# SUBDIRECTLY IRREDUCIBLE NON-IDEMPOTENT LEFT SYMMETRIC LEFT DISTRIBUTIVE GROUPOIDS* 

Emil Jeřábek ${ }^{1}$, Tomáš Kepka ${ }^{2}$ and David Stanovský ${ }^{2}$<br>${ }^{1}$ Mathematical Institute, Academy of Sciences<br>Prague, Czech Republic<br>${ }^{2}$ Charles University in Prague, Czech Republic

e-mail: jerabek@math.cas.cz
e-mail: kepka@karlin.mff.cuni.cz
e-mail: stanovsk@karlin.mff.cuni.cz


#### Abstract

We study groupoids satisfying the identities $x \cdot x y=y$ and $x \cdot y z=x y \cdot x z$. Particularly, we focus our attention at subdirectly irreducible ones, find a description and charecterize small ones.


Keywords: groupoid, left distributive, left symmetric, subdirectly irreducible.

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## 1. Introduction

A left symmetric left distributive groupoid (shortly an $L S L D$ groupoid) is a non-empty set equipped with a binary operation (usually denoted multiplicatively) satisfying the equations:

[^0](left symmetry)
\[

$$
\begin{aligned}
x \cdot x y & =y \\
x \cdot y z & =x y \cdot x z
\end{aligned}
$$
\]

(left distributivity)
An LSLDI groupoid is an idempotent LSLD groupoid, i.e., an LSLD groupoid satisfying the equation $x x=x$. For example, given a group $G$, the derived operation $x * y=x y^{-1} x$, usually called the core of $G$, is left symmetric, left distributive and idempotent. LSLDI groupoids were introduced in [10] and they (and their applications) were studied by several authors mainly in 1970's and 1980's. A reader is referred to the survey [8] for details. For a long time, it seemed that the non-idempotent case did not play any significant role in self-distributive structures (whether symmetric or not). This was certainly true for the two-sided case, but recently, due to the book [2] of P. Dehornoy, one-sided non-idempotent selfdistributive groupoids enjoyed certain attention. The purpose of the present note is to continue the investigations of non-idempotent LSLD groupoids started in [4] and, in particular, to get a better insight into the structure of subdirectly irreducible ones. Our main results are Theorems 4.2, 4.3 and 5.9.

As far as we know, the only papers concerning non-idempotent LSLD groupoids are [4] and [9]. Subdirectly irreducible idempotent left symmetric medial groupoids were characterized by B. Roszkowska [7] and simple idempotent LSLD groupoids by D. Joyce [3].

Our notation is rather standard and usually follows the book [1]. A reader can look at [5] for various notions concerning groupoids (i.e., sets with a single binary operation).

Let $G$ be a groupoid. For every $a \in G$, we denote $L_{a}$ the selfmapping of $G$ defined by $L_{a}(x)=a x$ for all $x \in G$ and call it the left translation by $a$ in $G$. By an involution we mean a permutation of order two.

Lemma 1.1. Let $G$ be a groupoid. Then

1. $G$ is $L S L D$, iff every left translation in $G$ is either the identity, or an involutive automorphism of $G$;
2. if $G$ is $L S L D$, then $L_{\varphi(a)}=\varphi L_{a} \varphi^{-1}$ for every $a \in G$ and every automorphism $\varphi$ of $G$;
3. if $G$ is LSLD, then the mapping $\lambda: a \mapsto L_{a}$ is a homomorphism of $G$ into the core of the symmetric group over $G$.

Proof. (1) Left symmetry says that every left translation $L_{a}$ satisfies $L_{a}^{2}=i d_{G}$. Left distributivity says that every $L_{a}$ is an endomorphism.
(2) Since $\varphi L_{a}(b)=\varphi(a b)=\varphi(a) \varphi(b)=L_{\varphi(a)} \varphi(b)$ for every $a, b \in G$, we have $\varphi L_{a}=L_{\varphi(a)} \varphi$ and thus $L_{\varphi(a)}=\varphi L_{a} \varphi^{-1}$.
(3) It follows from (2) for $\varphi=L_{a}$ that $L_{a b}=L_{a} L_{b} L_{a}^{-1}=L_{a} L_{b} L_{a}$.

Example. The following are all (up to an isomorphism) two-element LSLD groupoids (one idempotent, the other not).

| $\mathbf{S}$ | 0 | 1 |
| :---: | :--- | :--- |
| 0 | 0 | 1 |
| 1 | 0 | 1 |


| $\mathbf{T}$ | 0 | 1 |
| :---: | :---: | :---: |
| 0 | $\widetilde{0}$ | 0 |
| $\widetilde{0}$ | $\widetilde{0}$ | 0 |

Example. The following are all (up to an isomorphism) three-element idempotent LSLD groupoids. $\mathbf{S}_{1}$ is a right zero groupoid, $\mathbf{S}_{2}$ is a dual differential groupoid and $\mathbf{S}_{3}$ is a commutative distributive quasigroup and it forms the smallest Steiner triple system. $\mathbf{S}_{3}$ is simple and $\mathbf{S}_{2}$ is subdirectly irreducible.

| $\mathbf{S}_{1}$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | 0 | 1 | 2 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 1 | 2 |


| $\mathbf{S}_{2}$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | 0 | 2 | 1 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 1 | 2 |


| $\mathbf{S}_{3}$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | 0 | 2 | 1 |
| 1 | 2 | 1 | 0 |
| 2 | 1 | 0 | 2 |

Example. The following are all (up to an isomorphism) three-element non-idempotent LSLD groupoids. Both are subdirectly irreducible.

| $\mathbf{T}_{1}$ | $e$ | 0 | $\widetilde{0}$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $e$ | $e$ | 0 | $\widetilde{0}$ | $\mathbf{T}_{2}$ | $e$ | 0 | $\widetilde{0}$ |
| $0, \widetilde{0}$ | $e$ | $\widetilde{0}$ | 0 | $e$ | $\widetilde{0}$ | 0 |  |
| $0, \widetilde{0}$ | $e$ | $\widetilde{0}$ | 0 |  |  |  |  |

Example. We define an operation $\circ$ on the Prüfer 2-group $\mathbb{Z}_{2 \infty}(+)$ by $x \circ y=$ $2 x-y+a$, where $a \in \mathbb{Z}_{2 \infty}$ is an element satisfying $a \neq 0=2 a$. The groupoid $\mathbb{Z}_{2^{\infty}}(\circ)$ is an infinite subdirectly irreducible idempotent-free LSLD groupoid.

A non-empty subset $J$ of a groupoid $G$ is called a left ideal of $G$, if $a b \in J$ for every $a \in G$ and $b \in J$. Note that the set consisting of all left ideals in a left symmetric groupoid and the empty set is closed under intersection, union and complements. If $\{a\}$ is a left ideal of $G$, we call the element $a$ right zero.

Let $G$ be an LSLD groupoid. We put

$$
I d_{G}=\{x \in G: x x=x\} \quad \text { and } \quad K_{G}=\{x \in G: x x \neq x\}
$$

Each of $I d_{G}$ and $K_{G}$ is either empty or a left ideal of $G$. Further, we define relations

$$
\begin{aligned}
p_{G} & =\left\{(x, y) \in G \times G: L_{x}=L_{y}\right\} \\
q_{G} & =\left\{(a, b) \in I d_{G} \times I d_{G}:\left.L_{a}\right|_{K_{G}}=\left.L_{b}\right|_{K_{G}}\right\} \cup i d_{G} \\
i p_{G} & =\{(x, x x): x \in G\} \cup i d_{G}
\end{aligned}
$$

and a mapping $o_{G}: G \rightarrow G$ by $o_{G}(x)=x x$.

Lemma 1.2. Let $G$ be an $L S L D$ groupoid. Then

1. $p_{G}$ and $q_{G}$ are congruences of $G$ and $i p_{G} \subseteq p_{G}$;
2. $i p_{G}$ is a congruence of $G, G / i p_{G}$ is idempotent and $i p_{G}$ is the smallest congruence such that the corresponding factor is idempotent; moreover, every non-trivial block of $i p_{G}$ is isomorphic to $\mathbf{T}$;
3. $o_{G}$ is either the identity, or an involutive automorphism of $G$.

Proof. (1) The relation $p_{G}$ is the kernel of the homomorphism $\lambda$ from Lemma 1.1(3), hence it is a congruence.

The relation $q_{G}$ is an equivalence, so consider $a, b \in I d_{G}$ such that $\left.L_{a}\right|_{K_{G}}=\left.L_{b}\right|_{K_{G}}$. Then $\left.L_{a z}\right|_{K_{G}}=\left.L_{b z}\right|_{K_{G}}$ for all $z \in G$, since for every $k \in K_{G}$ we have $a z \cdot k=a(z \cdot a k)=a(z \cdot b k)=b(z \cdot b k)=b z \cdot k$ (because $z \cdot b k \in K_{G}$ ). And also $\left.L_{z a}\right|_{K_{G}}=\left.L_{z b}\right|_{K_{G}}$ for all $z \in G$, because for every $k \in K_{G}$ we have $z a \cdot k=z(a \cdot z k)=z(b \cdot z k)=z b \cdot k$ (because $z k \in K_{G}$ ). Consequently, $q_{G}$ is a congruence.

