Discussiones Mathematicae
General Algebra and Applications 26 (2006) 85–109

REGULAR ELEMENTS AND GREEN'S RELATIONS IN MENGER ALGEBRAS OF TERMS

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Abstract

Defining an $(n+1)$-ary superposition operation S^n on the set $W_\tau(X_n)$ of all n-ary terms of type τ, one obtains an algebra $n-clone\ \tau := (W_\tau(X_n); S^n, x_1, \ldots, x_n)$ of type $(n+1, 0, \ldots, 0)$. The algebra $n-clone\ \tau$ is free in the variety of all Menger algebras ([9]). Using the operation S^n there are different possibilities to define binary associative operations on the set $W_\tau(X_n)$ and on the cartesian power $W_\tau(X_n)^n$. In this paper we study idempotent and regular elements as well as Green's relations in semigroups of terms with these binary associative operations as fundamental operations.

Keywords: term, superposition of terms, Menger algebra, regular element, Green's relations.

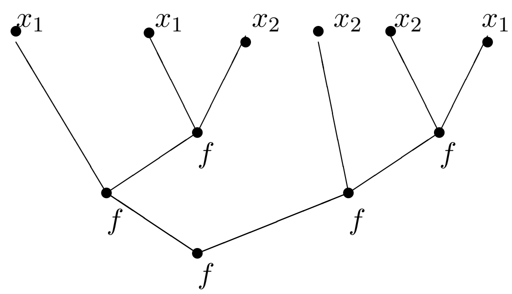
2000 Mathematics Subject Classification: 08A35, 08A40, 08A70.

1. Preliminaries

Let $\tau = (n_i)_{i\in I}$ be a type of algebras with an n_i-ary operation symbol f_i for every i in some index set I. For each $n \geq 1$ let $X_n = \{x_1, \dots x_n\}$ be an n-element alphabet. We denote by $W_\tau(X_n)$ the set of all n-ary terms of type τ. It is very common to illustrate terms by tree diagrams. Consider for example the type $\tau = (2)$ with a binary operation symbol f. Then the term

$$t = f(f(x_1, f(x_1, x_2)), f(x_2, f(x_2, x_1)))$$

corresponds to the tree diagram below.



On the set $W_\tau(X_n)$ one can define the following $(n+1)$-ary superposition operation

$$S^n : W_\tau(X_n)^{n+1} \to W_\tau(X_n)$$

by

$S^n(x_i, t_1, \dots, t_n) := t_i$, for every $1 \leq i \leq n$, and

$S^n(f_i(r_1, \dots, r_{n_i}), t_1, \dots, t_n) := f_i(S^n(r_1, t_1, \dots, t_n), \dots, S^n(r_{n_i}, t_1, \dots, t_n)).$

Together with the nullary operations $x_1, \dots, x_n$ one obtains an algebra

$$n - clone\ \tau := (W_\tau(X_n); S^n, x_1, \dots, x_n),$$

which satisfies the identities

(C1) $\widetilde{S}^n\ (\widetilde{Z}, \widetilde{S}^n\ (\widetilde{Y}_1, \widetilde{X}_1, \ldots, \widetilde{X}_n), \ldots, \widetilde{S}^n\ (\widetilde{Y}_n, \widetilde{X}_1, \ldots, \widetilde{X}_n))$

$\approx\ \widetilde{S}^n\ (\widetilde{S}^n\ (\widetilde{Z}, \widetilde{Y}_1, \ldots, \widetilde{Y}_n), \widetilde{X}_1, \ldots, \widetilde{X}_n),$

(C2) $\widetilde{S}^n\ (\lambda_i, \widetilde{X}_1, \ldots, \widetilde{X}_n) \approx\ \widetilde{X}_i$ for $1 \leq i \leq n$,

(C3) $\widetilde{S}^n\ (\widetilde{X}_i, \lambda_1, \ldots, \lambda_n)\ \approx\ \widetilde{X}_i$ for $1 \leq i \leq n$.

Here $\widetilde{S}^n$ is an $(n+1)$-ary operation symbol, $\lambda_1, \ldots, \lambda_n$ are nullary operation symbols and $\widetilde{Z}$, $\widetilde{Y}_1$, $\ldots$, $\widetilde{Y}_n$, $\widetilde{X}_1$, $\ldots$, $\widetilde{X}_n$ are new variables. The algebra n-clone τ is an example of a *unitary Menger algebra of rank* n. Without the nullary operations one speaks of a *Menger algebra of rank* n.

Now we consider a type τ_n consisting of n-ary operation symbols only. Let X be an arbitrary countably infinite alphabet of variables and let $W_\tau(X)$ be the set of all terms of type τ. On $W_\tau(X)$ we consider a generalized superposition operation S_n^g which is defined for any $n \geq 1, n \in \mathbb{N}^+$, inductively by the following steps:

Definition 1.1.

(i) If $t = x_i, 1 \leq i \leq n$, then $S_g^n(x_i, t_1, \ldots, t_n) := t_i$ for $t_1, \ldots, t_n \in W_{\tau_n}(X)$.

(ii) If $t = x_i,\ n < i$, then $S_g^n(x_i, t_1, \ldots, t_n) := x_i$.

(iii) If $t = f_i(s_1, \ldots, s_n)$, then

$$S_g^n(t, t_1, \ldots, t_n) := f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S_g^n(s_n, t_1, \ldots, t_n)).$$

Then we may consider the algebraic structure

$$clone_g\tau_n := \left(W_{\tau_n}(X); S_g^n, (x_i)_{i\in\mathbb{N}^+}\right)$$

with the universe $W_{\tau_n}(X)$, with one $(n+1)$-ary operation and infinitely many nullary operations. This algebra is called a *Menger algebra with infinitely many nullary operations.* Without the nullary operations we have a Menger algebra of rank n. It is not difficult to see ([1]) that this algebra satisfies the axioms

(Cg1) $\tilde{S}^n_g(T, \tilde{S}^n_g(F_1, T_1, \ldots, T_n), \ldots, \tilde{S}^n_g(F_n, T_1, \ldots, T_n))$

$\approx \tilde{S}^n_g(\tilde{S}^n_g(T, F_1, \ldots, F_n), T_1, \ldots, T_n)$.

(Cg2) $\tilde{S}^n_g(T, \lambda_1, \ldots, \lambda_n) = T$.

(Cg3) $\tilde{S}^n_g(\lambda_i, T_1, \ldots, T_n) = T_i$ for $1 \leq i \leq n$.

(Cg4) $\tilde{S}^n_g(\lambda_j, T_1, \ldots, T_n) = \lambda_j$ for $j > n$.

(Here $\tilde{S}^n_g, \lambda_i$ are operation symbols corresponding to the operations S^n_g and $x_i, i \in \mathbb{N}^+$, respectively and T, T_j, F_i are new variables.)

In any Menger algebra $(G; S^n)$ of rank n a binary operation $+$ can be defined by

$$x + y := S^n(x, y, \ldots, y).$$

It is easy to see that the operation $+$ is associative. The algebra $(G; +)$ is called *diagonal semigroup* (see e.g., [10]).

On the cartesian power G^n one may define a binary operation $*$ by $(x_1, \ldots, x_n)*(y_1, \ldots, y_n) := (S^n(x_1, y_1, \ldots, y_n), \ldots, S^n(x_n, y_1, \ldots, y_n))$. Then $(G^n; *)$ is also a semigroup.

We notice that $(G; +)$ can be embedded into $(G^n; *)$. Actually the subsemigroup $(\triangle_G; *|_{\triangle_G})$ of $(G^n; *)$ where $\triangle_G := \{(x, \ldots, x) \mid x \in G\}$ is the diagonal of G, is isomorphic to $(G; +)$.

An element x of a semigroup $(S; \cdot)$ is called *regular* if there is an element y of the same semigroup such that $x \cdot y \cdot x = x$. Clearly, every idempotent element is regular. Further, we recall the definition of Green's relations.

