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# EXTENSION OF CLASSICAL SEQUENCES TO NEGATIVE INTEGERS

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**Abstract**

We give a method to extend Bell exponential polynomials to negative indices. This generalizes many results of this type such as the extension to negative indices of Stirling numbers or of Bernoulli numbers.

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## 1. Introduction

Several classical sequences have a "natural" extension to negative indices which preserves algebraic relations. For example, the binomial polynomials

$$\binom{x}{k} = \frac{x(x-1)...(x-k+1)}{k!}$$

allows to define the binomial coefficients $\binom{n}{k}$ for $n \in \mathbb{Z}$, $k \in \mathbb{N}$.

The sequence $(x)_n = x(x-1)...(x-n+1), n \in \mathbb{N}$, is extended to negative integers by

$$(x)_{-n} = \frac{1}{(x+1)...(x+n)}$$

so that the relation

$$(x)_n(x-n)_m = (x)_{n+m}$$

remains valid for $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$.

The factorial sequence $\gamma(n) = n!$ is classically extended in [3] by

$$\gamma(-n) = \frac{(-1)^{n-1}}{(n-1)!}\,,\ n > 0\,.$$

In [1], extensions of the Stirling numbers of the second kind, $S(n,k)$, and of the first kind, $s(n,k)$, are obtained for negative $n$. We remark that Stirling numbers are values of Bell exponential polynomials, $B_{n,k}(a_1, a_2, ...)$, $n, k \in \mathbb{N}$, on particular sequences. We give an extension of the Bell polynomials for $n, k \in \mathbb{Z}$. This allows us to recover Branson's result and much more. We thank the referee for his remarks.

## 2. Notations and definitions

$C$ is a commutative field, of characteristic zero. For a sequence $u : \mathbb{Z} \to C$, let us note:

$$\operatorname{supp} u = \{n, u(n) \neq 0\}\,,$$

$$\operatorname{ord} u = \inf \operatorname{supp} u\,,$$

$$s(C) = \{u, \operatorname{ord} u > -\infty\}\,,$$

$$s_0(C) = \{u, \operatorname{ord} u \geq 0\}\,,$$

$$e_k \text{ the sequence defined by } e_k(n) = \delta_{n,k}\,,\ k \in \mathbb{Z}\,.$$

For $u \in s_0(C)$, let us denote:

$$g_u(X) = \sum_{n=0}^{\infty} u(n) \frac{X^n}{n!} \tag{1}$$

the associated Hurwitz series (or exponential) to $u$.

For $u \in s_0(C)$ and $v \in s_0(C)$, the product $g_u(X) \cdot g_v(X) = g_\omega(X)$ defines the Hurwitz product $\omega = u \,ш\, v$ of sequences $u$ and $v$, and

$$(u \,ш\, v)(n) = \sum_{j=0}^{n} \binom{n}{j} u(j) v(n-j)\,. \tag{2}$$

Let us denote by $\mathcal{A} = \mathcal{A}(C)$ the Hurwitz algebra of sequences of $s_0(C)$ provided with the usual addition and Hurwitz product. The order, ord, is a valuation on $\mathcal{A}$.

Let us denote $T$ the shift operator on $\mathcal{A}$:

$$(Tu)(n) = u(n+1) \tag{3}$$

and $q$ the operator of multiplication by $n$:

$$(qu)(n) = nu(n)\,. \tag{4}$$

Then

$$g_{Tu}(X) = \frac{d}{dX} g_u(X)\,, \tag{5}$$

$$g_{qu}(X) = X \frac{d}{dX} g_u(X) \tag{6}$$

where $\dfrac{d}{dX}$ stands for the operator of formal differentiation.