Finally, $x y=x(x \cdot x y)=x x \cdot(x \cdot x y)=x x \cdot y$ for every $x, y \in G$ and thus $i p_{G} \subseteq p_{G}$.
(2) Since $x x \cdot x x=x \cdot x x=x$ for every $x \in G$, the relation $i p_{G}$ is symmetric and transitive and every non-trivial block of $i p_{G}$ consists of two elements and thus is isomorphic to T. Further, $x z=x x \cdot z$ for every $z \in G$ due to (1) and $(z x, z \cdot x x) \in i p_{G}$ because $z \cdot x x=z x \cdot z x$; hence $i p_{G}$ is a congruence. Clearly, $G / i p_{G}$ is idempotent and $i p_{G}$ is the smallest congruence with this property.
(3) $o_{G}$ is an involution (or the identity) according to (2) and $o_{G}(x y)=$ $x y \cdot x y=x \cdot y y=x x \cdot y y=o_{G}(x) o_{G}(y)$ for all $x, y \in G$.

Corollary 1.3. T is the only (up to an isomorphism) simple non-idempotent LSLD groupoid.

Let $G$ be a groupoid, $e \notin G$ and $\varphi: G \rightarrow G$. We denote $G[\varphi]$ the groupoid defined on the set $G \cup\{e\}$ so that $G$ is a subgroupoid of $G[\varphi], e$ is a right zero and $e x=\varphi(x)$ for every $x \in G$.

Lemma 1.4. Let $G$ be an LSLD groupoid, $e \notin G$ and $\varphi: G \rightarrow G$. Then

1. $G[\varphi]$ is an LSLD groupoid, iff $\varphi=i d_{G}$ or $\varphi$ is an involutive automorphism of $G$ with $L_{x}=L_{\varphi(x)}$ for all $x \in G$;
2. $G\left[i d_{G}\right]$ and $G\left[o_{G}\right]$ are $L S L D$ groupoids and $G\left[o_{G}\right]\left[i d_{G\left[o_{G}\right]}\right], G\left[i d_{G}\right]$ $\left[o_{G\left[i d_{G}\right]}\right]$ are isomorphic.

Proof. This is a straightforward calculation.
Note that the three-element non-idempotent LSLD groupoids are isomorphic to $\mathbf{T}\left[i d_{\mathbf{T}}\right]$ and $\mathbf{T}\left[o_{\mathbf{T}}\right]$, respectively. One can check that $\left(\mathbf{T}\left[i d_{\mathbf{T}}\right]\right)\left[o_{\mathbf{T}\left[i d_{\mathbf{T}}\right]}\right]$ is the only four-element subdirectly irreducible non-idempotent LSLD groupoid.

The following technical lemmas become useful later.

Lemma 1.5. Let $G$ be an $L S L D$ groupoid and $\varphi \in\left\{i d_{G}, o_{G}\right\}$. Then the set $A_{\varphi}=\left\{a \in G: L_{a}=\varphi\right\}$ is either empty, or a left ideal of $G$.

Proof. Let $a \in A_{\varphi}$. By Lemma 1.1 $L_{x a}=L_{x} L_{a} L_{x}$ for every $x \in G$. If $L_{a}=\varphi=i d_{G}$, then $L_{x a}=L_{x} L_{x}=i d_{G}=\varphi$. If $L_{a}=\varphi=o_{G}$, then $L_{x a}(y)=x o_{G}(x y)=x(x y \cdot x y)=x(x \cdot y y)=o_{G}(y)$ for every $y \in G$ and thus $L_{x a}=o_{G}=L_{a}$. Hence $A_{\varphi}$ is a left ideal.

Lemma 1.6. Let $G$ be an $L S L D$ groupoid and $J$ a left ideal of $G$. Then the relation $\rho_{J}=\left(\left.\left(i p_{G}\right)\right|_{J}\right) \cup i d_{G}$ is a congruence of $G$.

Proof. The claim follows from Lemma 1.2.

Lemma 1.7. Let $G$ be an LSLD groupoid and $a \in G$ a right zero. Then

1. $x \cdot a y=a \cdot x y$ and $x y=a x \cdot y$ for all $x, y \in G$;
2. the relation $\nu_{a}=\{(x, a x): x \in G\} \cup i d_{G}$ is a congruence of $G$; moreover, every non-trivial block of $\nu_{a}$ has two elements.

Proof. (1) is calculated as follows: $x \cdot a y=x a \cdot x y=a \cdot x y$ and $a x \cdot$ $y=(a x)(a \cdot a y)=a(x \cdot a y)=a(a \cdot x y)=x y$. (2) Clearly, $\nu_{a}$ is both reflexive and symmetric and it follows from (1) that $\nu_{a}$ is compatible with the multiplication of $G$. We show that $\nu_{a}$ is transitive. If $(x, y) \in \nu_{a}$, $(y, z) \in \nu_{a}, x \neq y \neq z$, then $y=a x$ and $z=a y=a \cdot a x=x$ and thus $(x, z) \in \nu_{a}$. The rest becomes clear now.

Lemma 1.8. Let $G$ be an LSLD groupoid and let $\rho$ be a congruence of $K_{G}$ such that $(u, v) \in \rho$ implies $(a u, a v) \in \rho$ and $(u a \cdot z, v a \cdot z) \in \rho$ for all $a \in I d_{G}$ and $z \in K_{G}$. Define a relation $\sigma$ on $\operatorname{Id} d_{G} b y(a, b) \in \sigma$ iff $(a u, b v) \in \rho$ for every pair $(u, v) \in \rho$. Then $\rho \cup \sigma$ is a congruence of $G$.

Proof. This straightforward calculation is omitted.

## 2. Basic facts about subdirectly irreducible LSLD groupoids

It is well known that a groupoid $G$ is subdirectly irreducible (shortly $S I$ ), if and only if $G$ possesses a smallest non-trivial congruence (called the monolith of $G$ ), i.e., a congruence $\mu_{G} \neq i d_{G}$ such that $\mu_{G} \subseteq \nu$ for every congruence $\nu \neq i d_{G}$ on $G$.

Lemma 2.1. Let $G$ be an $S I$ non-idempotent $L S L D$ groupoid. Then

1. if $J \subseteq K_{G}$ is a left ideal, then $J=K_{G}$;
2. $i p_{G}$ is the monolith of $G$;
3. $\left.L_{a}\right|_{K_{G}} \neq\left. L_{b}\right|_{K_{G}}$ for every $a, b \in I d_{G}$ with $a \neq b$; in other words, $q_{G}=i d_{G} ;$
4. $\left.\varphi\right|_{K_{G}} \neq\left.\psi\right|_{K_{G}}$ for all automorphisms $\varphi, \psi$ of $G$ with $\varphi \neq \psi$.

Proof. (1) Let $J \subset K_{G}$ be a left ideal. Then $J^{\prime}=K_{G} \backslash J$ is a left ideal too and $\rho_{J}, \rho_{J^{\prime}}$ are non-trivial congruences, since both $J$ and $J^{\prime}$ contain at least two elements. However, $\rho_{J} \cap \rho_{J^{\prime}}=i d_{G}$ yields a contradiction with subdirect irreducibility of $G$.
(2) We have $\mu_{G} \subseteq i p_{G}$. Put $J=\left\{u \in K_{G}:(u, u u) \in \mu_{G}\right\}$. Then $J$ is a left ideal, because $\mu_{G}$ is a congruence, and thus $J=K_{G}$ and $\mu_{G}=i p_{G}$.
(3) According to Lemma 1.2(1), $q_{G}$ is a congruence. It is trivial, because $q_{G} \cap i p_{G}=i d_{G}$.
(4) Assume that $\left.\varphi\right|_{K_{G}}=\left.\psi\right|_{K_{G}}$ and we show that $\left.\varphi\right|_{I d_{G}}=\left.\psi\right|_{I d_{G}}$ too. Observe that $\left.\varphi\right|_{K_{G}}=\left.\psi\right|_{K_{G}}$ iff $\left.\varphi^{-1}\right|_{K_{G}}=\left.\psi^{-1}\right|_{K_{G}}$, because every automorphism of $G$ maps $K_{G}$ onto itself. Now, given $a \in I d_{G}$ and $u \in K_{G}$, we have $\varphi(a) u=\varphi(a) \varphi \varphi^{-1}(u)=\varphi\left(a \varphi^{-1}(u)\right)$ and, because $a \varphi^{-1}(u)=$ $a \psi^{-1}(u) \in K_{G}$, we have also $\varphi\left(a \varphi^{-1}(u)\right)=\psi\left(a \psi^{-1}(u)\right)=\psi(a) u$. Thus $\left.L_{\varphi(a)}\right|_{K_{G}}=\left.L_{\psi(a)}\right|_{K_{G}}$ and, by (3), $\varphi(a)=\psi(a)$.