Green's relations are special equivalence relations which can be defined on any semigroup or monoid, using the idea of mutual divisibility of elements. Let $\mathcal{S}$ be a semigroup and let $\mathcal{S}^+$ be the monoid which arises from S by adding a neutral element. For any semigroup $\mathcal{S}$ and any elements a, b of S, we say $a\mathcal{L}b$ if and only if there are c and d in $\mathcal{S}^+$ such that $c \cdot a = b$ and $d \cdot b = a$. Dually, $a\mathcal{R}b$ if and only if there are c and d in $\mathcal{S}^+$ such that $a \cdot c = b$ and $b \cdot d = a$. It follows easily from these definitions that $\mathcal{L}$ is always a right congruence, while $\mathcal{R}$ is always a left congruence. The relation $\mathcal{H}$ is defined as the intersection of $\mathcal{R}$ and $\mathcal{L}$, and the relation $\mathcal{D}$ is the join $\mathcal{L} \vee \mathcal{R}$.

It is easy to see that $\mathcal{L} \vee \mathcal{R} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$, where $\circ$ here refers to the usual composition of relations. Finally, the relation $\mathcal{J}$ is defined by $a\mathcal{J}b$ if and only if there exist elements c, d, p and q in $\mathcal{S}^+$ such that $a = c \cdot b \cdot d$ and $b = p \cdot a \cdot q$. For Green's relations we will use the following notation, $(a, b) \in \mathcal{R}$ and $a\mathcal{R}b$ and similarly for the other relations. For more information about Green's relations in general, we refer the reader to [7].

Using Green's relation $\mathcal{R}$ ($\mathcal{L}$) the set of all regular elements of a semigroup $\mathcal{S}$ can be described as follows (see e.g., [7]):

Theorem 1.2. *Let x be an element of the universe of a semigroup $\mathcal{S}$. Then the following are equivalent:*

(i) *x is regular.*

(ii) *$[x]_{\mathcal{R}}$ contains an idempotent element.*

(iii) *$[x]_{\mathcal{L}}$ contains an idempotent element.*

Then for the set $Reg(S)$ of all regular elements of the semigroup $\mathcal{S}$ we have

$$Reg(S) = \bigcup \Big\{ [e]_{\mathcal{R}} \mid e \in E(S) \Big\} = \bigcup \Big\{ [a]_{\mathcal{R}} \mid [a]_{\mathcal{R}} \cap E(S) \neq \emptyset \Big\}.$$

2. Idempotent elements

In this section we study idempotent terms of type τ with respect to the operations $+$ and $*$. To determine all idempotent elements we need some lemmas. The first lemma answers to the following question:

Let $s, t_1, \ldots t_n \in W_\tau(X_n)$. Under which conditions does there exist an n-ary term q such that $s = S^n(q, t_1, t_2, \ldots, t_n)$?

For a term $t \in W_\tau(X_n)$ we denote by $var(t)$ the set of all variables occurring in t.

Lemma 2.1. *Let $s, t_1, \ldots, t_n \in W_\tau(X_n)$. Then $s = S^n(s, t_1, \ldots, t_n)$ if and only if for each $i, 1 \leq i \leq n$, if $x_i \in var(s)$, then $t_i = x_i$.*

Proof. "$\Rightarrow$" Assume that $s = S^n(s, t_1, \ldots, t_n)$ and that $x_i \in var(s)$, but $t_i \neq x_i$. Then we have to substitute in s for x_i a term different from x_i and obtain $S^n(s, t_1, \ldots, t_n) \neq s$, a contradiction.

"⇐" We assume now that from $x_i \in var(s)$ there follows $t_i = x_i$. We will show $s = S^n(s, t_1, \ldots, t_n)$ by induction on the complexity of the term s. If $s = x_j \in X_n$, then $t_j = x_j$ and so $s = x_j = S^n(x_j, t_1, \ldots, x_j, \ldots, t_n)$. Now assume that for $s = f(s_1, \ldots, s_n)$ for all $1 \leq i \leq n$ we have $s_i = S^n(s_i, t_1, \ldots, t_n)$. Then

$$\begin{aligned}
&S^n(s, t_1, \ldots, t_n) \\
&= S^n(f(s_1, \ldots, s_n), t_1, \ldots, t_n) \\
&= f(s^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n)) \\
&= f(s_1, \ldots s_n) = s.
\end{aligned}$$

■

Now we solve the equation $x_i = S^n(q, t_1, t_2, \ldots, t_n)$.

Lemma 2.2. *Let $q, t_1, \ldots, t_n \in W_\tau(X_n)$. For each $i \in \{1, 2, \ldots, n\}$ we have $x_i = S^n(q, t_1, \ldots, t_n)$ if and only if there is an integer j for $1 \leq j \leq n$, such that $q = x_j$ and $t_j = x_i$.*

Proof. "⇐" This direction is clear.

"⇒" For the proof of this direction we use the following formula for the operation symbol count $op(t)$ of the term t ([3]). We denote by $vb_j(s)$ the number of occurrences of the variable x_j in the term s.

$$op(S^n(q, t_1, \ldots, t_n)) = \sum_{k=1}^{n} vb_k(q) op(t_k) + op(q).$$

(We mention that this formula is also valid for S_g^n and for terms $q, t_1, \ldots, t_n \in W_{\tau_n}(X)$). Then from $0 = op(x_i) = \sum_{k=1}^{n} vb_k(q) op(t_k) + op(q)$ we obtain $op(q) = 0$ and $q \in X_n$. Let $q = x_j$ for some $x_j \in X_n$. Then from $x_i = S^n(q, t_1, \ldots, t_n) = S^n(x_j, t_1, \ldots, t_n)$, we have $t_j = x_i$. ■

Now we want to determine all vectors of terms which are idempotent with respect to the operation $*$. Clearly, an n-vector $(q_1, \ldots, q_n)$ of n-ary terms is an idempotent element with respect to the operation $*$ if $S^n(q_i, q_1, \ldots, q_n) = q_i$ for all $1 \leq i \leq n$.

Then we have:

Theorem 2.3. *For $n \geq 1$ an n-vector $(q_1, \ldots, q_n)$ is idempotent if and only if the following condition* (ID) *is satisfied:*

$$\text{(ID)} \qquad x_j \in \bigcup_{i=1}^{n} var(q_i) \Rightarrow q_j = x_j.$$

Proof. Assume that $(q_1, \ldots, q_n)$ satisfies condition (ID). Let $var(q_i) = \{x_{i_1}, \ldots, x_{i_k}\} \subseteq X_n$. By condition (ID) we have $q_{i_j} = x_{i_j}$ for all $1 \leq j \leq k$ and then

$$\begin{aligned}
&S^n(q_i, q_1, \ldots, q_n) \\
&= S^n(q_i, q_1, \ldots, q_{i_1-1}, q_{i_1}, q_{i_1+1} \ldots, q_{i_k-1}, q_{i_k}, q_{i_k+1}, \ldots, q_n) \\
&= S^n(q_i, q_1, \ldots, q_{i_1-1}, x_{i_1}, q_{i_1+1} \ldots, q_{i_k-1}, x_{i_k}, q_{i_k+1}, \ldots, q_n) \\
&= q_i
\end{aligned}$$

by Lemma 2.1. Therefore, $(q_1, \ldots, q_n)$ is idempotent.

