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HYPER BCI-ALGEBRAS

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Abstract

We introduce the concept of a hyper BCI-algebra which is a generalization of a BCI-algebra, and investigate some related properties. Moreover we introduce a hyper BCI-ideal, weak hyper BCI-ideal, strong hyper BCI-ideal and reflexive hyper BCI-ideal in hyper BCI-algebras, and give some relations among these hyper BCI-ideals. Finally we discuss the relations between hyper BCI-algebras and hyper groups, and between hyper BCI-algebras and hyper H_v-groups.

Keywords: hyper BCI-algebra, hyper group, hyper H_v-group.

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1. Introduction

The study of BCK/BCI-algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK/BCI-algebras. The hyperstructure theory (called also multialgebras) was introduced in 1934 by F. Marty [8] at the 8th congress of Scandinavian Mathematicians. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy,

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Russia and Japan. Over the following decades, many important results appeared, but above all since the 70's onwards the most luxuriant flourishing of hyperstructures has been seen. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [7], Y.B. Jun *et al.* applied the hyperstructures to BCK-algebras, and introduced the concept of a hyper BCK-algebra, and investigated some related properties. In this note, we introduce the concept of a hyper BCI-algebra which is a generalization of a BCI-algebra, and investigate some related properties. Moreover we introduce a hyper BCI-ideal, weak hyper BCI-ideal, strong hyper BCI-ideal and reflexive hyper BCI-ideal in hyper BCI-algebras, and give some relations among these hyper BCI-ideals. Finally we discuss the relations between hyper BCI-algebras and hyper groups, and between hyper BCI-algebras and hyper H_v-groups.

2. Preliminaries

An algebra $(X;*,0)$ of type $(2,0)$ is said to be a BCI-algebra if it satisfies: for all $x,y,z \in X$,

(I) $((x*y)*(x*z))*(z*y)=0$,

(II) $(x*(x*y))*y=0$,

(III) $x*x=0$,

(IV) $x*y=0$ and $y*x=0$ imply $x=y$.

If a BCI-algebra $(X;*,0)$ satisfies the following

(V) $0*x=0$,

we call it a BCK-algebra. In any BCI/BCK-algebra X one can define a partial order $\leq$ by putting $x \leq y$ if and only if $x*y=0$.

Note that an algebra $(X,*,0)$ of type (2,0) is a BCI-algebra if and only if

(i) $((x*z)*(y*z))*(x*y)=0$,

(ii) $(z*x)*y=(z*y)*x$,

(iii) $x*x=0$,

(iv) $x * y = 0$ and $y * x = 0$ imply that $x = y$,

(vi) $(0 * (0 * x)) * x = 0$,

for all $x, y \in X$.

A non-empty subset I of a BCI-algebra X is called an ideal of X if $0 \in I$, and $x * y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

Let H be a non-empty set and "$\circ$" a function from $H \times H$ to $\wp(H) \setminus \{\emptyset\}$, where $\wp(H)$ denotes the power set of H. For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup\limits_{a \in A, b \in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

Definition 2.1 (Jun *et al.* [7]). By a *hyper BCK-algebra* we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HK3) $x \circ H \ll \{x\}$,

(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$.

Proposition 2.2 (Jun *et al.* [7]). *In a hyper BCK-algebra H, the condition* (HK3) *is equivalent to the condition:*

(i) $x \circ y \ll \{x\}$ *for all* $x, y \in H$.

3. Hyper BCI-algebras

Let H be a nonempty set and $\circ$ a function from $H \times H$ to $\wp^*(H)$, where $\wp^*(H)$ denotes the power set of $H \setminus \{0\}$. For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup\limits_{a \in A, b \in B} a \circ b$. Then we call $(H, \circ)$ a hyper groupoid and $\circ$ a hyperoperration. Also we define $x \ll y$ by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ means that for all $a \in A$ there is $b \in B$ such that $a \ll b$.

Definition 3.1. By a *hyper BCI-algebra* we mean a hyper groupoid $(H, \circ)$ that contains a constant 0 and satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HI3) $x \ll x$,

(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

(HI5) $0 \circ (0 \circ x) \ll x$,

for all $x, y, z \in H$.