Proposition 2.2. Let $G$ be a non-idempotent LSLD groupoid and $H$ a subgroupoid of $G$ such that $K_{G} \subseteq H$. Assume that $H$ is subdirectly irreducible. Then $G$ is subdirectly irreducible, iff $q_{G}=i d_{G}$.

Proof. The direct implication was proved in Lemma 2.1(3). So assume $q_{G}=i d_{G}$ and let $\rho$ be a non-trivial congruence on $G$. If $\left.\rho\right|_{H} \neq i d_{H}$, then $\left.i p_{H} \subseteq \rho\right|_{H}$. But $i p_{G}=i p_{H} \cup i d_{G}$ and thus $i p_{G} \subseteq \rho$. Hence assume that $\left.\rho\right|_{H}=$ $i d_{H}$. If $(a, b) \in \rho$ for some $a, b \in I d_{G}, a \neq b$, then $a u \neq b u$ for some $u \in K_{G}$ because $q_{G}=i d_{G}$ and we have $\left.(a u, b u) \in \rho\right|_{K_{G}}=i d_{K_{G}}$, a contradiction. If $(a, u) \in \rho$ for some $a \in I d_{G}$ and $u \in K_{G}$, then $(a, u u)=(a a, u u) \in \rho$ and, again, $\left.(u, u u) \in \rho\right|_{K_{G}}=i d_{K_{G}}$, a contradiction. Consequently, $G$ is subdirectly irreducible.

Corollary 2.3. Let $G$ be a non-idempotent LSLD groupoid such that $K_{G}$ is subdirectly irreducible. Then $G$ is subdirectly irreducible, iff $q_{G}=i d_{G}$.

Lemma 2.4. Let $G$ be an SI non-idempotent $L S L D$ groupoid and $a, b \in G$ right zeros. Then

1. $L_{a} \in\left\{i d_{G}, o_{G}\right\}$;
2. $a=b$, iff $L_{a}=L_{b}$;
3. $G$ contains at most two right zeros.

Proof. (1) Let $\nu_{a}$ be the congruence from Lemma 1.7. If $\nu_{a}=i d_{G}$, then $L_{a}=i d_{G}$. If $\nu_{a} \neq i d_{G}$, then $\mu_{G}=i p_{G} \subseteq \nu_{a}$ and thus $\left.L_{a}\right|_{K_{G}}=\left.o_{G}\right|_{K_{G}}$. Hence $L_{a}=o_{G}$ according to Lemma 2.1(4).

The statement (2) follows from Lemma 2.1(3) and (3) is an immediate consequence of (1) and (2).

Lemma 2.5. Let $G$ be an SI non-idempotent LSLD groupoid and let $a \in G$ be a right zero. Then $H=G \backslash\{a\}$ is an SI non-idempotent LSLD groupoid and it contains no right zero $b$ with $L_{b}=\left.L_{a}\right|_{H}$.

Proof. Clearly, $H$ is a left ideal of $G$ and thus a subgroupoid of $G$. Moreover, if $\rho$ is a non-trivial congruence of $H$, then $\sigma=\rho \cup\{(a, a)\}$ is a (non-trivial) congruence of $G$ (because $L_{a} \in\left\{i d_{G}, o_{G}\right\}$ ) and thus $i p_{G}=$ $\mu_{G} \subseteq \sigma$. So $i p_{H} \subseteq \rho$ and $H$ is subdirectly irreducible. Finally, if $b$ is a right zero in $H$, then it is also a right zero in $G$ and so $L_{b} \neq\left. L_{a}\right|_{H}$ by Lemma 2.4.

Lemma 2.6. Let $G$ be an $S I$ non-idempotent $L S L D$ groupoid and $\varphi \in$ $\left\{i d_{G}, o_{G}\right\}$. Then $G[\varphi]$ is subdirectly irreducible, iff $G$ contains no right zero a with $L_{a}=\varphi$.

Proof. The direct implication follows from Lemma 2.5. On the contrary, if $G$ contains no right zero $a$ with $L_{a}=\varphi$, then $A_{\varphi}=\emptyset$ (by Lemmas 1.5 and $2.1(3)\left|A_{\varphi}\right| \leq 1$, hence any element $b$ with $L_{b}=\varphi$ is a right zero), so $q_{G[\varphi]}=i d$ and Proposition 2.2 applies.

Corollary 2.7. Let $G$ be an SI non-idempotent LSLD groupoid with no right zero. Then

$$
G, G\left[i d_{G}\right], G\left[o_{G}\right] \text { and } G\left[i d_{G}\right]\left[o_{G\left[i d_{G}\right]}\right]
$$

are pairwise non-isomorphic SI LSLD groupoids.

Corollary 2.8. Let $G$ be an SI non-idempotent LSLD groupoid and let $A$ be the set of right zeros in $G$. Then $|A| \leq 2, H=G \backslash A$ is a left ideal of $G, H$ is an SI non-idempotent LSLD groupoid with no right zero and $G$ is isomorphic to exactly one of

$$
H, H\left[i d_{H}\right], H\left[o_{H}\right] \text { and } H\left[i d_{H}\right]\left[o_{H\left[i d_{H}\right]}\right]
$$

## 3. Groupoids of involutions

Let $\varepsilon$ be a binary relation on a non-empty set $X$. We denote $\operatorname{Inv}(X, \varepsilon)$ the set of all permutations $\varphi$ of $X$ such that $\varphi^{2}=i d_{X}$ and $(x, y) \in \varepsilon$ implies $(\varphi(x), \varphi(y)) \in \varepsilon$. It is easy to see that $\operatorname{Inv}(X, \varepsilon)$ is a subgroupoid of the core of the symmetric group over $X$ and thus it is an idempotent LSLD groupoid.

An equivalence $\varepsilon$ is called a pairing (a semipairing, resp.), if every block of $\varepsilon$ consists of (at most, resp.) two elements. Let $\alpha(m)=|\operatorname{Inv}(m, \varepsilon)|$, where $\varepsilon$ is a pairing on a cardinal number $m(\alpha(m)$ is defined for even and infinite cardinals only).

Proposition 3.1. $\alpha(2)=2, \alpha(4)=6$ and $\alpha(m)=2 \alpha(m-2)+(m-2)$ $\alpha(m-4)$ for every even $6 \leq m<\omega$. Further, $\alpha(m)=2^{m}$ for every infinite $m$.

Proof. Assume that $m$ is finite even and the blocks of $\varepsilon$ are the sets $\{2 k, 2 k+1\}^{2}, k=0, \ldots, \frac{m}{2}-1$. The claim is trivial for $m \in\{2,4\}$, so assume $m \geq 6$. Let $I_{k}=\{\varphi \in \operatorname{Inv}(m, \varepsilon): \varphi(0)=k\}$ for $0 \leq k \leq m-1$. Then $\operatorname{Inv}(m, \varepsilon)=\bigcup_{k=0}^{m-1} I_{k}$ and $I_{k}$ 's are pairwise disjoint. If $\varphi \in I_{0}$, then $\varphi(1)=1$. If $\varphi \in I_{1}$, then $\varphi(1)=0$. Consequently, $\left|I_{0}\right|=\left|I_{1}\right|=\alpha(m-2)$. On the other hand, if $\varphi \in I_{k}$ for $k \geq 2$, then $\varphi(1)=k^{\prime}$, where $k^{\prime} \neq k$ is such that $\left(k, k^{\prime}\right) \in \varepsilon$, and thus $\varphi(k)=0, \varphi\left(k^{\prime}\right)=1$. Hence $\left|I_{k}\right|=\alpha(m-4)$ and $|\operatorname{Inv}(m, \varepsilon)|=2 \alpha(m-2)+(m-2) \alpha(m-4)$.

If $m$ is infinite, consider all involutions of the form $\left(x_{1} y_{1}\right)\left(x_{2} y_{2}\right) \ldots$, where $\left\{x_{1}, y_{1}\right\},\left\{x_{2}, y_{2}\right\}, \ldots$ are pairwise different blocks of $\varepsilon$. They belong to $\operatorname{Inv}(m, \varepsilon)$ and thus $\alpha(m) \geq 2^{m}$. Hence $\alpha(m)=2^{m}$.

|  | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\alpha(m)$ | 2 | 6 | 20 | 76 | 312 | 1384 | 6512 | 32400 | 168992 | 921184 |

For every semipairing $\varepsilon$ on $X$ there is a unique mapping $o_{\varepsilon} \in \operatorname{Inv}(X, \varepsilon)$ such that $\left(x, o_{\varepsilon}(x)\right) \in \varepsilon$ and $o_{\varepsilon}(x)=x$ iff $\{x\}$ is a one-element block of $\varepsilon$. It is easy to see that $i d_{X}$ and $o_{\varepsilon}$ are right zeros $\operatorname{in} \operatorname{Inv}(X, \varepsilon)$ and that $i d_{X} * \varphi=\varphi$ and $o_{\varepsilon} * \varphi=\varphi$ for every $\varphi \in \operatorname{Inv}(X, \varepsilon)$. Let $\operatorname{Inv}^{-}(X, \varepsilon)=\operatorname{Inv}(X, \varepsilon) \backslash\left\{i d_{X}, o_{\varepsilon}\right\}$. Clearly, it is either empty, or a left ideal of $\operatorname{Inv}(X, \varepsilon)$.