Conversely, assume that $(q_1, \ldots, q_n)$ is idempotent. Suppose $(q_1, \ldots, q_n)$ does not satisfy the condition (ID). Then there exist integers $i, j, 1 \leq i$, $j \leq n$ such that $x_j \in var(q_i)$, but $q_j \neq x_j$. Then by Lemma 2.1 we have $S^n(q_i, q_1, \ldots, q_j, \ldots, q_n) \neq q_i$ which contradicts the idempotency of the vector $(q_1, \ldots, q_n)$. Hence $(q_1, \ldots q_n)$ satisfies the condition (ID). ∎

Considering idempotent elements with respect to the operation $+$, we obtain:

Corollary 2.4. *An element $t \in W_\tau(X_n)$ is idempotent with respect to $+$ if and only if $t = x_i$ for some $1 \leq i \leq n$.*

Now we consider a type τ_n consisting of n-ary operation symbols and the binary associative operations $*_g : (W_{\tau_n}(X)^n)^2 \to W_{\tau_n}(X)^n$ and $+_g : W_{\tau_n}(X)^2 \to W_{\tau_n}(X)$ defined by

$$x +_g y := S_g^n(x, y, \ldots, y)$$

and

$$(x_1, \ldots, x_n) *_g (y_1, \ldots, y_n) := (S_g^n(x_1, y_1, \ldots, y_n), \ldots, S_g^n(x_n, y_1, \ldots, y_n)).$$

Let $X' := X \setminus X_n$. Instead of Lemma 2.1 we now have:

Lemma 2.5. *Let $s, t_1, \ldots, t_n \in W_{\tau_n}(X)$. If $var(s) \not\subseteq X'$, then $s = S_g^n(s, t_1, \ldots, t_n)$ if and only if for each i with $1 \leq i \leq n$ we have: if $x_i \in var(s)$, then $t_i = x_i$.*

If $var(s) \subseteq X'$, then $s = S_g^n(s, t_1, \ldots, t_n)$.

Proof. The proof is clear for $var(s) \subseteq X'$. If $var(s) \cap X_n \neq \emptyset$, then we conclude similar as in Lemma 2.1. ■

It is clear that instead of Lemma 2.2. we have now:

Lemma 2.6. *Let $q, t_1, \ldots, t_n \in W_{\tau_n}(X)$. For $1 \leq i \leq n$ we have $x_i = S_g^n(q, t_1, \ldots, t_n)$ if and only if there is an integer j with $1 \leq j \leq n$ such that $q = x_j$ and $t_j = x_i$.*

Then we obtain:

Theorem 2.7. *An n-vector $(q_1, \ldots, q_n)$ of elements from $W_{\tau_n}(X)$ is idempotent with respect to $*_g$ if and only if the following condition (ID*) is satisfied*

$$\text{(ID)}^* \qquad x_j \in \bigcup_{i=1}^{n} var(q_i) \cap X_n \Rightarrow q_j = x_j.$$

Proof. If $(q_1, \ldots, q_n)$ is idempotent and if $x_j \in \bigcup\limits_{i=1}^{n} var(q_i) \cap X_n$, then by Lemma 2.5, $q_j = x_j$.

Assume that (ID*) is satisfied. If $\bigcup_{i=1}^{n} var(q_i) \cap X_n = \emptyset$, then application of Lemma 2.5 gives $S_g^n(q_j, q_1, \ldots, q_n) = q_j$ and $(q_1, \ldots, q_n)$ is idempotent. If $\bigcup_{i=1}^{n} var(q_i) \cap X_n \neq \emptyset$ and $x_j \in \bigcup_{i=1}^{n} var(q_i)$ for some $1 \leq j \leq n$, then by (ID^*) we obtain $q_j = x_j$ and Lemma 2.5 gives $S_g^n(q_j, q_1, \ldots, q_n) = q_j$ and $(q_1, \ldots, q_n)$ is idempotent. ■

Further we have:

Corollary 2.8. *If $var(t) \subseteq X'$, then the term $t \in W_{\tau_n}(X')$ is idempotent with respect to $+_g$. If $var(t) \cap X_n \neq \emptyset$, then an element $t \in W_{\tau_n}(X)$ is idempotent with respect to $+_g$ if and only if $t = x_i$ for some $1 \leq i \leq n$.*

3. Regular elements

A vector $(q_1,\ldots,q_n)$ is a regular element with respect to $*$ if there exists a vector $(s_1,\ldots,s_n)$ such that $(q_1,\ldots,q_n)=(q_1,\ldots,q_n)*(s_1,\ldots,s_n)*(q_1,\ldots,q_n)$.

By definition of $*$, a vector $(q_1,\ldots,q_n)$ is regular with respect to $*$ if and only if

$$q_i = S^n(S^n(q_i,s_1,\ldots,s_n),q_1,\ldots,q_n) \text{ for } 1\le i\le n.$$

Moreover we define (see [10]):

Definition 3.1. A vector $(q_1,\ldots,q_n)$ is called a v-regular element if there exists an s such that $(q_1,\ldots,q_n)=(q_1,\ldots,q_n)*(s,\ldots,s)*(q_1,\ldots,q_n)$.

By definition of $+$ an element $t\in W_\tau(X_n)$ is regular with respect to $+$ if there exists a term $s\in W_\tau(X_n)$ such that $t=S^n(S^n(t,s,\ldots,s),t,t,\ldots,t)$.

We define also (see [9]):

Definition 3.2. An element $t\in W_\tau(X_n)$ is called weakly regular if there exist $s_1,\ldots,s_n\in W_\tau(X_n)$ such that $t=S^n(S^n(t,s_1,s_2,\ldots,s_n),t,t,\ldots,t)$.

To determine regular vectors with respect to $*$ we need the following lemma:

Lemma 3.3. *Let $q_1,q_2,\ldots,q_n,s_1,s_2,\ldots,s_n\in W_\tau(X_n)$. Then*

$$(q_1,\ldots,q_n)=(q_1,\ldots,q_n)*(s_1,\ldots,s_n)*(q_1,\ldots,q_n)$$

if and only if for all $i,j\in\{1,2,\ldots,n\}$ we have:

$$x_j\in var(q_i)\Rightarrow \exists\ l\in\{1,2,\ldots,n\}\ (s_j=x_l \text{ and } q_l=x_j).$$

Proof.
$(q_1,\dots,q_n)*(s_1,\dots,s_n)*(q_1,\dots,q_n)=(q_1,\dots,q_n)$

$\Leftrightarrow (S^n(q_1,S^n(s_1,q_1,\dots,q_n),\dots,S^n(s_n,q_1,\dots,q_n)),\dots,$

$S^n(q_n,S^n(s_1,q_1,\dots,q_n),\dots,S^n(s_n,q_1,\dots,q_n)))=(q_1,\dots,q_n)$(using (C1))

$\Leftrightarrow \forall i\in\{1,2,\dots,n\}(S^n(q_i,S^n(s_1,q_1,\dots,q_n),\dots,S^n(s_n,q_1,\dots,q_n))=q_i)$

$\Leftrightarrow \forall i\in\{1,2,\dots,n\}(\forall j\in\{1,2,\dots,n\}$

$(x_j\in var(q_i)\Rightarrow S^n(s_j,q_1,\dots,q_n)=x_j))$ (by Lemma 2.1)

$\Leftrightarrow \forall i,j\in\{1,2,\dots,n\}(x_j\in var(q_i)\Rightarrow S^n(s_j,q_1,\dots,q_n)=x_j)$

$\Leftrightarrow \forall i,j\in\{1,2,\dots,n\}(x_j\in var(q_i)\Rightarrow s_j=x_l$ and $q_l=x_j$

for some $l\in\{1,2,\dots,n\}$) (by Lemma 2.2).

■

Then we obtain the following result:

Theorem 3.4. *Let $q_1,\dots,q_n\in W_\tau(X_n)$. Then $(q_1,\dots,q_n)$ is a regular vector in $(W_\tau(X_n)^n;*)$ if and only if for any $i,j\in\{1,2,\dots,n\}$ we have: if $x_j\in var(q_i)$ then there exists $l\in\{1,2,\dots,n\}$ such that $q_l=x_j$.*

Proof. "$\Leftarrow$" Follows from the previous lemma.