Example 3.2. (1) Let $(H, *, 0)$ be a BCI-algebra and define a hyper operation "$\circ$" on H by $x \circ y = \{x * y\}$ for all $x, y \in H$. Then $(H, \circ)$ is a hyper BCI-algebra.

(2) Define a hyper operation "$\circ$" on $H := [0, \infty)$ by

$$x \circ y := \begin{cases} [0, x] & \text{if} x \leq y \\ (0, y] & \text{if} x > y \neq 0 \\ \{x\} & \text{if} y = 0 \end{cases}$$

for all $x, y \in H$. Then $(H, \circ)$ is a hyper BCI-algebra.

(3) Let $H = \{0, 1, 2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Then $(H, \circ)$ is a hyper BCI-algebra but it is not a hyper BCK-algebra since $0 \circ 1 = \{0, 1\} \neq \{0\}$.

Proposition 3.3. *Let $(H, \circ)$ be a hyper BCK-algebra, then $(H, \circ)$ is also a hyper BCI-algebra. The converse is not true.*

Proof. It follows from Definition 2.1, Definition 3.1 and Example 3.2(3). ■

Proposition 3.4. *Let* $(H, \circ)$ *be a hyper* BCI*-algebra. Then*

(ii) $(A \circ B) \circ C = (A \circ C) \circ B$, *for every non-empty subsets* A, B *and* C *of* H.

Proof. Straightforward. ■

Proposition 3.5. *In any hyper* BCI*-algebra, the following hold:*

(i) $x \ll 0$ *implies* $x = 0$,

(ii) $0 \in x \circ (x \circ 0)$,

(iii) $x \ll x \circ 0$,

(iv) $0 \circ (x \circ y) \ll y \circ x$,

(v) $A \ll A$,

(vi) $A \subseteq B$ *implies* $A \ll B$,

(vii) $A \ll \{0\}$ *implies* $A = \{0\}$,

(viii) $x \circ 0 \ll \{y\}$ *implies* $x \ll y$,

(ix) $y \ll z$ *implies* $x \circ z \ll x \circ y$,

(x) $x \circ y = \{0\}$ *implies* $(x \circ z) \circ (y \circ z) = \{0\}$ *and* $x \circ z \ll y \circ z$,

(xi) $A \circ \{0\} = \{0\}$ *implies* $A = \{0\}$,

for all $x, y, z \in H$ *and for all non-empty subsetes* A *and* B *of* H.

Proof.

(i) Let $x \ll 0$. Then $0 \in x \circ 0$ and so $0 \in 0 \circ (x \circ 0) \subseteq (0 \circ 0) \circ (x \circ 0) \ll 0 \circ x$. This means that $0 \ll 0 \circ x$. By (HI5),$0 \in 0 \circ (0 \circ x) \ll x$. Then $0 \ll x$. Combining $x \ll 0$, we get $x = 0$.

(ii) Note that $0 \in (x \circ 0) \circ (x \circ 0) = (x \circ (x \circ 0)) \circ 0$, we have that there exists $c \in x \circ (x \circ 0)$ such that $c \ll 0$. By (i),$c = 0$ and so $0 \in x \circ (x \circ 0)$.

(iii) It follows from (ii).

(iv) By (HI3) and (HK1), $0 \circ (x \circ y) \subseteq (y \circ y) \circ (x \circ y) \ll y \circ x$. This shows that $0 \circ (x \circ y) \ll y \circ x$.

(v) It is by (HI3).

(vi) Assume that $A \subset B$ and let $a \in A$. Taking $b = a$, then $b \in B$ and $a \ll b$ by (HI3). Therefore $A \ll B$.

(vii) Assume that $A \ll \{0\}$ and let $a \in A$. Then $a \ll 0$ and so $a = 0$. Therefore $A = \{0\}$.

(viii) Note that $0 \in (x \circ 0) \circ y = (x \circ y) \circ 0$, so that there exists $c \in x \circ y$ such that $0 \in c \circ 0$, i.e., $c \ll 0$. It follows that $c = 0 \in x \circ y$ by (i). That is $x \ll y$.