Finally, let $\operatorname{Aut}_{2}(G)=\left\{\varphi \in \operatorname{Aut}(G): \varphi^{2}=i d\right\}$. If $G$ is an LSLD groupoid, then $\operatorname{Aut}_{2}(G)$ is a subgroupoid of $\operatorname{Inv}\left(G, i p_{G}\right), L_{x} \in \operatorname{Aut}_{2}(G)$ for every $x \in G$ and the mapping $x \mapsto L_{x}$ is a homomorphism of $G$ into $\operatorname{Aut}_{2}(G)$. Let $\operatorname{Aut}_{2}^{-}(G)=\operatorname{Aut}_{2}(G) \cap \operatorname{Inv}^{-}\left(G, i p_{G}\right)$.

Proposition 3.2. Let $G$ be an SI non-idempotent LSLD groupoid with at least one idempotent element. Then the mapping

$$
\eta: I d_{G} \rightarrow \operatorname{Aut}_{2}\left(K_{G}\right),\left.\quad a \mapsto L_{a}\right|_{K_{G}}
$$

is an injective homomorphism.

Proof. It follows from Lemmas 1.1 and 2.1(3).

Corollary 3.3. Let $G$ be an SI LSLD groupoid with $\left|K_{G}\right|=m \neq 0$. Then

$$
\left|I d_{G}\right| \leq \alpha(m) \quad \text { and } \quad|G| \leq \alpha(m)+m
$$

It will be shown in the next section that the upper bound on $\left|I d_{G}\right|$ is best possible.

## 4. A DESCRIPTION OF SUBDIRECTLY IRREDUCIBLE LSLD GROUPOIDS

Lemma 4.1. Let $K$ be an idempotent-free $L S L D$ groupoid and $I$ a subgroupoid of $\operatorname{Aut}_{2}(K)$. Put $G=I \cup K$. Then the following conditions are equivalent.

1. The operations of $I$ and $K$ can be extended onto $G$ so that $G$ becomes an $L S L D$ groupoid with $\varphi \cdot u=\varphi(u)$ for all $\varphi \in I, u \in K$.
2. $L_{u} \varphi L_{u} \in I$ for all $\varphi \in I, u \in K$.

Moreover, if the conditions are satisfied, the operation of $G$ is uniquely determined and $u \cdot \varphi=L_{u} \varphi L_{u}$ for all $\varphi \in I, u \in K$.

Proof. Clearly, $u \varphi \in I=I d_{G}$ for every $u \in K, \varphi \in I$. Since $u(\varphi v)=$ $(u \varphi)(u v)$ for every $u, v \in K, \varphi \in I$, we have $L_{u}(\varphi(v))=(u \varphi)\left(L_{u}(v)\right)$ and thus $u \varphi=L_{u} \varphi\left(L_{u}\right)^{-1}=L_{u} \varphi L_{u}$. Indeed, this is possible, iff $L_{u} \varphi L_{u} \in I$ for all $\varphi \in I, u \in K$. We omit the straightforward calculation showing that the resulting groupoid $G$ is LSLD.
The groupoid $G$ from Lemma 4.1 will be denoted by $I \sqcup K$. The groupoid $\operatorname{Aut}_{2}(K) \sqcup K$ will be called the full extension of $K$ and denoted Full( $(K)$.

| $I \sqcup K$ | $\psi$ | $v$ |
| :---: | :---: | :---: |
| $\varphi$ | $\varphi \psi \varphi$ | $\varphi(v)$ |
| $u$ | $L_{u} \psi L_{u}$ | $u v$ |

Theorem 4.2. Let $G$ be an SI non-idempotent LSLD groupoid. Then there exists an injective homomorphism $\eta: G \rightarrow \operatorname{Full}\left(K_{G}\right)$ such that

$$
\eta(u)=u \text { for every } u \in K_{G} \quad \text { and } \quad \eta(a)=\left.L_{a}\right|_{K_{G}} \text { for every } a \in I d_{G} .
$$

Thus $G$ is isomorphic (via $\eta$ ) to the subgroupoid $\eta\left(I d_{G}\right) \sqcup K_{G}$ of $\operatorname{Full}\left(K_{G}\right)$.
Proof. It is straightforward to check that $\eta$ is a homomorphism and it is injective according to Proposition 3.2.

Remark. Let $K$ be an idempotent-free LSLD groupoid and assume the set $\mathcal{S}$ of SI subgroupoids $G$ of $\operatorname{Full}(K)$ with $K_{G}=K$. The set $\mathcal{S}$ is non-empty, iff $\operatorname{Full}(K) \in \mathcal{S}$; in this case, the set $\mathcal{S}$ has minimal elements, say $H_{1}, \ldots, H_{k}$, and it follows from Proposition 2.2 that $G \in \mathcal{S}$, iff $G$ is a subgroupoid of Full $(K)$ and $H_{i} \subseteq G$ for at least one $1 \leq i \leq k$.

Theorem 4.3. The following conditions are equivalent for an idempotentfree LSLD groupoid $K$ :

1. There exists an SI LSLD groupoid $G$ with $K_{G}=K$.
2. The groupoid Full( $K$ ) is $S I$.
3. The groupoid Full $^{-}(K)$ is $S I$.
4. If $\rho$ is a non-trivial $\operatorname{Aut}_{2}(K)$-invariant congruence of $K$, then $i p_{K} \subseteq \rho$.

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition $2.2,(2) \Rightarrow(3)$ follows from Lemma 2.5 and $(3) \Rightarrow(1)$ is trivial.

Now, assume that (4) is true and let $\sigma$ be a non-trivial congruence of $\operatorname{Full}(K)$. If $\left.\sigma\right|_{K} \neq i d_{K}$, then $i p_{K} \subseteq \sigma$ by (4) and thus $\operatorname{Full}(K)$ is SI. So assume that $\rho=\left.\sigma\right|_{K}=i d_{K}$. If $(\varphi, \psi) \in \sigma$ for some $\varphi, \psi \in \operatorname{Aut}_{2}(K)$, $\varphi \neq \psi$, then there is at least one $u \in K$ with $\varphi(u) \neq \psi(u)$ and we have $(\varphi(u), \psi(u)) \in \rho$, a contradiction. Thus $(\varphi, u) \in \sigma$ for some $\varphi \in \operatorname{Aut}_{2}(K)$, $u \in K$. In this case, $(\varphi, u u) \in \sigma$ and so $(u, u u) \in \rho$, a contradiction again.

Finally, assume (2) and consider a non-trivial Aut ${ }_{2}(K)$-invariant congruence $\rho$ of $K$. Define a relation $\sigma$ on $\operatorname{Aut}_{2}(K)$ by $(\varphi, \psi) \in \sigma$ iff $(\varphi(u), \psi(v)) \in \rho$ for every pair $(u, v) \in \rho$. According to Lemma 1.8, $\rho \cup \sigma$ is a congruence of $\operatorname{Full}(K)$ and so $i p_{K} \subseteq \rho$.
A groupoid $K$ satisfying the conditions of Theorem 4.3 will be called pre-SI.
Example. Let $\varepsilon$ be a pairing on a non-empty set $K$. We equip the set $K$ with an operation such that $L_{u}=o_{\varepsilon}$ for every $u \in K$. Clearly, $K$ is an idempotent-free LSLD groupoid and $\operatorname{Aut}_{2}(K)=\operatorname{Inv}(K, \varepsilon)$. Using Theorem 4.3, we prove that $K$ is pre-SI and thus $G=\operatorname{Full}(K)$ is an SI LSLD groupoid of size $\alpha\left(\left|K_{G}\right|\right)+\left|K_{G}\right|($ cf. Corollary 3.3).

Let $\rho$ be a non-trivial $\operatorname{Aut}_{2}(K)$-invariant congruence on $K$. We claim that $i p_{K}=o_{\varepsilon} \subseteq \rho$. Indeed, if $\left(u, o_{K}(u)\right) \in \rho$ for some $u \in K$, then for every $v \in K$ the involution $\varphi=(u v)\left(o_{K}(u) o_{K}(v)\right)$ belongs to $\operatorname{Aut}_{2}(K)$ and thus $\left(v, o_{K}(v)\right) \in \rho$. Thus $i p_{K} \subseteq \rho$. On the other hand, if $(u, v) \in \rho$, $u \neq v \neq o_{K}(u)$, then the involution $\psi=\left(v o_{K}(v)\right)$ belongs to $\operatorname{Aut}_{2}(K)$ and thus $(u, o(v))=(\psi(u), \psi(v)) \in \rho$ and so $(v, o(v)) \in \rho$.