"$\Rightarrow$" Assume that for each $i,j\in\{1,2,\dots,n\}$ we have: if $x_j\in var(q_i)$ then there exists an element $l\in\{1,2,\dots,n\}$ such that $q_l=x_j$. For each $1\le j\le n$ we obtain: if $x_j\in\bigcup\limits_{i=1}^{n}var(q_i)$, then there exists an element l with $1\le l\le n$ such that $q_l=x_j$. We select an index l_j, so that $q_{l_j}=x_j$. Since $\bigcup\limits_{i=1}^{n}var(q_i)\neq\emptyset$, let $x_k\in\bigcup\limits_{i=1}^{n}var(q_i)$ be fixed. Then we define s_j for $1\le j\le n$ as follows:

$$s_j=\begin{cases} x_{l_j} & \text{if} \quad x_j\in\bigcup\limits_{i=1}^{n}var(q_i)\\ x_k & \text{otherwise.}\end{cases}$$

It is not difficult to see that for each $1 \leq i, j \leq n$ we get: if $x_j \in var(q_i)$, then there exists an element $1 \leq l \leq n$ such that $s_j = x_l$ and $q_l = x_j$. By Lemma 3.3 we have that $(q_1, \ldots, q_n)$ is a regular element. ∎

Now we determine regular elements with respect to the operation $+$. We need the following lemma:

Lemma 3.5. *Let $s, t \in W_\tau(X_n)$. Then*

$$S^n(s, t, t, \ldots, t) = t \quad \textit{if and only if} \quad s = x_i \ \textit{ for some } \ 1 \leq i \leq n.$$

(That means, $s + t = t$ if and only if $s = x_i$ for some $1 \leq i \leq n$).

Proof. If $s = x_i$ for some $1 \leq i \leq n$ then $S^n(s, t, t, \ldots, t) = t$. Now we assume $S^n(s, t, t, \ldots, t) = t$ and $s \notin X_n$. Then $op(s) \geq 1$ and we get

$$\begin{aligned} op(t) &= op(S^n(s, t, t, \ldots, t)) \\ &= \sum_{j=1}^{n} vb_j(s) op(t) + op(s) \\ &\geq op(t) + op(s) \\ &> op(t) \end{aligned}$$

which is a contradiction. Hence $s \in X_n$. ∎

From Theorem 3.4 using the embedding described in section 1, for regular elements with respect to $+$ we obtain the following result:

Corollary 3.6. *A term $t \in W_\tau(X_n)$ is regular with respect to $+$ if and only if it is idempotent.*

By definition of the operation $*$ on $W_\tau(X_n)^n$ a vector $(q_1, \ldots, q_n)$ is v-regular if and only if there exists an element $s \in W_\tau(X_n)$ such that

$$q_i = S^n(S^n(q_i, s, \ldots, s), q_1, \ldots, q_n)$$

for $1 \leq i \leq n$. By (C1) this means that

$$q_i = S^n(q_i, S^n(s, q_1, \ldots, q_n), \ldots, S^n(s, q_1, \ldots, q_n)),$$

for $1 \leq i \leq n$. By Lemma 2.1 this is satisfied if and only if there is a j with $1 \leq j \leq n$ such that $q_i = x_j$ for all $1 \leq i \leq n$. Therefore, $(q_1, \ldots, q_n)$ is v-regular if and only if there is an integer j with $1 \leq j \leq n$ such that $q_i = x_j$ for every i with $1 \leq i \leq n$.

By Lemma 3.5 the term $t \in W_\tau(X_n)$ is weakly regular if and only if there exists an integer i with $1 \leq i \leq n$ and terms $s_1, \ldots, s_n$ such that $x_i = S^n(t, s_1, \ldots s_n)$. By Lemma 2.2 this is satisfied if and only if there is an integer j for $1 \leq j \leq n$ such that $t = x_j$ and $s_j = x_i$. Altogether, this means that with respect to the binary operation $+$ on $W_\tau(X_n)$ the concepts of regular and of weakly regular elements are equal.

Now we consider regularity of terms of type τ_n with respect to the operations $*_g$ and $+_g$, respectively, derived from the generalized superposition operation S^n_g.

Let $X' := X \setminus X_n$. If $q_1, \ldots, q_n \in W_{\tau_n}(X')$, then for any $s_1, \ldots s_n \in W_{\tau_n}(X)$

$$\begin{aligned} &S^n_g(S^n_g(q_i, s_1, \ldots, s_n), q_1, \ldots, q_n) \\ &\quad = S^n_g(q_i, q_1, \ldots q_n) \\ &\quad = q_i \text{ for } 1 \leq i \leq n, \end{aligned}$$

i.e. every $(q_1, \ldots, q_n) \in W_{\tau_n}(X')^n$ is regular with respect to $*_g$.

If $t \in W_{\tau_n}(X')$, then for every $s \in W_{\tau_n}(X)$ we have

$$S^n_g(S^n_g(t, s, \ldots, s), t, \ldots, t) = t.$$

Therefore, every $t \in W_{\tau_n}(X')$ is regular with respect to $+_g$. This gives the following results:

Theorem 3.7. *Let $q_1, \ldots, q_n \in W_{\tau_n}(X)$. Then $(q_1, \ldots, q_n)$ is a regular vector in $(W_{\tau_n}(X)^n; *_g)$ if and only if for any $i, j \in \{1, 2, \ldots, n\}$ we have: if $x_j \in var(q_i)$ then there exists $l \in \{1, 2, \ldots, n\}$ such that $q_l = x_j$.*

Theorem 3.8. *A term $t \in W_{\tau_n}(X)$ is regular with respect to $+_g$ if and only if it is idempotent.*

4. Ideals in menger algebras and Green's relations

In the next sections we will study Green's relations in Menger algebras of terms. Green's relations are studied for semigroups or for monoids. Therefore, it is quite natural to study Green's relations for $(W_\tau(X_n); +)$ and for $(W_{\tau_n}(X); +_g)$. For Menger algebras of rank n in [7] Green's relations were defined by using of ideals.

In [9], different kinds of ideals in Menger algebras of rank n were defined as follows:

Definition 4.1. Let $(M; S^n)$ be a Menger algebra of rank n. A nonempty subset H of M is called an s-ideal, if from $h \in H$ there follows $S^n(h, t_1, \ldots, t_n) \in H$ for all $t_1, \ldots, t_n \in M$. A set H is called a v-ideal, if from $h_1, \ldots, h_n \in H$ there follows $S^n(t, h_1, \ldots, h_n) \in H$ for all $t \in M$. The set H is called an l-ideal, if at least one of $t, h_1, \ldots, h_n$ belongs to H, then $S^n(t, h_1, \ldots, h_n)$ belongs to H.

Definition 4.2. Let $(M; S^n)$ be a Menger algebra of rank n and let $a, b \in M$.

(i) $a\mathcal{L}b$ if either $a = b$ or if there are elements $s_1, \ldots, s_n, t_1, \ldots, t_n \in M$ such that $S^n(a, s_1, \ldots, s_n) = b$ and $S^n(b, t_1, \ldots, t_n) = a$.

(ii) $a\mathcal{R}b$ if either $a = b$ or if there are elements $s, t \in W_\tau(X_n)$ such that $S^n(s, a, a, \ldots, a) = b$ and $S^n(t, b, b, \ldots, b) = a$.

(iii) $\mathcal{D} = \mathcal{R} \circ \mathcal{L} (= \mathcal{L} \circ \mathcal{R})$.

(iv) $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$.

(v) $a\mathcal{J}b$ if either $a = b$ or if there are elements $s, s_1, \ldots, s_n \in W_\tau(X_n)$ with $a = S^n(s, s_1, \ldots, s_n)$ such that at least one of the factors is equal to b and there are elements $t, t_1, \ldots, t_n \in M$ with $b = S^n(t, t_1, \ldots, t_n)$ such that at least one of the factors is equal to a.

It can be proved that $(a, b) \in \mathcal{L}$ if and only if a and b generate the same s-ideal, $(a, b) \in \mathcal{R}$ if and only if a and b generate the same v-ideal, and also $(a, b) \in \mathcal{J}$ if and only if a and b generate the same l-ideal; $\mathcal{L}$ and $\mathcal{R}$ are commuting equivalence relations ([9]).