(ix) Assume that $y \ll z$. Then $(x \circ z) \circ 0 \subseteq (x \circ z) \circ (y \circ z) \ll x \circ y$ and hence $(x \circ z) \circ 0 \ll x \circ y$. This means that for each $a \in x \circ z$ there exists $b \in x \circ y$ such that $a \circ 0 \ll \{b\}$. Hence, by (viii), we have $a \ll b$ and so $x \circ z \ll x \circ y$.

(x) Assume that $x \circ y = \{0\}$. Then $(x \circ z) \circ (y \circ z) \ll x \circ y = \{0\}$ and so $(x \circ z) \circ (y \circ z) = \{0\}$ by (vii), which implies that $x \circ z \ll y \circ z$.

(xi) Straightforward. This completes the proof.

■

Proposition 3.6. *Let A be a subset of a hyper BCK-algebra $(H, \circ)$ and let $x, y, z \in H$. If $(x \circ y) \circ z \ll A$, then $a \circ z \ll A$ for all $a \in x \circ y$.*

Proof. Straightforward.

■

Definition 3.7. Let $(H, \circ)$ be a hyper BCI-algebra and let S be a subset of H containing 0. If S is a hyper BCI-algebra with respect to the hyper operation "$\circ$" on H, we say that S is a *hyper subalgebra* of H.

Proposition 3.8. *Let S be a non-empty subset of a hyper BCI-algebra $(H, \circ)$. If $x \circ y \subseteq S$ for all $x, y \in S$, then $0 \in S$.*

Proof. Assume that $x \circ y \subseteq S$ for all $x, y \in S$ and let $a \in S$. Since $a \ll a$, we have $0 \in a \circ a \subseteq S$ and we are done.

■

Theorem 3.9. *Let S be a non-empty subset of a hyper BCI-algebra $(H, \circ)$. Then S is a hyper subalgebra of H if and only if $x \circ y \subseteq S$ for all $x, y \in S$.*

Proof. $(\Rightarrow)$ Clear.

$(\Leftarrow)$ Assume that $x \circ y \subseteq S$ for all $x, y \in S$. Then $0 \in S$ by Proposition 3.8. For any $x, y, z \in S$, we have $x \circ z \subseteq S$, $y \circ z \subseteq S$ and $x \circ y \subseteq S$. Hence

$$(x \circ z) \circ (y \circ z) = \bigcup_{\substack{a \in x \circ z \\ b \in y \circ z}} a \circ b \subseteq S$$

and so (HK1) holds in S. Similarly we can prove that the axioms (HK2), (HI3), (HK4) and (HI5) are true in S. Therefore S is a hyper subalgebra of H. ■

Example 3.10. (1) Let $(H, \circ)$ be the hyper BCI-algebra in Example 3.2(1) and let S be a subalgebra of a BCI-algebra $(H, *, 0)$. Then S is a hyper subalgebra of $(H, \circ)$.

(2) Let $(H, \circ)$ be the hyper BCI-algebra in Example 3.2(2) and let $S = [0, a]$ for every $a \in [0, \infty)$. Then S is a hyper subalgebra of $(H, \circ)$.

(3) Let $(H, \circ)$ be the hyper BCI-algebra in Example 3.2(3) and let $S_1 = \{0, 1\}$ and $S_2 = \{0, 2\}$. Then S_1 is a hyper subalgebra of H, but S_2 is not a hyper subalgebra of H since $2 \circ 2 = \{0, 1, 2\} \not\subseteq S_2$.

Theorem 3.11. *Let $(H, \circ)$ be a hyper BCI-algebra. Then the set $S(H) := \{x \in H | 0 \circ x = \{0\}\}$ is a hyper subalgebra of H whenever $S(H)$ is non-empty.*

Proof. Let $x, y \in S(H)$ and $a \in x \circ y$. Then $0 \circ (x \circ y) = (0 \circ y) \circ (x \circ y) \ll 0 \circ x = 0$ and hence by proposition 3.5(vii) $0 \circ (x \circ y) = \{0\}$. Therefore $x \circ y \subseteq S(H)$. By Theorem 3.9, we end the proof. ■

Theorem 3.12. *Let $(H, \circ)$ be a hyper BCI-algebra and $S_K := \{x \in H | x \circ (x \circ 0) = 0\}$. If S_K is non-empty, then we have*

(i) *S_K is a hyper subalgebra of H,*

(ii) *$(S_K, \circ)$ forms a hyper BCK-algebra.*

Proof. (i) Let $x, y \in S_K$. Using Proposition 3.5(x) and $x \circ (x \circ 0) = 0$, we have $(x \circ y) \circ ((x \circ y) \circ 0) = (x \circ y) \circ ((x \circ 0) \circ y) \ll x \circ (x \circ 0) = 0$ and hence $(x \circ y) \circ ((x \circ y) \circ 0) = 0$. This shows that $x \circ y \in S_K$. Combining Theorem 3.9, we have that S_K is a hyper subalgebra of H.