Example. Consider the following four-element groupoid $K$.

$$
\begin{array}{c|cccc}
K & 0 & \widetilde{0} & 1 & \widetilde{1} \\
\hline 0, \widetilde{0} & \widetilde{0} & 0 & \widetilde{1} & 1 \\
1, \widetilde{1} & 0 & \widetilde{0} & \widetilde{1} & 1
\end{array}
$$

One can check that $K$ is an LSLD groupoid, $\operatorname{Aut}_{2}(K)=$ $\left\{i d_{K},(0 \widetilde{0}),(1 \widetilde{1}),(0 \widetilde{0})(1 \widetilde{1})\right\}$ and the relation $\rho=\{(0, \widetilde{0}),(\widetilde{0}, 0)\} \cup i d_{K}$ is an $\operatorname{Aut}_{2}(K)$-invariant congruence of $K$. However, $i p_{K} \nsubseteq \rho$ and thus $K$ is not pre-SI.

## 5. Few idempotent elements

In this section, let $G$ be a finite SI non-idempotent LSLD groupoid with $I d_{G} \neq \emptyset$ and $r, s, \alpha, \beta$ will denote non-negative integers.

Let $n=\left|I d_{G}\right|$ and $2 m=\left|K_{G}\right|$. We put $K_{1}(a)=\left\{u \in K_{G}: a u=u\right\}$, $K_{2}(a)=\left\{u \in K_{G}: a u=u u\right\}$ and $K_{3}(a)=K_{G} \backslash\left(K_{1}(a) \cup K_{2}(a)\right)$ for every $a \in I d_{G}$.

Lemma 5.1. $\left|K_{1}(a)\right|,\left|K_{2}(a)\right|$ are even numbers and $\left|K_{3}(a)\right|$ is divisible by 4 .

Proof. $\left|K_{1}(a)\right|$ is even, because $u \in K_{1}(a)$, iff $u u \in K_{1}(a)$ (and analogously for $\left.\left|K_{2}(a)\right|\right)$. Furthermore, the sets $\{v, v v, a v, a \cdot v v\}, v \in K_{3}(a)$, are fourelement and pairwise disjoint.
Let $r(a)=\frac{1}{2}\left|K_{1}(a)\right|$ and $s(a)=\frac{1}{2}\left|K_{2}(a)\right|$. Hence $m-r(a)-s(a)$ is a (non-negative) even number.

Lemma 5.2. $r(x a)=r(a)$ and $s(x a)=s(a)$ for all $a \in I d_{G}, x \in G$.
Proof. If $v \in K_{1}(a)$, then $x a \cdot x v=x \cdot a v=x v$ and so $x v \in K_{1}(x a)$. Conversely, if $w \in K_{1}(x a)$, then $x w=x(x a \cdot w)=(x \cdot x a)(x w)=a \cdot x w$ and so $x w \in K_{1}(a)$. Thus $L_{x}$ maps bijectively $K_{1}(a)$ onto $K_{1}(x a)$ and, in particular, $r(a)=\left|K_{1}(a)\right|=\left|K_{1}(x a)\right|=r(x a)$. Analogously, $s(a)=s(x a)$.

Let $I(r, s)=\left\{a \in I d_{G}: r(a)=r, s(a)=s\right\}$. Indeed, if $I(r, s) \neq \emptyset$, then $m-r-s$ is a non-negative even number. It follows from Lemma 5.2 that $I(r, s)$ is either empty, or a left ideal of $G$.

## Lemma 5.3.

1. If $r \geq m$ and $I(r, s) \neq \emptyset$, then $r=m, s=0$ and $|I(r, s)|=1$.
2. If $s \geq m$ and $I(r, s) \neq \emptyset$, then $r=0, s=m$ and $|I(r, s)|=1$.

Proof. (1) Since $m \geq r+s$, we have $r=m$ and $s=0$. Consequently, $I(r, s)=I(m, 0)=\left\{a \in I d_{G}: a u=u\right.$ for every $\left.u \in K_{G}\right\}$, and hence $|I(r, s)|=1$ by Lemma $2.1(3)$. (2) is analogous.
Let $K(r, s, \alpha, \beta)$ be the set of all $u \in K_{G}$ such that $\mid\{a \in I(r, s): u \in$ $\left.K_{1}(a)\right\} \mid=\alpha$ and $\left|\left\{a \in I(r, s): u \in K_{2}(a)\right\}\right|=\beta$.

Lemma 5.4. Either $K(r, s, \alpha, \beta)=\emptyset$, or $K(r, s, \alpha, \beta)=K_{G}$.
Proof. Assume that $J=K(r, s, \alpha, \beta) \neq \emptyset$. We prove that $J$ is a left ideal. Since $a \cdot x u=x u$ iff $x a \cdot u=u$ for every $u \in J, x \in G, a \in I d_{G}$, we have $L_{x}(\{b \in I(r, s): b \cdot x u=x u\})=\{c \in I(r, s): c u=u\}$ (use the fact that $I(r, s)$ is a left ideal) and, in particular, $\left|\left\{b \in I(r, s): x u \in K_{1}(b)\right\}\right|=\alpha$. Similarly, $\left|\left\{b \in I(r, s): x u \in K_{2}(b)\right\}\right|=\beta$ and thus $x u \in J$. Consequently, $J=K_{G}$ by Lemma 2.1(1).

Consequently, for every $r, s$ there is a unique pair $(\alpha, \beta)$ such that $K(r, s, \alpha, \beta)$ $=K_{G}$ and $K\left(r, s, \alpha^{\prime}, \beta^{\prime}\right)=\emptyset$ for all $\left(\alpha^{\prime}, \beta^{\prime}\right) \neq(\alpha, \beta)$.

Lemma 5.5. If $K(r, s, \alpha, \beta)=K_{G}$, then $\alpha m=r t$ and $\beta m=s t$, where $t=|I(r, s)|$.

Proof. Since $|\{a \in I(r, s): a u=u\}|=\alpha$ and $|\{a \in I(r, s): a u=u u\}|=\beta$ for every $u \in K_{G}$, we have $|L|=2 \alpha m$, where $L=\left\{(a, u) \in I(r, s) \times K_{G}\right.$ : $a u=u\}$. On the other hand, $|L|=2 r t$ by the definition of $I(r, s)$. Thus $\alpha m=r t$. Considering the set $\left\{(a, u) \in I(r, s) \times K_{G}: a u=u u\right\}$, a similar proof yields $\beta m=s t$.

Lemma 5.6. If $K(r, s, \alpha, \beta)=K_{G}, I(r, s) \neq \emptyset$ and the numbers $m$ and $t=|I(r, s)|$ are relatively prime, then just one of the following cases takes place:

1. $r=s=\alpha=\beta=0$.
2. $r=m, s=0, \alpha=1, \beta=0$ and $t=1$.
3. $r=0, s=m, \alpha=0, \beta=1$ and $t=1$.

Proof. By Lemma 5.5, $\alpha m=r t$ and $\beta m=s t$. If $r=s=0$, then obviously $\alpha=\beta=0$. If $r \geq 1$, then $m$ divides $r$ and thus $r \geq m$. If $s \geq 1$, then $m$ divides $s$ and thus $s \geq m$. In both cases, Lemma 5.3 applies.

Proposition 5.7. If $I(r, s) \neq \emptyset, r+s \geq 1$ and the numbers $m$ and $t=$ $|I(r, s)|$ are relatively prime, then $G$ contains a right zero.

Proof. Choose $\alpha, \beta$ such that $K(r, s, \alpha, \beta)=K_{G}$. It follows from Lemma 5.6 that $t=1$ and thus $I(r, s)$ consists of a right zero.

Proposition 5.8. If $m$ is not divisible by any prime number $p \in\{2, \ldots, n-$ $2, n\}$, then either $G$ contains a right zero, or $n=3, m$ is even and $u \neq a u \neq$ uu for all $a \in I d_{G}, u \in K_{G}$.

Proof. If $n=1$, then $I d_{G}=\{a\}$ and $a$ is a right zero; so we may assume that $n \geq 2$. Obviously, if $I(r, s)=\emptyset$ for all $r$, $s$ with $r+s \geq 1$, then $u \neq a u \neq u u$ for all $a \in I d_{G}, u \in K_{G}$, and thus $m$ is divisible by 2 according to Lemma 5.1. Consequently, $2=n-1$ and thus $n=3$.

So assume that there are $r, s$ such that $r+s \geq 1$ and $t=|I(r, s)| \geq 1$. If $m$ and $t$ are relatively prime, then Lemma 5.7 yields the result. If $p$ is a prime dividing both $m$ and $t$, then $p \leq t \leq n$, and therefore $p=n-1$, $t=n-1$ and the only $a \in I d_{G} \backslash I(r, s)$ is a right zero.