Let $\triangle_{W_\tau(X_n)}$ be the set of all pairs (s, s) of terms from $W_\tau(X_n)$ (the diagonal of $W_\tau(X_n)$).

Theorem 4.3. *Let* $a, b \in W_\tau(X_n)$. *Then* $a\mathcal{R}b$ *iff* $a = b$, *i.e.,* $\mathcal{R} = \triangle_{W_\tau(X_n)}$.

Proof. If $a = b$, then by definition of $\mathcal{R}$ we have $a\mathcal{R}b$. Assume that $a\mathcal{R}b$. Then there exist $s, t \in W_\tau(X_n)$ such that $a = S^n(s, b, b, \ldots, b)$ and $b = S^n(t, a, a, \ldots, a)$. This implies

$$a = S^n(s, S^n(t, a, a, \ldots, a), S^n(t, a, a, \ldots, a), \ldots, S^n(t, a, a, \ldots, a))$$

and then

$$a = S^n(S^n(s, t, t, \ldots, t), a, a, \ldots, a) \text{ by (C1).}$$

By Lemma 3.3 and 2.2, we have $S^n(s, t, \ldots, t) = x_i$ for some $1 \le i \le n$ and so $s = x_j$ and $t = x_i$ for some $1 \le j \le n$. We obtain $a = S^n(x_j, b, \ldots, b) = b$. ∎

Then we get:

Corollary 4.4. $\mathcal{H} = \mathcal{R}$ *and* $\mathcal{L} = \mathcal{D}$.

For Green's relation $\mathcal{L}$ we have:

Theorem 4.5. *Let* $a, b \in W_\tau(X_n)$. *Then* $a\mathcal{L}b$ *if and only if there exists a permutation* r *on the set* $\{1, 2, \ldots, n\}$ *such that* $b = S^n(a, x_{r(1)}, x_{r(2)}, \ldots, x_{r(n)})$.

Proof. Assume that there exists a permutation r on $\{1, 2, \ldots, n\}$ such that $b = S^n(a, x_{r(1)}, x_{r(2)}, \ldots, x_{r(n)})$. Then

$$\begin{aligned}
&S^n(b, x_{r^{-1}(1)}, x_{r^{-1}(2)}, \ldots, x_{r^{-1}(n)}) \\
&\quad = S^n(S^n(a, x_{r(1)}, x_{r(2)}, \ldots, x_{r(n)}), x_{r^{-1}(1)}, x_{r^{-1}(2)}, \ldots, x_{r^{-1}(n)}) \\
&\quad = S^n(a, S^n(x_{r(1)}, x_{r^{-1}(1)}, x_{r^{-1}(2)}, \ldots, x_{r^{-1}(n)}), \ldots, \\
&\qquad S^n(x_{r(n)}, x_{r^{-1}(1)}, x_{r^{-1}(2)}, \ldots, x_{r^{-1}(n)})) \text{ (by (C1))} \\
&\quad = S^n(a, x_{r^{-1}(r(1))}, \ldots, x_{r^{-1}(r(n))}) \\
&\quad = S^n(a, x_1, x_2, \ldots, x_n) = a.
\end{aligned}$$

Therefore $a\mathcal{L}b$.

Conversely, assume that $a\mathcal{L}b$. Then there exist elements $t_1, \ldots, t_n$ and $s_1, \ldots, s_n \in W_\tau(X_n)$ such that $S^n(a, t_1, \ldots, t_n) = b$ and $S^n(b, s_1, \ldots, s_n) = a$. Then we have:

$$S^n(S^n(a, t_1, \ldots, t_n), s_1, \ldots, s_n) = a$$

$$\Rightarrow S^n(a, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)) = a \ \ (\text{by } \ (\text{C1})).$$

If $var(a) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ then $S^n(t_{i_j}, s_1, \ldots, s_n) = x_{i_j}$ for all $j = 1, 2, \ldots, k$ and then for each i_j there exists an l_j such that $t_{i_j} = x_{l_j}$ and $s_{l_j} = x_{i_j}$.

We notice that for $1 \leq p, q \leq k$ we have: if $l_p = l_q$ then $i_p = i_q$ because of $s_{l_p} = s_{l_q} \Rightarrow x_{i_p} = x_{i_q} \Rightarrow i_p = i_q$.

Therefore $|\{i_1, i_2, \ldots, i_k\}| = |\{l_1, l_2, \ldots, l_k\}|$ implies that there exists a permutation r on the set $\{1, 2, \ldots, n\}$ such that $r(i_j) = l_j$ for all $1 \leq j \leq k$. Then we have

$$\begin{aligned} b &= S^n(a, t_1, \ldots, t_{i_1}, \ldots, t_{i_2}, \ldots, t_{i_k}, \ldots, t_n) \\ &= S^n(a, t_1, \ldots, x_{l_1}, \ldots, x_{l_2}, \ldots, x_{l_k}, \ldots, t_n) \\ &= S^n(a, t_1, \ldots, x_{r(i_1)}, \ldots, x_{r(i_2)}, \ldots, x_{r(i_k)}, \ldots, t_n) \\ &= S^n(a, x_{r(1)}, \ldots, x_{r(i_1)}, \ldots, x_{r(i_2)}, \ldots, x_{r(i_k)}, \ldots, x_{r(n)}). \end{aligned}$$

(Since $x_{r(p)} \notin var(a)$ if $p \notin \{i_1, i_2, \ldots, i_k\}$). ■

Then for Green's relation $\mathcal{J}$ we have:

Theorem 4.6.

(i) *If* $n = 1$ *then* $\mathcal{J} = \triangle_{W_\tau(X_1)}$.

(ii) *If* $n \geq 2$ *then* $\mathcal{J} = W_\tau(X_n) \times W_\tau(X_n)$.

Proof.

(i) Let $a, b \in W_\tau(X_1)$ such that $a\mathcal{J}b$. Then we have the following four cases.

(1) $a = S^1(b, t)$ and $b = S^1(a, s)$ for some $t, s \in W_\tau(X_1)$.

(2) $a = S^1(b, t)$ and $b = S^1(s, a)$ for some $t, s \in W_\tau(X_1)$.

(3) $a = S^1(t, b)$ and $b = S^1(a, s)$ for some $t, s \in W_\tau(X_1)$.

(4) $a = S^1(t, b)$ and $b = S^1(s, a)$ for some $t, s \in W_\tau(X_1)$.

In all cases we obtain $op(a) = op(b) + op(t)$ and $op(b) = op(a) + op(s)$. These imply $op(a) = op(a) + op(s) + op(t)$ and then $op(s) = op(t) = 0$ and hence $s = x_1 = t$. Thus, we have $a = b$ in all four cases.

(ii) Let $a, b \in W_\tau(X_n)$ where $n \geq 2$. Then

$$a = S^n(x_1, a, b, b, \ldots, b) \text{ and } b = S^n(x_1, b, a, a, \ldots, a).$$

This means $a\mathcal{J}b$ for all $a, b, \in W_\tau(X_n)$. Hence $\mathcal{J} = W_\tau(X_n) \times W_\tau(X_n)$. ■

Now for a type τ_n we consider the generalized superposition operation S_g^n and the Menger algebra $clone_g\tau_n$ with infinitely many nullary operations. Corresponding to Definition 4.2 we define Green's relations $\mathcal{L}^g, \mathcal{R}^g, \mathcal{D}^g, \mathcal{H}^g$ and $\mathcal{J}^g$. Let $X' = X \setminus X_n$. For Green's relation $\mathcal{L}^g$ we have:

Theorem 4.7. *Let $a, b \in W_{\tau_n}(X)$. Then $a\mathcal{L}^g b$ if and only if there exists a permutation r on the set $\{1, 2, \ldots, n\}$ such that $b = S_g^n(a, x_{r(1)}, \ldots, x_{r(n)})$.*

Proof. By definition we have $a\mathcal{L}^g b$ if either $a = b$ or if there are elements $s_1, \ldots, s_n, t_1, \ldots, t_n \in W_{\tau_n}(X)$ such that $S^n(a, s_1, \ldots, s_n) = b$ and $S^n(b, t_1, \ldots, t_n) = a$.