Let us define for $k \in \mathbb{Z}$, $g_{e_k}(X) = \dfrac{X^k}{\gamma(k)}$. If we impose the validity of (5) and $\gamma(-1) = 1$, we obtain

$$\gamma(n) = \begin{cases} n! & \text{for} \quad n \geq 0 \\ \dfrac{(-1)^{-n-1}}{(-n-1)!} & \text{for} \quad n < 0 \end{cases} \tag{7}$$

what allows us to define the Hurwitz series $\displaystyle\sum_n u(n) \frac{X^n}{\gamma(n)}$ of a sequence $u$ of finite order (positive or negative), and to define the Hurwitz product of two sequences $u$ and $v$ of $s(C)$

$$(u \sqcup\!\sqcup\, v)(n) = \sum_{i+j=n} \frac{\gamma(n)}{\gamma(i)\gamma(j)} u(i)v(j)\,, \tag{8}$$

actually $\left\lceil \begin{matrix} n \\ k \end{matrix} \right\rfloor = \dfrac{\gamma(n)}{\gamma(k)\gamma(n-k)}$, $n \in \mathbb{Z}$, $k \in \mathbb{Z}$ is the Roman coefficient [3].

$s(C)$, provided with the generalized Hurwitz product (8) is the fraction fields of the ring $\mathcal{A}$.

Let $u$ be a sequence of strictly positive order; the composition of series $(g_v \circ g_u)(X) = g_{\overline{\omega}}(X)$ allows to define the composition of sequences, $\overline{\omega} = v \circ u$.

For $k \in \mathbb{N}$

$$(e_k \circ u)(n) = \mathcal{B}_{n,k}(u) \tag{9}$$

is Bell partial exponential polynomial [2]. It is a polynomial in $u(1)$, $u(2), \ldots, u(n), \ldots$ with coefficients in $\mathbb{Z}$.

For $v \in s_0(C)$ ,

$$(v \circ u)(n) = \sum_{k=1}^{n} \mathcal{B}_{n,k}(u) v(k)\,.$$

**Proposition 2.1.** *The set $\Omega$ of sequences of order one is a group for the composition. The inverse $\overline{u}$ of $u$ corresponds to the series $g_{\overline{u}}(X)$ reciprocal of the series $g_u(X)$.*

**Examples 2.2.** Let "$a$" be the sequence defined by $g_a(X) = e^X - 1$ ; then $g_{\overline{a}}(X) = \log(1+X)$ then

$$\begin{cases} \mathcal{B}_{n,k}(a) & = S(n,k) \\ \mathcal{B}_{n,k}(\overline{a}) & = s(n,k) \end{cases}$$

are the Stirling numbers.

Let $(t)_q$ be the sequence $(t)_q(n) = t(t-1)...(t-n+1)$ and $t^q$ be the sequence $t^q(n) = t^n$. Then

$$t^q = (t)_q \circ a\,,$$

$$(t)_q = t^q \circ \overline{a}$$

$Y_q(u,t) = t^q \circ u$ is the sequence of Bell exponential polynomials [2] and

$$Y_n(a,t) = \sum_{k=1}^{n} S(n,k) t^k = P_n(t)$$

is the *nth* Bell polynomial.

**Remark 2.3.** By application of the operators $T$ and $q$ (they are derivations in the Hurwitz algebra $\mathcal{A}(C)$), we can obtain various classical relations on the Bell exponential polynomials and the Stirling numbers.

## 3. Extension of Bell partial exponential polynomials

Let $u$ be a sequence of order one and $k$ a rational integer; let us define for $k \in \mathbb{N}$

$$g_{(e_k \circ u)}(X) = \begin{cases} \dfrac{g_u^k(X)}{\gamma(k)} \\ \dfrac{1}{\gamma(-k)X^k}\left[\dfrac{X}{g_u(X)}\right]^k \end{cases}$$

and so

$$g_{(e_{-k} \circ u)}(X) = \sum_n \mathcal{B}_{n,-k}(u)\frac{X^n}{\gamma(n)}\,.$$

Let us recall the definition of the generalized Bernoulli numbers:

$$\left[\frac{X}{g_u(X)}\right]^k = \sum_{n=0}^{\infty} b_n^{(k)}(u)\frac{X^n}{n!}$$

then:

- for $n < -k$, $\qquad \mathcal{B}_{n,-k}(u) = 0$
- for $n \geq 0$, $\qquad \mathcal{B}_{n,-k}(u) = \dfrac{(-1)^{k-1}(k-1)!n!}{(n+k)!}b_{n+k}^{(k)}(u)$
- for $0 < n \leq k$, $\qquad \mathcal{B}_{-n,-k}(u) = \dfrac{(-1)^{n+k}(n-1)!(k-1)!}{(k-n)!}b_{k-n}^{(k)}(u)$