Theorem 5.9. Let $G$ be a finite SI non-idempotent LSLD groupoid with $\left|K_{G}\right|=2 m \geq 4$ and let $p$ be the least prime divisor of $m$. If $\left|I d_{G}\right|<p$, then either $I d_{G}$ contains precisely three elements which are not right zeros, or every element of $I d_{G}$ is a right zero and thus $\left|I d_{G}\right| \leq 2$ and $K_{G}$ is subdirectly irreducible.

Proof. Let $H=G \backslash A$, where $A$ is the set of all right zeros of $G$. According to Corollary $2.8, H$ is an SI LSLD groupoid with no right zeros. However, if $I d_{H} \neq \emptyset$, then $H$ contains a right zero by Proposition 5.8, a contradiction. The rest follows from Corollary 2.8 too.

## 6. Small subdirectly irreducible LSLD groupoids

In this section we apply the theory developed above to search for small SI non-idempotent LSLD groupoids. The procedure for finding all SI LSLD groupoids $G$ with $m>0$ non-idempotent elements follows.

1. We find all $\frac{m}{2}$-element LSLDI groupoids.
2. We find all $m$-element idempotent-free LSLD groupoids by extending groupoids found in the first step and check which of them are pre-SI (using Theorem 4.3).
3. For each pre-SI groupoid $K$ found in the second step, we characterize subgroupoids $I$ of $\mathrm{Aut}_{2}^{-} K$ with the property $4.1(2)$ and check which $I \sqcup K$ are subdirectly irreducible.
4. Each SI LSLD groupoid found in the third step can be extended by $i d_{G}, o_{G}$, none or both (see Corollary 2.7).

Two non-idempotents. Let $G$ be an SI LSLD groupoid with $\left|K_{G}\right|=2$. Then $K_{G} \simeq \mathbf{T}$ and $I d_{G}$ is either empty, or isomorphic to a subgroupoid of $\operatorname{Aut}_{2}(\mathbf{T})=\operatorname{Inv}\left(\mathbf{T}, i p_{\mathbf{T}}\right)=\left\{i d_{\mathbf{T}}, o_{\mathbf{T}}\right\}$. Hence

$$
\mathbf{T}, \mathbf{T}\left[i d_{\mathbf{T}}\right], \mathbf{T}\left[o_{\mathbf{T}}\right] \text { and } \mathbf{T}\left[i d_{\mathbf{T}}\right]\left[o_{\mathbf{T}\left[i d_{\mathbf{T}}\right]}\right]
$$

are the only (up to an isomorphism) SI LSLD groupoids with two nonidempotent elements.

Four non-idempotents. Let $G$ be an SI LSLD groupoid with $\left|K_{G}\right|=4$. Then $K_{G} / i p_{K_{G}}$ is isomorphic to $\mathbf{S}$, the only two-element LSLDI groupoid. Clearly, the following groupoids $K_{1}, K_{2}, K_{3}$ are the only (up to an isomorphism) 4-element idempotent-free LSLD groupoids:

| $K_{1}$ | 0 | $\widetilde{0}$ | 1 | $\widetilde{1}$ |
| :---: | :---: | :---: | :---: | :---: |
| $0, \widetilde{0}$ | $\widetilde{0}$ | 0 | $\widetilde{1}$ | 1 |
| $1, \widetilde{1}$ | $\widetilde{0}$ | 0 | $\widetilde{1}$ | 1 |


| $K_{2}$ | 0 | $\widetilde{0}$ | 1 | $\widetilde{1}$ |
| :---: | :---: | :---: | :---: | :---: |
| $0, \widetilde{0}$ | $\widetilde{0}$ | 0 | 1 | $\widetilde{1}$ |
| $1, \widetilde{1}$ | 0 | $\widetilde{0}$ | $\widetilde{1}$ | 1 |


| $K_{3}$ | 0 | $\widetilde{0}$ | 1 | $\widetilde{1}$ |
| :---: | :---: | :---: | :---: | :---: |
| $0, \widetilde{0}$ | $\widetilde{0}$ | 0 | $\widetilde{1}$ | 1 |
| $1, \widetilde{1}$ | 0 | $\widetilde{0}$ | $\widetilde{1}$ | 1 |

$K_{1}$ and $K_{2}$ are pre-SI, $K_{3}$ is not (see the last example in the fourth section). Hence $K_{G}$ is isomorphic to one of $K_{\widetilde{1}}, K_{2}$. Now, we designate $a=(0 \widetilde{0})$, $b=\left(\begin{array}{ll}1 & \widetilde{1}\end{array}\right), c=\left(\begin{array}{ll}0 & 1\end{array}\right)(\widetilde{0} \widetilde{1}), d=\left(\begin{array}{ll}0 & \widetilde{1})(\widetilde{0} 1) \text { the elements of } I=\operatorname{Aut}_{2}^{-}\left(K_{1}\right)= \\ =\end{array}\right.$ $\operatorname{Aut}_{2}^{-}\left(K_{2}\right)$. The multiplication table of $I$ is

| $I$ | $a$ | $b$ | $c$ | $d$ |
| :---: | :--- | :--- | :--- | :--- |
| $a$ | $a$ | $b$ | $d$ | $c$ |
| $b$ | $a$ | $b$ | $d$ | $c$ |
| $c$ | $b$ | $a$ | $c$ | $d$ |
| $d$ | $b$ | $a$ | $c$ | $d$ |

Thus $I$ contains three non-trivial subgroupoids $I_{1}=\{a, b\}, I_{2}=\{c, d\}$ and $I_{3}=\{a, b, c, d\}$. Neither $K_{1}$ nor $K_{2}$ is SI. Since both $I_{1} \sqcup K_{1}, I_{1} \sqcup K_{2}$ contain the left ideal $\{0, \widetilde{0}\}$, they are not SI. In $I_{2} \sqcup K_{1}$, the element $c$ is a right zero, because $L_{x}=o_{K_{1}}$ for every $x \in K_{1}$, and thus $L_{x} c L_{x}=c$; so $I_{2} \sqcup K_{1}$ is not SI by Corollary 2.8. On the other hand, it is easy to check that $I_{2} \sqcup K_{2}$, $I_{3} \sqcup K_{1}$ and $I_{3} \sqcup K_{2}$ are SI.

Proposition 6.1. There are 12 (up to an isomorphism) SI LSLD groupoids with four non-idempotent elements:

$$
I_{3} \sqcup K_{1}, \quad I_{2} \sqcup K_{2}, \quad I_{3} \sqcup K_{2}
$$

and their extensions by right zeros.

Six non-idempotents. Let $G$ be an SI LSLD groupoid with $\left|K_{G}\right|=6$. Then $K_{G} / i p_{K_{G}}$ is isomorphic to one of $\mathbf{S}_{1}, \mathbf{S}_{2}, \mathbf{S}_{3}$ (see the list of threeelement LSLDI groupoids in the introduction). $\mathbf{S}_{2}$ cannot be isomorphic to $K_{G} / i p_{K_{G}}$, because the $i p_{K_{G}}$-block corresponding to the element 0 of $\mathbf{S}_{2}$ is always a proper left ideal inside $K_{G}$ (every automorphism of $G$ preserves this block), a contradiction with Lemma 2.1(1). Now, one can check that the following groupoids $K_{4}, K_{5}, K_{6}, K_{7}$ are the only (up to an isomorphism) 6 -element idempotent-free LSLD groupoids such that their factorgroupoid over $i p$ is one of $\mathbf{S}_{1}, \mathbf{S}_{3}$.

| $K_{4}$ | 0 | $\widetilde{0}$ | 1 | $\widetilde{1}$ | 2 |  | $\widetilde{2}$ | $K_{5}$ | 0 | $\widetilde{0}$ | 1 | 1 | 2 |  | 2 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0, ${ }_{0}$ | $\widetilde{0}$ | 0 | Ĩ | 1 | $\widetilde{2}$ |  | 2 | 0, ${ }_{0}$ | O | 0 | 1 | $\widetilde{1}$ | 2 |  | $\widetilde{2}$ |
| 1, $\widetilde{1}$ | $\widetilde{0}$ | 0 | $\widetilde{1}$ | 1 | $\widetilde{2}$ |  | 2 | $1, \widetilde{1}$ | 0 | $\widetilde{0}$ | $\widetilde{1}$ | 1 | 2 |  | $\widetilde{2}$ |
| $2, \widetilde{2}$ | O | 0 | $\widetilde{1}$ | 1 | $\widetilde{2}$ |  | 2 | $2, \widetilde{2}$ | 0 | $\widetilde{0}$ | 1 | $\widetilde{1}$ | $\widetilde{2}$ |  | 2 |
| $K_{6}$ | 0 | $\widetilde{0}$ | 1 | İ | 2 |  | $\widetilde{2}$ | $K_{7}$ | 0 | 0 | 1 | 1 | 2 |  | $\widetilde{2}$ |
| 0, ${ }_{0}$ | $\widetilde{0}$ | 0 | $\widetilde{1}$ | 1 | 2 |  | $\widetilde{2}$ | 0, ${ }^{\text {a }}$ | 0 | 0 | $\widetilde{2}$ | 2 | IT |  | 1 |
| 1, | 0 | $\widetilde{0}$ | $\widetilde{1}$ | 1 | $\widetilde{2}$ |  | 2 | 1, 1 | $\widetilde{2}$ | 2 | $\widetilde{1}$ | 1 | O |  | 0 |
| $2, \widetilde{2}$ | $\widetilde{0}$ | 0 | 1 | $\widetilde{1}$ | $\widetilde{2}$ |  | 2 | $2, \widetilde{2}$ | $\widetilde{1}$ | 1 | $\widetilde{0}$ | 0 | $\widetilde{2}$ |  | 2 |

