| and not(Stop) do
   j := 1
  While j ≤ |X| and not(Stop) do
   If x_i * y_j ∈ I and y_j ∈ I then
    If x_i ∉ I then
      Stop:=true;
    EndIf
   EndIf
  EndWhile
 EndWhile
 If Stop then
  Output("I is an ideal of X")
 Else
  (1.) Output("I is not an ideal of X")
 EndIf
End
```

Algorithm for n-fold commutative ideals

```
Input(X: BCK-algebra, I: subset of X, n ∈ ℕ);
Output("I is an n-fold commutative ideal of X or not");
Begin
 If I = ∅ then
   go to (1.);
 EndIf
 If 0 ∉ I then
   go to (1.);
 EndIf
 Stop:=false;
 i := 1;
 While i ≤ |X| and not(Stop) do
   j := 1
  While j ≤ |X| and not(Stop) do
    k := 1
   While k ≤ |X| and not(Stop) do
    If (x_i * y_j) * z_k ∈ I and z_k ∈ I then
     If x_i * (y_j * (y_j * x_i^n)) ∉ I then
       Stop:=true;
     EndIf
    EndIf
   EndWhile
  EndWhile
 EndWhile
 If Stop then
  Output("I is an n-fold commutative ideal of X")
 Else
  (1.)Output("I is not an n-fold commutative ideal of X")
 EndIf
End
```

```
Algorithm for n-fold weak commutative ideals
Input(X: BCK-algebra, I: subset of X, n ∈ ℕ);
Output("I is an n-fold weak commutative ideal of X or not");
Begin
 If I = ∅ then
   go to (1.);
 EndIf
 If 0 ∉ I then
   go to (1.);
 EndIf
 Stop:=false;
 i := 1;
 While i ≤ |X| and not(Stop) do
   j := 1
  While j ≤ |X| and not(Stop) do
    k := 1
   While k ≤ |X| and not(Stop) do
    If (x_i * (x_i * y_j^n)) * z_k ∈ I and z_k ∈ I then
     If y_j * (y_j * x_i) ∉ I then
       Stop:=true;
     EndIf
    EndIf
   EndWhile
  EndWhile
 EndWhile
 If Stop then
  Output("I is an n-fold weak commutative ideal of X")
 Else
  (1.) Output("I is not an n-fold weak commutative ideal of X")
 EndIf
End
```

```
Algorithm for fuzzy subsets
Input(X : BCI-algebra, A : X ⟶ [0,1]);
Output("A is a fuzzy subset of X or not");
Begin
 Stop:=false;
 i := 1;
 While i ≤ |X| and not(Stop) do
  If (A(x_i) < 0) or (A(x_i) > 1) then
   Stop:=true;
  EndIf
 EndWhile
 If Stop then
  Output("A is a fuzzy subset of X")
 Else
  Output("A is not a fuzzy subset of X")
 EndIf
End
```

```
Algorithm for fuzzy n-fold commutative ideals
Input(X: BCK-algebra, ∗: binary operation, A: fuzzy subset of X);
Output("A is a fuzzy n-fold commutative ideal of X or not");
Begin
 Stop:=false;
 i := 1;
 While i ≤ |X| and not(Stop) do
  If A(0) < A(x_i) then
   Stop:=true;
  EndIf
  j := 1
  While j ≤ |X| and not(Stop) do
   k := 1;
   While k ≤ |X| and not(Stop) do
    If A(x_i ∗ (y_j ∗ (y_j ∗ x_i^n))) < Min(A(x_i ∗ y_j) ∗ z_k), A(z_k)) then
     Stop:=true;
    EndIf
   EndWhile
  EndWhile
 EndWhile
 If Stop then
  Output("A is not a fuzzy n-fold commutative ideal of X")
 Else
  Output("A is a fuzzy n-fold commutative ideal of X")
 EndIf
End
```

```
Algorithm for fuzzy n-fold weak commutative ideals
Input(X: BCK-algebra, ∗: binary operation, A: fuzzy subset of X);
Output("A is a fuzzy n-fold weak commutative ideal of X or not");
Begin
 Stop:=false;
 i := 1;
 While i ≤ |X| and not(Stop) do
  If A(0) < A(x_i) then
   Stop:=true;
  EndIf
  j := 1
  While j ≤ |X| and not(Stop) do
   k := 1;
   While k ≤ |X| and not(Stop) do
    If A(y_j ∗ (y_j ∗ x_i)) < Min(A((x_i ∗ (x_i ∗ y_j^n) ∗ z_k)), A(z_k)) then
     Stop:=true;
    EndIf
   EndWhile
  EndWhile
 EndWhile
 If Stop then
  Output("A is not a fuzzy n-fold weak commutative ideal of X")
 Else
  Output("A is a fuzzy n-fold weak commutative ideal of X")
 EndIf
End
```

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