(ii) It is sufficient to show that $x \circ y \ll x$ for all $x, y \in S_K$. Let $x, y \in S_K$. Then $x \circ (x \circ 0) = 0$. It follows that $x \ll c$ for any $c \in x \circ 0$. On the other hand, $0 \in (x \circ x) \circ 0 = (x \circ 0) \circ x$ and hence there exists $c \in x \circ 0$ such that $c \ll x$. Therefore $x = c \in x \circ 0$. Moreover $(x \circ y) \circ x \subseteq (x \circ y) \circ (x \circ 0) = (x \circ (x \circ 0)) \circ y = 0 \circ y = 0$ and so $(x \circ y) \circ x = 0$ or $x \circ y \ll x$. Now we get that S_K is a hyper BCK-algebra. ■

Theorem 3.13. *Let $(H, \circ)$ be a hyper BCI-algebra. Then the set*

$$S_I := \{x \in H \mid x \circ x = \{0\}\}$$

is a hyper subalgebra of H whenever $S_I \neq \emptyset$.

Proof. Let $x, y \in S_I$ and $a \in x \circ y$. Then $(x \circ y) \circ (x \circ y) \ll x \circ x = \{0\}$ and hence by Proposition 3.5(vii) $(x \circ y) \circ (x \circ y) = \{0\}$, and $a \circ a \subseteq (x \circ y) \circ (x \circ y)$ $= \{0\}$. Thus $a \circ a = \{0\}$ or equivalently $a \in S_I$. Therefore $x \circ y \subseteq S_I$. By Theorem 3.9, S_I is a hyper subalgebra of H, ending the proof. ■

Theorem 3.14. *Let $(H, \circ)$ be a hyper BCI-algebra. Then $(S_I, \circ, 0)$ is a BCI-algebra whenever S_I is not empty set.* We then call S_I the *BCI-part* of a hyper BCI-algebra H.

Proof. It is sufficient to show that $x \circ y$ is a singleton subset of S_I for all $x, y \in S_I$. Let $x, y \in S_I$ and let $a, b \in x \circ y$. Note that $a \circ b \subseteq (x \circ y) \circ (x \circ y) \ll x \circ x = 0$, we have $a \circ b = \{0\}$, i.e., $a \ll b$. Similarly we have $b \ll a$ and thus $a = b$ which means that $x \circ y$ is singleton. Now by some calculations we get that S_I is a BCI-algebra. ■

Corollary 3.15. *Let $(H, \circ)$ be a hyper BCI-algebra. Then $(H, \circ, 0)$ is a BCI-algebra if and only if $H = S_I$.*

Proof. Straightforward. ■

4. Hyper BCI-ideals of hyper BCI-algebras

Definition 4.1. Let I be a non-empty subset of a hyper BCI-algebra H. Then I is said to be a *hyper BCI-ideal* of H if

(HI1) $0 \in I$,

(HI2) $x \circ y \ll I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Example 4.2. (1) Let $(H, \circ)$ be the hyper BCI-algebra in Example 3.2(1). Then every ideal I of a BCI-algebra $(H, *, 0)$ is a hyper BCI-ideal of H.

(2) Let $(H, \circ)$ be the hyper BCK-algebra in Example 3.2(2). Then $(H, \circ)$ have no proper hyper BCK-ideals, i.e., there are only two hyper BCK-ideals $\{0\}$ and H itself. In fact, if I is a hyper BCK-ideal of H and $I \neq \{0\}$, then there is $a \in I$ such that $a \neq 0$. For any $b \in [0, a]$, we have $b \circ a = [0, b] \ll \{a\}$ and so $b \circ a \ll I$. Since $a \in I$, it follows from (HI2) that $b \in I$ so that $[0, a] \subseteq I$. Moreover for every $a < c$, we get $c \circ a = (0, a] \ll I$ and so $c \in I$. Therefore $(a, \infty) \subseteq I$, i.e., $I = H$.