We consider the following cases:

1. $var(a)$ or $var(b) \subseteq X'$: In this case from $a\mathcal{L}^g b$ we get $b = S_g^n(a, s_1, \ldots, s_n) = a$.

2. $var(a)$, $var(b) \not\subseteq X'$: In this case we have $var(a) \cap X_n \neq \emptyset$ and $var(b) \cap X_n \neq \emptyset$. Now we proceed as in the proof of Theorem 4.5.

■

For $\mathcal{R}^g$ we have:

Theorem 4.8.
$$\mathcal{R}^g = W_{\tau_n}(X')^2 \cup \triangle_{W_{\tau_n}(X)}.$$

Proof. By definition we have $a\mathcal{R}^g b$ if either $a = b$ or if there are elements $s, t \in W_{\tau_n}(X)$ such that $S^n_g(s, a, \ldots, a) = b$ and $S^n_g(t, b, \ldots, b) = a$. We consider again the following cases:

1. $var(a)$ or $var(b) \subseteq X'$: If $var(a) \subseteq X'$, then $var(S^n_g(s, a, \ldots, a)) = var(b) \subseteq X'$ and conversely, if $var(b) \subseteq X'$, then $var(a) \subseteq X'$. So, we may assume that both, $var(a)$ and $var(b)$ are subsets of X'. Let $\mathcal{R}^g|_{W_{\tau_n}(X')^2} \subseteq W_{\tau_n}(X')^2$ be the restriction of $\mathcal{R}^g$ to $W_{\tau_n}(X')$. If $var(a)$, $var(b) \subseteq X'$, then $S^n_g(b, a, \ldots, a) = b$ and $S^n_g(a, b, \ldots, b) = a$ and therefore $a\mathcal{L}^g b$. This shows $\mathcal{R}^g|_{W_{\tau_n}(X')^2} = W_{\tau_n}(X')^2$.

2. $var(a) \not\subseteq X'$ and $var(b) \not\subseteq X'$: In this case we have $var(s) \cap X_n \neq \emptyset, var(t) \cap X_n \neq \emptyset, var(S^n_g(s, t, \ldots, t)) \cap X_n \neq \emptyset$ and $var(S^n_g(t, s, \ldots, s)) \cap X_n \neq \emptyset$. From $a\mathcal{R}^g b$ we obtain by substitution and by (Cg1), $S^n_g(S^n_g(s, t, \ldots, t), b, \ldots, b) = b$ and $S^n_g(S^n_g(t, s, \ldots, s), a, \ldots, a) = a$. Similar to Lemma 3.5 there follows
$$S^n_g(s, t, \ldots, t) = x_i, S^n_g(t, s, \ldots, s) = x_j$$
for some $1 \leq i, j \leq n$. But then similar as in Lemma 2.2 we conclude that $s = x_j, t = x_i$ and $t = x_i, s = x_j$, i.e. $s = t = x_i$. Then we have $a = b$. The converse is clear.

■

For Green's relation $\mathcal{J}^g$ we have:

Theorem 4.9. *If $n = 1$, then $\mathcal{J}^g = W_{\tau_1}(X') \times W_{\tau_1}(X') \cup \triangle_{W_{\tau_1}(X)}$.*

Proof. We consider the following two cases:

1. $var(a) \subseteq X'$ or $var(b) \subseteq X'$: In this case from $var(a) \subseteq X'$ we have $a = S^1_g(a, b)$ and then $var(b) \subseteq X'$ and from $var(b) \subseteq X'$ and $b = S^1_g(b, a)$, we obtain $var(a) \subseteq X'$. Therefore in this case we get $\mathcal{J}^g|W_{\tau_1}(X')^2 = W_{\tau_1}(X')^2$.

2. $var(a) \cap X_n \neq \emptyset$ and $var(b) \cap X_n \neq \emptyset$: Assume that $a \neq b$ and $a\mathcal{J}^g b$, then the following cases are possible:

(1) $a = S^1_g(b,t)$ and $b = S^1_g(a,s)$ for some $t, s \in W_{\tau_1}(X)$.

(2) $a = S^1(b,t)$ and $b = S^1(s,a)$ for some $t, s \in W_{\tau_1}(X)$.

(3) $a = S^1(t,b)$ and $b = S^1(a,s)$ for some $t, s \in W_{\tau_1}(X)$.

(4) $a = S^1(t,b)$ and $b = S^1(s,a)$ for some $t, s \in W_{\tau_1}(X)$.

Since in the first case $a = S^1_g(b,t)$ and $var(a) \cap X_1 \neq \emptyset$ we have $var(t) \cap X_1 \neq \emptyset$ and similarly we have $var(s) \cap X_1 \neq \emptyset$. In the other three cases one has also $var(t) \cap X_1 \neq \emptyset$ and $var(s) \cap X_1 \neq \emptyset$. Using the formula

$$op(S^1_g(q,t)) = op(t) + op(q), \quad q, t \in W_{\tau_1}(X)$$

it is not difficult to see that for arbitrary terms $u, v, w \in W_{\tau_1}(X)$ from $u = S^1_g(v,w)$, $var(u) \cap X_1 \neq \emptyset$ and $var(v) \cap X_1 \neq \emptyset$ there follows $op(u) \geq op(v) + op(w)$. Using this inequality we obtain in all four cases $op(s) = 0$ and $op(t) = 0$ and thus by $var(t) \cap X_1 \neq \emptyset, var(s) \cap X_1 \neq \emptyset$ we get $s = t = x_1$ and then $a = b$, a contradiction. Altogether we have $\mathcal{J}^g = W_{\tau_1}(X')^2 \cup \triangle_{W_{\tau_1}(X)}$. ■

For $n \geq 2$ we obtain

Theorem 4.10. *If $n \geq 2$, then $\mathcal{J}^g = W_{\tau_n}(X) \times W_{\tau_n}(X)$.*

Proof. Let $(a,b) \in W_{\tau_n}(X)^2$. Then

$$a = S^n_g(x_1, a, b, \ldots, b) \text{ and } b = S^n_g(x_1, b, a, \ldots, a).$$

This means $a\mathcal{J}^g b$. ■

Moreover, we have $\mathcal{H}^g = \mathcal{R}^g \cap \mathcal{L}^g = \triangle_{W_{\tau_n}(X)}$ and $\mathcal{D}^g = \mathcal{R}^g \vee \mathcal{L}^g = W_{\tau_n}(X')^2 \vee \mathcal{L}^g$. For $\mathcal{D}^g$ we have even:

Corollary 4.11. *$\mathcal{D}^g = \mathcal{R}^g \cup \mathcal{L}^g = W_{\tau_n}(X')^2 \vee \mathcal{L}^g$.*

Proof. The inclusion $\mathcal{R}^g \cup \mathcal{L}^g \subseteq \mathcal{D}^g$ is clear. Let $(a,b) \in \mathcal{D}^g$. By $\mathcal{D}^g = \mathcal{L}^g \circ \mathcal{R}^g$, there is a term $c \in W_{\tau_n}(X)$ such that $(a,c) \in \mathcal{R}^g$ and $(c,b) \in \mathcal{L}^g$. We consider the following two cases:

1. $var(a) \subseteq X'$ or $var(b) \subseteq X'$: Since $(c,b) \in \mathcal{L}^g$, by Theorem 4.7 there is a permutation r on $\{1,2,\ldots,n\}$ such that $b = S_g^n(c, x_{r(1)}, \ldots, x_{r(n)})$. If $var(b) \subseteq X'$, we get $var(c) \subseteq X'$ and $b = c$. It follows $(a,b) = (a,c) \in \mathcal{R}^g$ and $var(a) \subseteq X'$. If $var(a) \subseteq X'$, then $(a,c) \in \mathcal{R}^g$ implies $var(c) \subseteq X'$ and then by $(c,b) \in \mathcal{L}^g$ also $var(b) \subseteq X'$ and we continue as we did in the first case. This shows also
$$var(a) \subseteq X' \text{ or } var(b) \subseteq X' \Leftrightarrow var(a) \subseteq X' \text{ and } var(b) \subseteq X'.$$
2. $var(a) \cap X_n \neq \emptyset$ and $var(b) \cap X_n \neq \emptyset$: Then by Theorem 4.8 we get $a = c$ and this implies $(a,b) = (c,b) \in \mathcal{L}^g$.