$K_{4}$ and $K_{5}$ are pre-SI, $K_{6}$ and $K_{7}$ aren't. Hence $K_{G}$ is isomorphic to one of $K_{4}, K_{5}$. One can compute that $I=\operatorname{Inv}^{-}\left(K_{4}, i p_{K_{4}}\right)=\operatorname{Aut}_{2}^{-}\left(K_{4}\right)=$ Aut $_{2}^{-}\left(K_{5}\right)$ contains the following non-trivial subgroupoids:

$$
\begin{aligned}
& I_{1}=\{(x \widetilde{x}): x=0,1,2\}, \\
& I_{2}=\{(x \widetilde{x})(y \widetilde{y}): x, y=0,1,2, x \neq y\}, \\
& I_{3,1}=\{(x y)(\widetilde{x} \widetilde{y}): x, y=0,1,2, x \neq y\}, \\
& I_{3,2}=\left\{\left(\begin{array}{ll}
0 & \left.\widetilde{1})(\widetilde{0} 1),(0 \widetilde{2})(\widetilde{0} 2),\left(\begin{array}{ll}
1 & 2
\end{array}\right)(\widetilde{1} \widetilde{2})\right\},
\end{array}\right.\right. \\
& I_{3,3}=\left\{\left(\begin{array}{ll}
0 & \left.\widetilde{1})(\widetilde{0} 1),\binom{1}{2}(\widetilde{1} 2),\left(\begin{array}{ll}
0 & 2
\end{array}\right)(\widetilde{0} \widetilde{2})\right\}, \\
\end{array}\right.\right. \\
& I_{3,4}=\left\{\left(\begin{array}{ll}
0 & \left.\widetilde{2})(\widetilde{0} 2),\left(\begin{array}{ll}
1 & \widetilde{2})(\widetilde{1} 2),(01)(\widetilde{0} \\
\hline
\end{array}\right)\right\}, ~
\end{array}\right.\right. \\
& I_{3}=\{(x y)(\widetilde{x} \widetilde{y}),(x \widetilde{y})(\widetilde{x} y): x, y=0,1,2, x \neq y\}=I_{3,1} \cup I_{3,2} \cup I_{3,3} \cup I_{3,4}, \\
& I_{4,1}=\{(x \widetilde{y})(\widetilde{x} y)(z \widetilde{z}):\{x, y, z\}=\{0,1,2\}\}, \\
& I_{4,2}=\left\{\left(\begin{array}{ll}
0 & \left.1)(\widetilde{0} \widetilde{1})(2 \widetilde{2}),\left(\begin{array}{ll}
0
\end{array}\right)(\widetilde{0} \widetilde{2})(1 \widetilde{1}),\binom{1}{2}(\widetilde{1} 2)(0 \widetilde{0})\right\},
\end{array}\right.\right. \\
& I_{4,3}=\left\{\left(\begin{array}{ll}
0 & \left.1)(\widetilde{0} \widetilde{1})(2 \widetilde{2}),\left(\begin{array}{ll}
1 & 2
\end{array}\right)(\widetilde{1} \widetilde{2})(0 \widetilde{0}),(0 \widetilde{2})(\widetilde{0} 2)(1 \widetilde{1})\right\},
\end{array}\right.\right. \\
& I_{4,4}=\left\{\left(\begin{array}{ll}
0 & 2)(\widetilde{0} \widetilde{2})(1 \widetilde{1}),(12)(\widetilde{1} \widetilde{2})(0 \widetilde{0}),(0 \widetilde{1})(\widetilde{0} 1)(2 \widetilde{2})\},
\end{array}\right.\right. \\
& I_{4}=\{(x \widetilde{y})(\widetilde{x} y)(z \widetilde{z}),(x y)(\widetilde{x} \widetilde{y})(z \widetilde{z}):\{x, y, z\}=\{0,1,2\}\}=I_{4,1} \cup \cdots \cup I_{4,4}, \\
& I_{3, i} \cup I_{4, i}, \quad i=1,2,3,4,
\end{aligned}
$$

all unions of $I_{1}, I_{2}, I_{3}, I_{4}$.
Clearly, $\left|I_{1}\right|=\left|I_{2}\right|=\left|I_{3, i}\right|=\left|I_{4, i}\right|=3, i=1, \ldots, 4$ and $\left|I_{3}\right|=\left|I_{4}\right|=6$. Now, none of $K_{4}, K_{5}$ is SI. The following table shows, which of $J \sqcup K_{4}, J \sqcup K_{5}$ ( $J$ a subgroupoid of $I$ ) are subdirectly irreducible. (An empty space means it does not satisfy the condition 4.1(2).)

| $\sqcup$ | $I_{1}$ | $I_{2}$ | $I_{3,1}$ | $I_{3,2}, I_{3,3}, I_{3,4}$ | $I_{3}$ | $I_{4,1}$ | $I_{4,2}, I_{4,3}, I_{4,4}$ | $I_{4}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $K_{4}$ | - | - | - | - | + | - | - | + |
| $K_{5}$ | - | - |  |  | + |  |  | + |


| $\sqcup I_{3,1} \cup I_{4,1}$ | $I_{3, i} \cup I_{4, i}$ | $I_{1} \cup I_{2}$ | $I_{i} \cup I_{j}$ | $I_{i} \cup I_{j} \cup I_{k}$ | $I$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $i=2,3,4$ |  | $i \neq j,\{i, j\} \neq\{1,2\}$ | $i \neq j \neq k \neq i$ |  |
| $K_{4}$ | - | - | - | + | + | + |
| $K_{5}$ |  |  | - | + | + | + |

Proposition 6.2. There are 96 (up to an isomorphism) SI LSLD groupoids with six non-idempotent elements: the 24 without right zeros described in the table above and their extensions by right zeros.

The following table displays the number of SI LSLD groupoids with 2, 4 and 6 non-idempotent elements and a respective number of idempotent elements.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 2 | 1 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 0 | 0 | 1 | 2 | 3 | 4 | 2 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 0 | 0 | 0 | 0 | 0 | 0 | 4 | 8 | 4 | 8 | 16 | 8 | 6 | 12 | 6 | 4 | 8 | 4 | 2 | 4 | 2 |

## More non-idempotents.

Lemma 6.3. Let $G$ be an SI LSLD groupoid with $\left|K_{G}\right|=8$. Then $K_{G} / i p_{K_{G}}$ is isomorphic to one of $R_{1}, R_{2}$.

| $R_{1}$ | 0 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 1 | 2 | 3 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 1 | 2 | 3 |
| 3 | 0 | 1 | 2 | 3 |


| $R_{2}$ | 0 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 1 | 3 | 2 |
| 1 | 0 | 1 | 3 | 2 |
| 2 | 1 | 0 | 2 | 3 |
| 3 | 1 | 0 | 2 | 3 |

Proof. For every $u \in K_{G}$, let $t(u)$ be the number of $v \in K_{G}$ such that $u v \in\{v, v v\}$. We have $t(u)=t(x u)$ for every $x \in G$ (because $x y \cdot z=z$ iff $y \cdot x z=x z$ ), hence the set $\left\{u \in K_{G}: t(u)=t\right\}$ is a left ideal of $G$ for every $t$. Consequently, there is $t$ such that $t(u)=t$ for every $u \in K_{G}$ (see Lemma 2.1(1)) and thus all left translations in $R=K_{G} / i p_{K_{G}}$ have the same number $\frac{t}{2}$ of fixed points. Let us denote the elements of $R$ by $0,1,2,3$. Clearly, $\frac{t}{2} \geq 1$ is an even number. If $\frac{t}{2}=4$, then $R$ is the right zero band $R_{1}$. Otherwise $\frac{t}{2}=2$ and we may assume that 0,1 are the only fix points of $L_{0}$, i.e., $L_{0}=(23)$. Then $1 \cdot 0=(0 \cdot 1)(0 \cdot 0)=0(1 \cdot 0)$ (left distributivity) and hence $1 \cdot 0$ is a fix point of $L_{0}$. Therefore $1 \cdot 0=0$ and so $L_{1}=L_{0}$. Now, $L_{2 \cdot 0}=L_{2} L_{0} L_{2}=L_{2} L_{1} L_{2}=L_{2 \cdot 1}$. Since $L_{2}(0), L_{2}(1) \neq 2$ and $L_{0}=L_{1} \neq L_{3}$ (because $\left.L_{0}(3) \neq L_{3}(3)\right)$, we have $\{2 \cdot 0,2 \cdot 1\}=\{0,1\}$. Hence $L_{2}=(01)$, because it has two fixed points. Analogously also $L_{3}=\binom{0}{1}$.