(3) Let $(H, \circ)$ be the hyper BCI-algebra in Example 8.2(3). Then $I_1 = \{0, 1\}$ is a hyper BCI-ideal of H, but $I_2 = \{0, 2\}$ is not a hyper BCI-ideal of H because $1 \circ 2 = \{0, 1\} \ll I_2$ and $2 \in I_2$, but $1 \notin I_2$.

Definition 4.3. Let I be a nonempty subset of a hyper BCI-algebra H. Then I is called a *weak hyper BCI-ideal* of H if

(HI1) $0 \in I$,

(WHI) $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Example 4.4. (1) Let $(H, \circ)$ be the hyper BCI-algebra in Example 3.2(1). Then every ideal I of a BCI-algebra $(H, *, 0)$ is a weak hyper BCI-ideal of H.

(2) Let $(H, \circ)$ be the hyper BCI-algebra in Example 3.2(2) and let $I = \{0\} \cup [a, \infty)$ for any $a \in H$. Then I is a weak hyper BCI-ideal of H. Indeed, let $x \circ y \subseteq I$ and $y \in I$. If $y = 0$, then $\{x\} = x \circ y = x \circ 0 \subseteq I$ and so $x \in I$. Assume that $y \neq 0$. Then $x \circ y \subseteq I$ implies $x = 0 \in I$ because if $x \neq 0$ then

$$x \circ y = \begin{cases} [0, x] & \text{if } x \leq y \\ (0, y] & \text{if } x > y \end{cases}$$

which is not contained in I. Hence I is a weak hyper BCI-ideal of H.

(3) Let $(H, \circ)$ be the hyper BCI-algebra in Example 3.2(3). Then $I_1 = \{0, 1\}$ and $I_2 = \{0, 2\}$ are weak hyper BCI-ideals of H.

Combining Proposition 3.5(vi) and Definition 4.3, we have the following theorem.

Theorem 4.5. *Let $(H, \circ)$ be a hyper BCI-algebra. Then every hyper BCI-ideal of H is a weak hyper BCI-ideal of H.*

The converse of Theorem 4.5 may not be true. In fact, the weak hyper BCI-ideal $I_2 = \{0, 2\}$ in Example 3.2(3) is not a hyper BCI-ideal (see Example 4.2(3)). In addition, this result shows that a weak hyper BCK-ideal may not be a hyper subalgebra.

Definition 4.6. Let I be a nonempty subset of a hyper BCI-algebra H. Then I is called a *strong hyper BCI-ideal* of H if it satisfies (HI1) and

(SHI) $(x \circ y) \cap I \neq \emptyset$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Example 4.7. (1) Let $(H, \circ)$ be the hyper BCI-algebra in Example 3.2(1). Then every ideal I of a BCI-algebra $(H, *, 0)$ is a strong hyper BCI-ideal of H.

(2) Let $(H, \circ)$ be the hyper BCI-algebra in Example 3.2(2). Then $(H, \circ)$ have no proper strong hyper BCI-ideals, i.e., there are only two strong hyper BCI-ideals $\{0\}$ and H itself.

(3) Let $(H, \circ)$ be the hyper BCI-algebra in Example 3.2(3). Then $\{0\}$ and H are only strong hyper BCI-ideals of H.

(4) Let $H = \{0, 1, 2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,1,2\}$

Then $(H, \circ)$ is a hyper BCI-algebra, and $I_1 := \{0, 1\}$ and $I_2 := \{0, 2\}$ are strong hyper BCI-ideals of H.