Altogether we have $\mathcal{D}^g \subseteq \mathcal{R}^g \cup \mathcal{L}^g$. ■

5. Green's Relations on $(W_\tau(X_n);+)$

Now we consider Green's relations with respect to the the semigroup $(W_\tau(X_n);+)$ where $+$ is defined by $a+b := S^n(a,b,b\ldots,b)$. Let $(W_\tau(X_n))^+$ be the monoid arising from $(W_\tau(X_n);+)$ by adding a neutral element 0. Corresponding to the usual definition for Green's relations $\mathcal{L}_+, \mathcal{R}_+, \mathcal{D}_+, \mathcal{J}_+$, and $\mathcal{H}_+$ we have:

$$a\mathcal{R}_+b :\Leftrightarrow a = b \text{ or } a = S^n(b,s,s,\ldots,s) \text{ and}$$
$$b = S^n(a,t,t,\ldots,t) \text{ for some } s,t \in W_\tau(X_n).$$
$$a\mathcal{L}_+b :\Leftrightarrow a = b \text{ or } a = S^n(s,b,b,\ldots,b) \text{ and}$$
$$b = S^n(t,a,a,\ldots,a) \text{ for some } s,t \in W_\tau(X_n).$$
$$a\mathcal{J}_+b :\Leftrightarrow a = S^n(s, S^n(b,t,t,\ldots,t),\ldots,S^n(b,t,t,\ldots,t)) \text{ and}$$
$$b = S^n(s', S^n(a,t',t',\ldots,t'),\ldots,S^n(a,t',t',\ldots,t'))$$
$$\text{for some } s,s',t,t' \in (W_\tau(X_n))^+.$$

$\mathcal{H}_+ = \mathcal{L}_+ \cap \mathcal{R}_+$ and $\mathcal{D}_+ = \mathcal{R}_+ \circ \mathcal{L}_+ (= \mathcal{L}_+ \circ \mathcal{R}_+)$.

By definition and Theorem 4.3 we have $\mathcal{L}_+ = \mathcal{R} = \triangle_{W_\tau(X_n)}, \mathcal{H}_+ = \mathcal{L}_+$, and $\mathcal{D}_+ = \mathcal{R}_+$. It is left to determine $\mathcal{R}_+$ and $\mathcal{J}_+$.

Theorem 5.1. *Let $a, b \in W_\tau(X_n)$. Then*

$$a\mathcal{R}_+b :\Leftrightarrow a = b \text{ or } a = S^n(b, x_i, x_i, \ldots, x_i) \quad \text{and } b = S^n(a, x_j, x_j, \ldots, x_j)$$

for some $i, j \in \{1, 2, \ldots, n\}$.

Proof. Assume that $a\mathcal{R}_+b$. If $a \neq b$ then $a = S^n(b, s, s, \ldots, s)$ and $b = S^n(a, t, t, \ldots, t)$ for some $s, t \in W_\tau(X_n)$, so we have

$$\begin{aligned} a &= S^n(b, s, s, \ldots, s) \\ &= S^n(S^n(a, t, t, \ldots, t), s, \ldots, s) \\ &= S^n(a, S^n(t, s, s, \ldots, s), \ldots, S^n(t, s, s, \ldots, s)) \text{ (by (C1))}. \end{aligned}$$

By Lemma 2.2, we get $S^n(t, s, \ldots, s) = x_j$ for some j and then $var(a) = \{x_j\}$ because of $a = S^n(a, x_j, \ldots, x_j)$. Since $S^n(t, s, \ldots, s) = x_j$, by Lemma 2.2 we have $t = x_i$ and $s = x_j$ for some $1 \leq i \leq n$. Then $b = S^n(a, t, \ldots, t) = S^n(a, x_i, \ldots, x_i)$ and $a = S^n(b, x_j, \ldots, x_j)$.

Conversely, if $a = b$ or $a = S^n(b, x_i, \ldots, x_i)$ and $b = S^n(a, x_j, \ldots, x_j)$ for some $i, j \in \{1, 2, \ldots, n\}$, then by definition of $\mathcal{R}_+$ we have $a\mathcal{R}_+b$. ■

This means, if $a\mathcal{R}_+b$ and $a \neq b$, then $var(a) = x_i$ and $var(b) = x_j$ and a arises from b by exchanging x_i and x_j.

For $\mathcal{J}_+$ we have:

Theorem 5.2. $\mathcal{J}_+ = \mathcal{R}_+$.

Proof. We will show that $\mathcal{J}_+ \subseteq \mathcal{R}_+$. Let $a, b \in W_\tau(X_n)$ such that $a\mathcal{J}_+b$. Then $a = s + b + t$ and $b = s' + a + t'$ for some $s, s', t, t' \in (W_\tau(X_n))^+$. So we have

$$\begin{aligned} op(a) &= op(s + b + t) \\ &= op(S^n(s, S^n(b, t, \ldots, t), \ldots, S^n(b, t, \ldots, t)) \\ &\geq op(s) + op(S^n(b, t, \ldots, t)) \\ &\geq op(s) + op(b) + op(t) \\ &= op(s) + op(s' + a + t') + op(t) \\ &\geq op(s) + op(s') + op(a) + op(t') + op(t). \end{aligned}$$

It follows that $op(s) + op(s') + op(t') + op(t) = 0$, and then s, s', t, t' are variables or the neutral element of the monoid $(W_\tau(X_n))^+$. It is not difficult to see that in all of the cases we get

$$a = b + t \text{ and } b = a + t'.$$

This means $a\mathcal{R}_+b$. Since $\mathcal{R}_+ \subseteq \mathcal{J}_+$, we get $\mathcal{J}_+ = \mathcal{R}_+$.

■

Altogether we have $\mathcal{H}_+ = \mathcal{L}_+ = \triangle_{W_\tau(X_n)}$, $\mathcal{R}_+ = \mathcal{D}_+ = \mathcal{J}_+$.

If we consider only unary terms, then $S^1 = +$. In this case, all of Green's relations are equal.

Proposition 5.3. *In the diagonal Menger algebra* $(W_\tau(X_1); +)$ *(in the monoid* $(W_\tau(x_1); S^1, x_1)$*) , we have*

$$\mathcal{H}_+ = \mathcal{L}_+ = \mathcal{R}_+ = \mathcal{D}_+ = \mathcal{J}_+ = \triangle_{W_\tau(X_1)}.$$

Proof. It is enough to show that $\mathcal{J} \subseteq \triangle_{W_\tau(X_1)}$. Let $a, b \in W_\tau(X_1)$ and $a\mathcal{J}b$. Then there are elements $s, t, s', t' \in (W_\tau(X_1))^1$ such that $a = s + b + t$ and $b = s' + a + t'$. Consider

$$\begin{aligned} a &= s + b + t \\ &= S^1(s, S^1(b, t)) \\ &= S^1(s, S^1(s' + a + t', t)) \\ &= S^1(s, S^1(S^1(s', S^1(a, t')), t)). \end{aligned}$$

Then $op(a) = op(s) + op(s') + op(a) + op(t') + op(t)$.