**Theorem 3.1.** *Let $u$ be a sequence of order one and $0 < n \leq k$, then*

$$\mathcal{B}_{-n,-k}(u) = (-1)^{n+k}\mathcal{B}_{k,n}(\overline{u})$$

*where $\overline{u}$ is the inverse (for composition) of $u$.*

***Proof.*** The $\mathcal{B}_{n,k}(u)$ are rational functions in the variables $u_1, u_2, ...$ with coefficients in $\mathbb{Q}$. One can always suppose the $u_n$ algebraically free on $\mathbb{Q}$ and $\mathbb{Q}(u_1, u_2, ...)$ embedded in the field of complex numbers $\mathbb{C}$. One can also suppose that $\overline{\lim}\, |u_n|^{1/n} < +\infty$, so that one can represent the $b_n^{(k)}(u)$ by Cauchy's formula:

$$\frac{b_{k-n}^{(k)}}{(k-n)!} = \frac{1}{2i\pi} \int_{\mathcal{C}} \left( \frac{z}{g_u(z)} \right)^k \frac{dz}{z^{k-n+1}}, \qquad 0 < n \leq k$$

$\mathcal{C}$ circle with radius $\varepsilon > 0$ and center 0.

By the change of variable $z = g_{\overline{u}}(t)$, and after integration, we obtain

$$\begin{aligned} \frac{b_{k-n}^{(k)}}{(k-n)!} &= \frac{k}{2i\pi n} \int_{\mathcal{C}'} g_{\overline{u}}^n(t) \frac{dt}{t^{k+1}} \\ &= \frac{k}{n} n! \frac{\mathcal{B}_{k,n}(\overline{u})}{k!} \end{aligned}$$

from which we get the relation of the theorem. ■

**Corollary 3.2.** *For* $0 < n \leq k$ ,

$$S(-n, -k) = (-1)^{n+k} s(k, n)$$

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**Remark 3.3.** We check that:

$$\begin{aligned} (X)_{-n} &= \frac{1}{(X+1)...(X+n)} \\ &= \sum_{k \geq 0} s(-n, -k) X^{-k} = \sum_{k \geq 0} S(-n, -k)(X)_{-k}\,. \end{aligned}$$

More generally, if one considers the Bell exponential polynomials associated with a sequence $u$ of order one:

$$Y_n(u,t) = \sum_{k\geq 0} \mathcal{B}_{n,k}(u)t^k$$

one can extend them to negative integers by the series:

$$Y_{-n}(u,t) = \sum_{k\geq 0} \mathcal{B}_{-n,-k}(u)t^{-k} = \sum_{k\geq n}(-1)^{n+k}\mathcal{B}_{k,n}(\overline{u})t^{-k}\,.$$

For Bernoulli polynomials, with $D = \dfrac{d}{dX}$ and $B_q$ the sequence of the classical Bernoulli numbers:

$$B_n(X) = g_{B_q}(D)\cdot X^n = (B_q \,\text{Ш}\, X^q)(n)\,,$$

$$B_{-n}(X) = g_{B_q}(D)\cdot X^{-n} = \sum_{k\geq 0}\binom{-n}{k} B_k X^{-n-k}$$

since $g_{B_q}(D) = \dfrac{D}{e^D-1}$, $B_{-n}(X)$ satisfies:

$$B_{-n}(X+1) - B_{-n}(X) = -nX^{-n-1}\,.$$

**Remark 3.4.** For $k,n$ strictly positive integers

$$S(q,k) = \frac{1}{k!}a^{\text{Ш}k} = \frac{1}{k!}(1^q - e_0)^{\text{Ш}k}$$

where $a^{\text{Ш}k}$ denote powers calculated in the Hurwitz algebra. From this we get:

$$S(n,k) = \frac{1}{k!}\sum_{j=1}^{k}\binom{k}{j}(-1)^{k-j}j^n$$

which allows to define an extension to the negative integers $n$ and positive integers $k$.

To obtain an analogue for the Stirling numbers of second kind $s(n,k)$, we can consider

$$(X)_{-n} = \frac{1}{(X+1)...(X+n)} = \sum_{k\geq 0} s(-n,k)X^k$$

and hence

$$s(-n,k) = (-1)^{n+k-1} S(-k,n)\,.$$

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