Theorem 4.8. *Let I be a strong hyper BCI-ideal of a hyper BCI-algebra H. Then*

(i) *I is a weak hyper BCI-ideal of H,*

(ii) *I is a hyper BCI-ideal of H.*

Proof. We only need to prove (ii). Let $x, y \in H$ be such that $x \circ y \ll I$ and $y \in I$. Then for each $a \in x \circ y$ there exists $b \in I$ such that $a \ll b$, i.e., $0 \in a \circ b$. It follows that $(a \circ b) \cap I \neq \emptyset$ so from (SH1) that $a \in I$. Thus $x \circ y \subseteq I$ and so $(x \circ y) \cap I \neq \emptyset$, and using (SHI) we get $x \in I$. Hence I is a hyper BCI-ideal of H, ending the proof. ■

Note that $I = \{0, 1\}$ in Example 3.2(3) is a (weak) hyper BCI-ideal of H. But it is not a strong hyper BCI-ideal of H since $(2 \circ 1) \cap I = \{1\} \neq \emptyset$ and $1 \in I$, but $2 \notin I$. This shows that the converse of Theorem 4.8 may not be true.

Definition 4.9. A hyper BCI-ideal I of H is said to be *reflexive* if $x \circ x \subseteq I$ for all $x \in H$.

Example 4.10. (1) Let H be a hyper BCI-algebra in Example 3.2(1). Then every ideal I of a BCI-algebra $(H, *, 0)$ is a reflexive hyper BCI-ideal of H.

(2) Let $H = \{0, 1, 2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$

Then $(H, \circ)$ is a hyper BCI-algebra, and $I_2 := \{0, 2\}$ is a strong hyper BCI-ideal, and so a hyper BCI-ideal of H. Moreover, noticing that $x \circ x \subseteq I_2$ for all $x \in H$, we know that I_2 is reflexive. But $I_1 := \{0, 1\}$ is not reflexive.

Lemma 4.11. *Let A, B C and I be subsets of H.*

(i) *If $A \subseteq B \ll C$, then $A \ll C$.*

(ii) *If $A \circ x \ll I$ for $x \in H$, then $a \circ x \ll I$ for all $a \in A$.*

(iii) *If I is a hyper BCK-ideal of H and if $A \circ x \ll I$ for $x \in I$, then $A \ll I$.*

Proof. Straightforward. ■

Theorem 4.12. *Let I be a reflexive hyper BCI-ideal of a hyper BCI-algebra H. Then*

$$(x \circ y) \cap I \neq \emptyset \quad \textit{implies} \quad x \circ y \ll I \textit{ for all } x, y \in H.$$

Proof. Let $x, y \in H$ be such that $(x \circ y) \cap I \neq \emptyset$. Then there exists $a \in (x \circ y) \cap I$, and so

$$(x \circ y) \circ a \subseteq (x \circ y) \circ (x \circ y) \ll x \circ x \subseteq I,$$

whence $(x \circ y) \circ a \ll I$ by Lemma 4.11(i). It follows from Lemma 4.11(iii) that $x \circ y \ll I$, ending the proof. ■

Theorem 4.13. *Let I be a reflexive hyper BCI-ideal of hyper BCI-algebra H and let A be a subset of H. If $A \ll I$, then $A \subseteq I$.*

Proof. Assume that $A \ll I$ and let $a \in A$. Then there exists $x \in I$ such that $a \ll x$, i.e., $0 \in a \circ x$. Hence $0 \in (a \circ x) \cap I$, i.e., $(a \circ x) \cap I \neq \emptyset$, which implies $a \circ x \ll I$ by Theorem 4.12. It follows from (HI2) that $a \in I$ so that $A \subseteq I$. This completes the proof. ■

Corollary 4.14. *Let I be a reflexive hyper BCI-ideal of hyper BCI-algebra H. Then*

$$(x \circ y) \cap I \neq \emptyset \ \textit{ implies } \ x \circ y \subseteq I \textit{ for all } x, y \in H.$$

Proof. Straightforward. ■

Theorem 4.15. *Every reflexive hyper BCI-ideal of hyper BCI-algebra H is a strong hyper BCI-ideal of H.*

Proof. Let I be a reflexive hyper BCI-ideal of H and let $x, y \in H$ be such that $(x \circ y) \cap I \neq \emptyset$ and $y \in I$. Then $x \circ y \ll I$ by Theorem 4.12. It follows from (HI2) that $x \in I$. Hence I is a strong hyper BCK-ideal of H. ■

The converse of Theorem 4.15 may not be true. For example, in Example 4.10(2), $I_1 := \{0, 1\}$ is a strong hyper BCI-ideal of H, but it is not reflexive.