This implies $op(s) = op(s') = op(t') = op(t) = 0$, and so $s = s' = t = t' = x_1$. Since x_1 is a neutral element of $(W_\tau(X_1); +)$, we get $a = x_1 + b + x_1 = b$. Hence $\mathcal{J} = \triangle_{W_\tau(X_1)}$. ■

Now for a type τ_n we consider the generalized superposition operation S^n_g and the diagonal algebra $(W_{\tau_n}(X); +_g)$, where $a +_g b := S^n_g(a, b, \ldots, b)$. Then we define Green's relations $\mathcal{L}^g_+, \mathcal{R}^g_+, \mathcal{D}^g_+, \mathcal{H}^g_+$ and $\mathcal{J}^g_+$ in the usual way. Let $X' = X \setminus X_n$. For Green's relation $\mathcal{R}^g_+$ we have:

Theorem 5.4. *Let $a, b \in W_{\tau_n}(X)$. Then*

$$a\mathcal{R}_+^g b :\Leftrightarrow a = b \text{ or } a = S_g^n(b, x_i, x_i, \ldots, x_i) \text{ and } b = S_g^n(a, x_j, x_j, \ldots, x_j)$$

for some $i, j \in \{1, 2, \ldots, n\}$.

Proof. We consider the following two cases:

1. $var(a) \subseteq X'$ or $var(b) \subseteq X'$: If $var(a) \subseteq X'$, then $S_g^n(a, t, \ldots, t) = a$ for arbitrary $t \in W_{\tau_n}(X)$ and if $var(b) \subseteq X'$, then $S_g^n(b, s, \ldots, s) = b$ for arbitrary $s \in W_{\tau_n}(X)$. This shows that $\mathcal{R}_+^g|_{W_{\tau_n}(X')^2} = \triangle_{W_{\tau_n}(X')}$.
2. $var(a) \cap X_n \neq X_n$ and $var(b) \cap X_n \neq X_n$: If $a\mathcal{R}_+^g b$ and $a \neq b$, then $a = S_g^n(b, s, \ldots, s)$ and $b = S_g^n(a, t, \ldots, t)$ for some $s, t \in W_{\tau_n}(X)$. Then we obtain

$$\begin{aligned} a &= S_g^n(b, s, \ldots, s) \\ &= S_g^n(S_g^n(a, t, \ldots, t), s, \ldots, s) \\ &= S_g^n(a, S_g^n(t, s, \ldots, s), \ldots, S_g^n(t, s, \ldots, s)) \text{ by (Cg1).} \end{aligned}$$

By $var(a) \cap X_n \neq \emptyset$, similar to Lemma 2.1 we obtain $S_g^n(t, s, \ldots, s) = x_i$ for $x_i \in var(a) \cap X_n$ and then similar as in Lemma 2.2 we get $t = x_j$ and $s = x_i$ for some $1 \leq j \leq n$. ∎

By definition and Theorem 4.8 we have $\mathcal{L}_+^g = \mathcal{R}^g = W_{\tau_n}(X')^2 \cup \triangle_{W_{\tau_n}(X)}$ and then $\mathcal{H}_+^g = \mathcal{R}_+^g \cap \mathcal{L}_+^g = \triangle_{W_{\tau_n}(X)}$. For $\mathcal{D}_+^g$ we get $\mathcal{D}_+^g = \mathcal{R}_+^g \vee \mathcal{L}_+^g = \mathcal{R}_+^g \vee W_{\tau_n}(X')^2$. It is left to determine $\mathcal{J}_+^g$.

Theorem 5.5.

$$\mathcal{J}_+^g = \mathcal{R}_+^g \cup \mathcal{L}_+^g = \mathcal{R}_+^g \cup W_{\tau_n}(X')^2.$$

Proof. Let $a, b, a \neq b \in W_{\tau_n}(X)$ such that $a\mathcal{J}_+^g b$. Then $a = s +_g b +_g t$ and $b = s' +_g a +_g t'$ for some $s, s', t, t' \in W_{\tau_n}(X)^+$. We consider the following two cases:

1. $var(a) \subseteq X'$ or $var(b) \subseteq X'$: Then $a +_g t' = a$ and therefore $b = s' +_g a$. Similarly, from $var(b) \subseteq X'$ there follows $a = s +_g b$. Now we show that $var(a) \subseteq X'$ if and only if $var(b) \subseteq X'$. Indeed, if $var(a) \subseteq X'$, and if we assume that $var(s') \subseteq X'$, then $s' +_g a = s'$ and $b = s' +_g a$ implies $b = s'$ and thus $var(b) \subseteq X'$. If $var(s') \cap X_n \neq \emptyset$, then from $b = s' +_g a$ we obtain
$$var(b) = var(a) \cup (var(s') \cap X') \subseteq X' \cup X' = X'.$$
Similarly, from $var(b) \subseteq X'$ there follows $var(a) \subseteq X'$. This means, that $var(a) \subseteq X'$ or $var(b) \subseteq X'$ implies $a = s +_g b$ and $b = s' +_g a$ for some $s, s' \in W_{\tau_n}(X)$ and then $a\mathcal{L}^g_+ b$. From $\mathcal{L}^g_+|_{W_{\tau_n}(X')^2} = W_{\tau_n}(X')^2$ we obtain $\mathcal{J}^g_+|_{W_{\tau_n}(X')^2} = W_{\tau_n}(X')^2$.

2. $var(a) \cap X_n \neq \emptyset$ and $var(b) \cap X_n \neq \emptyset$: From $a = s +_g b +_g t$ we get $var(s) \cap X_n \neq \emptyset$, since from $var(s) \subseteq X'$, we obtained $a = s$ and so $var(a) \subseteq X'$, a contradiction. In a similar way we show that $var(s') \cap X_n \neq \emptyset$. The next step is to show that $var(t) \cap X_n \neq \emptyset$ and $var(t') \cap X_n \neq \emptyset$. Indeed, if $var(t) \cap X_n = \emptyset$, then $var(t) \subseteq X'$ and then
$$var(a) \subseteq var(t) \cup (var(s +_g b) \cap X') \subseteq X' \cup X' = X',$$
a contradiction. This proves $var(t) \cap X_n \neq \emptyset$. Similarly, we show that $var(t') \cap X_n \neq \emptyset$.
Using the formula
$$\begin{aligned} &op(S^n_g(q, t_1, \ldots, t_n)) \\ &= \sum_{k=1}^{n} vb_k(q) op(t_k) + op(q), \quad q, t_1, \ \ldots, t_n \in W_{\tau_n}(X) \end{aligned}$$
it is not difficult to see that for arbitrary terms $u, v, w \in W_{\tau_n}(X)$ from $u = v +_g w$ and $var(u) \cap X_n \neq \emptyset$ and $var(v) \cap X_n \neq \emptyset$ there follows $op(u) \geq op(v) + op(w)$. Using this inequality we obtain

$$
\begin{aligned}
op(a) &= op(s +_g b +_g t) \\
&= op(S_g^n(s, S_g^n(b, t, \ldots, t), \ldots, S_g^n(b, t, \ldots, t)) \\
&\geq op(s) + op(b) + op(t) \\
&= op(s) + op(s' +_g a +_g t') + op(t) \\
&\geq op(s) + op(s') + op(a) + op(t') + op(t).
\end{aligned}
$$

It follows that $op(s) + op(s') + op(t') + op(t) = 0$, and then s, s', t, t' are variables or the neutral element of the monoid $(W_\tau(X_n))^+$. It is not difficult to see that in all of the cases we get

$$a = b +_g t \text{ and } b = a +_g t',$$

i.e. $a\mathcal{R}_+^g b$ and because of $\mathcal{R}_+^g \subseteq \mathcal{J}_+^g$ we have $\mathcal{J}_+^g = \mathcal{R}_+^g$ in the second case and together with the first case we obtain $\mathcal{J}_+^g = \mathcal{R}_+^g \cup W_{\tau_n}(X')^2$.

■

As a final result we get $\mathcal{L}_+^g = \mathcal{D}_+^g = \mathcal{J}_+^g = \triangle_{W_{\tau_n}(X)} \cup W_{\tau_n}(X')^2$, $\mathcal{H}_+^g = \triangle_{W_{\tau_n}(X)}$ and $\mathcal{R}_+^g$ as in Theorem 5.4.

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Received July 2005
Revised August 2005