Proposition 6.4. There is no SI idempotent-free LSLD groupoid with 8 elements.

Proof. Since both $R_{1}, R_{2}$ contain proper left ideals, so does any 8 -element SI idempotent-free LSLD groupoid, a contradiction with Lemma 2.1(1).

Lemma 6.5. Let $G$ be an SI LSLD groupoid with $\left|K_{G}\right|=10$. Then $K_{G} / i p_{K_{G}}$ is isomorphic to one of $R_{3}, R_{4}$.

| $R_{3}$ | 0 | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 1 | 2 | 3 | 4 |
| 3 | 0 | 1 | 2 | 3 | 4 |
| 4 | 0 | 1 | 2 | 3 | 4 |


| $R_{4}$ | 0 | 1 | 2 | 3 | 4 |
| :---: | :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 2 | 1 | 4 | 3 |
| 1 | 3 | 1 | 4 | 0 | 2 |
| 2 | 4 | 3 | 2 | 1 | 0 |
| 3 | 2 | 4 | 0 | 3 | 1 |
| 4 | 1 | 0 | 3 | 2 | 4 |

Proof. Proceed similarly as in the proof of Lemma 6.3.

Proposition 6.6. There is no SI idempotent-free LSLD groupoid with 10 elements.

Proof. Assume that $K=\{0, \widetilde{0}, 1, \widetilde{1}, 2, \widetilde{2}, 3, \widetilde{3}, 4, \widetilde{4}\}$ is an idempotent-free LSLD groupoid, where blocks of $i p_{K}$ are the sets $\{k, k\}$ for every $k=0, \ldots, 4$. Then $K / i p_{K} \simeq R_{4}$ and without loss of generality we put $0 \cdot 1=\widetilde{2}, 0 \cdot 3=\widetilde{4}$, $1 \cdot 2=\widetilde{4}, 1 \cdot 0=\widetilde{3}$. Then $\widetilde{1} \cdot \widetilde{0}=3, \widetilde{1} \cdot \widetilde{2}=4$ and thus $2 \cdot 0=\widetilde{4}, 2 \cdot 1=\widetilde{3}$, because $L_{0}$ is an automorphism. Also $3 \cdot 0=\widetilde{2}, 2 \cdot 1=\widetilde{4}, 4 \cdot 0=\widetilde{1}, 4 \cdot 2=\widetilde{3}$, because $L_{2}$ is an automorphism, and the operation on $K$ is determined. We see that $\rho=\{0,1,2,3,4\}^{2} \cup\{\widetilde{0}, \widetilde{1}, \widetilde{2}, \widetilde{3}, \widetilde{4}\}^{2}$ is a congruence on $K$ and $\rho \cap i p_{K}=i d_{K}$. Hence $K$ is not subdirectly irreducible.

Proposition 6.7. The following groupoid is the smallest SI idempotent-free LSLD groupoid with more than two elements.

| $K_{8}$ | 0 | $\widetilde{0}$ | 1 | $\widetilde{1}$ | 2 | $\widetilde{2}$ | 3 | $\widetilde{3}$ | 4 | $\widetilde{4}$ | 5 | $\widetilde{5}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $0, \widetilde{0}$ | $\widetilde{0}$ | 0 | 1 | $\widetilde{1}$ | $\widetilde{4}$ | 4 | $\widetilde{5}$ | 5 | $\widetilde{2}$ | 2 | $\widetilde{3}$ | 3 |
| $1, \widetilde{1}$ | 0 | $\widetilde{0}$ | $\widetilde{1}$ | 1 | $\widetilde{5}$ | 5 | $\widetilde{4}$ | 4 | $\widetilde{3}$ | 3 | $\widetilde{2}$ | 2 |
| $2, \widetilde{2}$ | $\widetilde{4}$ | 4 | $\widetilde{5}$ | 5 | $\widetilde{2}$ | 2 | 3 | $\widetilde{3}$ | $\widetilde{0}$ | 0 | $\widetilde{1}$ | 1 |
| $3, \widetilde{3}$ | 5 | $\widetilde{5}$ | 4 | $\widetilde{4}$ | 2 | $\widetilde{2}$ | $\widetilde{3}$ | 3 | 1 | $\widetilde{1}$ | 0 | $\widetilde{0}$ |
| $4, \widetilde{4}$ | $\widetilde{2}$ | 2 | 3 | $\widetilde{3}$ | $\widetilde{0}$ | 0 | 1 | $\widetilde{1}$ | $\widetilde{4}$ | 4 | 5 | $\widetilde{5}$ |
| $5, \widetilde{5}$ | 3 | $\widetilde{3}$ | $\widetilde{2}$ | 2 | $\widetilde{1}$ | 1 | 0 | $\widetilde{0}$ | 4 | $\widetilde{4}$ | $\widetilde{5}$ | 5 |

Proof. Subdirect irreducibility of $K_{8}$ can be checked easily from the multiplication table and non-existence of a smaller one was proved above.

## 7. The group generated by left translations

In the last section, we find another criterion for recognizing that a groupoid is not SI or pre-SI.

Let $G$ be an LSLD groupoid. We denote $\mathrm{L}(G)$ the subgroup of $\operatorname{Aut}(G)$ generated by all left translations in $G$. For a subset $N$ of $\mathrm{L}(G)$ we define a relation $\rho_{N}$ by $(x, y) \in \rho_{N}$, iff there exists $\varphi \in N$ such that $\varphi(x)=y$.

Lemma 7.1. Let $G$ be an LSLD groupoid and $N$ a normal subgroup of $\mathrm{L}(G)$. Then $\rho_{N}$ is a congruence of $G$.

Proof. Clearly, $\rho_{N}$ is an equivalence on $G$. Let $(x, y) \in \rho_{N}$ and $z \in G$. We have $y z=\varphi(x) z=L_{\varphi(x)} L_{x}(x z)=\varphi L_{x} \varphi^{-1} L_{x}(x z)$, and so $(x z, y z) \in \rho_{N}$ via the automorphism $\varphi L_{x} \varphi^{-1} L_{x} \in N$. Further, $z y=z \varphi(x)=z \varphi(z \cdot z x)=$ $L_{z} \varphi L_{z}(z x)$, and so $(z x, z y) \in \rho_{N}$ via the automorphism $L_{z} \varphi L_{z} \in N$.

Proposition 7.2. Let $G$ be an SI non-idempotent or a pre-SI idempotentfree LSLD groupoid and let $N$ be a non-trivial normal subgroup of $\mathrm{L}(G)$. Then for every $u \in G$ there exists $\varphi \in N$ such that $\varphi(u)=u u$.

Proof. If $G$ is SI non-idempotent, then $i p_{G} \subseteq \rho_{N}$, because $\rho_{N}$ is a nontrivial congruence. If $G$ is pre-SI idempotent-free, one must check (in a view of Theorem 4.3) that $\rho_{N}$ is also $\operatorname{Aut}_{2}(G)$-invariant. If $(x, y) \in \rho_{N}$, $\varphi(x)=y$, and $\psi \in \operatorname{Aut}_{2}(G)$, then $\left(\psi \varphi \psi^{-1}\right)(\psi(x))=\psi \varphi(x)=\psi(y)$, and thus $(\psi(x), \psi(y)) \in \rho_{N}$ via the automorphism $\psi \varphi \psi^{-1} \in N$.

Example. Recall the groupoid $K_{3}$ from the previous section. It is easy to calculate that $\mathrm{L}\left(K_{3}\right)=\{i d,(0 \widetilde{0}),(1 \widetilde{1}),(0 \widetilde{0})(1 \widetilde{1})\}$, and thus $N=\{i d,(0 \widetilde{0})\}$ is a normal subgroup. However, there is no $\varphi \in N$ such that $\varphi(1)=\widetilde{1}$, hence $K_{3}$ is not pre-SI by Proposition 7.2.

Remark. Let $G$ be a simple LSLD groupoid. Then the subgroup of $\mathrm{L}(G)$ generated by all $L_{x} L_{y}, x, y \in G$, is a smallest non-trivial normal subgroup of $\mathrm{L}(G)$ and thus $\mathrm{L}(G)$ is subdirectly irreducible. This is a result of H. Nagao [6] and it can be proved similarly. However, due to Corollary 1.3, it is interesting in the idempotent case only.

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