5. HYPER *BCI*-ALGEBRAS AND HYPERGROUPS

Let H be a non-empty set and "$\cdot$" a function from $H \times H$ to $\wp(H) \setminus \{\emptyset\}$, where $\wp(H)$ denotes the power set of H. F. Marty [8] defined a hypregroup as a hyperstructure $(H, \cdot)$ such that the following axioms hold: (i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all x, y, z in H, (ii) $x \cdot H = H \cdot x = H$ for all x in H. A subset K of H is called a subhypergroup if $(K, \cdot)$ is a hypergroup. T. Vougiouklis [11] introduced an H_ν-group which is a hyperstructure $(H, \cdot)$ such that (i) $(x \cdot y) \cdot z \cap x \cdot (y \cdot z) \neq \emptyset$ for all $x, y, z \in H$, (ii) $x \cdot H = H \cdot x = H$ for all $x \in H$. If $(H, \cdot)$ satifies only the first axiom, it is called an H_ν-semigroup.

Now we study the relations between hyper BCI-algebras and hypergroups, and hyper BCI-algebras and H_ν-groups.

At first we define $x \cdot y$ by $x \cdot y = y \circ (0 \circ x)$ for all $x, y \in H$ in a hyper BCI-algebras.

Theorem 5.1. *Let $(H, \circ)$ be a hyper BCI-algebra and satisfy the following conditions:*

(i) $x \in a \circ (a \circ x)$ *for all* $x, a \in H$,

(ii) $x \cdot y \cap y \cdot x \neq \emptyset$,

(iii) $(x \cdot y) \cdot z = (x \cdot z) \cdot y$ *for all* $x, y, z \in H$.

Then $(H, \cdot)$ is a H_ν-group.

Proof. By (i), $x \in a \circ (a \circ x) \subseteq a \circ (0 \circ (0 \circ (a \circ x))) = (0 \circ (a \circ x)) \cdot a \subseteq H \cdot a$. Hence $H \subseteq H \cdot a$ or $H = H \cdot a$. By the other hand, $x \in a \circ (a \circ x) \subseteq (0 \circ (0 \circ a)) \circ (a \circ x) = (0 \circ (a \circ x)) \circ (0 \circ a) = a \cdot (0 \circ (a \circ x)) \subseteq a \cdot H$. Hence $x \cdot H = H \cdot x = H$.

Nextly note that

$$y \cdot (x \cdot z) = (z \circ (0 \circ x)) \circ (0 \circ y) = (z \circ (0 \circ y)) \circ (0 \circ x) = x \cdot (y \cdot z).$$

By (ii), we have $x \cdot z \cap z \cdot x \neq \emptyset$, and hence there exists $c_1 \in x \cdot z \cap z \cdot x$. Using (ii) again we obtain that there is $c_2 \in c_1 \cdot y \cap y \cdot c_1$ and thus $c_2 \in c_1 \cdot y \subseteq (x \cdot z) \cdot y = (x \cdot y) \cdot z$ by (iii). On the other hand, $c_2 \in y \cdot c_1 \subseteq y \cdot (x \cdot z) = x \cdot (y \cdot z)$. That is $c_2 \in (x \cdot y) \cdot z \cap x \cdot (y \cdot z)$.

Combining the above arguments we have that $(H, \cdot)$ is a H_ν-group. ■

Theorem 5.2. *Let $(H, \circ)$ be a hyper BCI-algebra and satisfy the following conditions:*

(i) $x \in a \circ (a \circ x)$,

(ii) $x \circ (0 \circ y) = y \circ (0 \circ x)$.

Then $(H, \cdot)$ is a hypergroup.

Proof. Similar to the proof of Theorem 5.1, we can get that $x \cdot H = H \cdot x = H$. Moreover we have

$$\begin{aligned}(x \cdot y) \cdot z &= z \circ (0 \circ (y \circ (0 \circ x)))\\ &= (y \circ (0 \circ x)) \circ (0 \circ z)\\ &= (y \circ (0 \circ z)) \circ (0 \circ x)\\ &= x \cdot (z \circ (0 \circ y))\\ &= x \cdot (y \cdot z).\end{aligned}$$

We complete the proof. ■

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