

PRESOLID VARIETIES OF n -SEMIGROUPS

AVAPA CHANTASARTRASSMEE

The University of the Thai Chamber of Commerce
Department of Mathematics
126/1 Vibhavadee-Rangsit Road
Din Daeng Bangkok 10400, Thailand
e-mail: avapa_a@hotmail.com

AND

JÖRG KOPPITZ

University of Potsdam, Institute of Mathematics
Am Neuen Palais, 14415 Potsdam, Germany
e-mail: koppitz@rz.uni-potsdam.de

Abstract

The class of all M -solid varieties of a given type τ forms a complete sublattice of the lattice $\mathcal{L}(\tau)$ of all varieties of algebras of type τ . This gives a tool for a better description of the lattice $\mathcal{L}(\tau)$ by characterization of complete sublattices. In particular, this was done for varieties of semigroups by L. Polák ([10]) as well as by Denecke and Koppitz ([4], [5]). Denecke and Hounnon characterized M -solid varieties of semirings ([3]) and M -solid varieties of groups were characterized by Koppitz ([9]). In the present paper we will do it for varieties of n -semigroups. An n -semigroup is an algebra of type (n) , where the operation satisfies the $[i, j]$ -associative laws for $1 \leq i < j \leq n$, introduced by Dörtnie ([2]). It is clear that the notion of a 2-semigroup is the same as the notion of a semigroup. Here we will consider the case $n \geq 3$.

Keywords: hypersubstitution, presolid, n -semigroup.

2000 Mathematics Subject Classification: 08B15, 08B25.

1. INTRODUCTION

Let τ be a fixed type of algebras, with fundamental operation symbols f_i of arity n_i , for $i \in I$. A hypersubstitution of type τ is a mapping which associates to every operation symbol f_i an n_i -ary term $\sigma(f_i)$ of type τ . Let $W_\tau(X)$ be the set of all terms of type τ on an alphabet $X := \{x_1, x_2, x_3, \dots\}$. By $W_\tau(X_n)$ ($X_n := \{x_1, \dots, x_n\}$) we denote the set of all n -ary terms, $n \geq 1$. For $1 \leq m, n \in \mathbb{N}$ we define an operation $S_m^n : W_\tau(X_n) \times W_\tau(X_m)^n \rightarrow W_\tau(X_m)$ inductively as follows: For $(t_1, \dots, t_n) \in W_\tau(X_m)^n$ we put:

- (i) $S_m^n(x_i, t_1, \dots, t_n) := t_i$ for $1 \leq i \leq n$;
- (ii) $S_m^n(f_i(s_1, \dots, s_{n_i}), t_1, \dots, t_n) := f_i(S_m^n(s_1, t_1, \dots, t_n), \dots, S_m^n(s_{n_i}, t_1, \dots, t_n))$ for $i \in I, s_1, \dots, s_{n_i} \in W_\tau(X_n)$ where $S_m^n(s_1, t_1, \dots, t_n), \dots, S_m^n(s_{n_i}, t_1, \dots, t_n)$ will be assumed to be already defined.

Any hypersubstitution σ can be uniquely extended to a mapping $\hat{\sigma}$ on $W_\tau(X)$ inductively as follows:

- (i) $\hat{\sigma}[w] := w$ for $w \in X$;
- (ii) $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] := S_m^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$ for $i \in I, t_1, \dots, t_{n_i} \in W_\tau(X_m)$ where $\hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}]$ will be assumed to be already defined.

A binary operation \circ_h can be defined on the set $Hyp(\tau)$ of all hypersubstitutions of type τ , by letting $\sigma_1 \circ_h \sigma_2 = \hat{\sigma}_1 \circ \sigma_2$, where \circ is the usual composition of functions. The set $Hyp(\tau)$ is closed under this associative operation. It also contains an identity element for \circ_h , namely the identity hypersubstitution σ_{id} which maps every f_i to $f_i(x_1, \dots, x_{n_i})$. Thus $Hyp(\tau)$ is a monoid.

Now let M be any submonoid of $Hyp(\tau)$. A variety V is called M -solid if for every $\sigma \in M$ and every identity $u \approx v$ in V , the identity $\hat{\sigma}[u] \approx \hat{\sigma}[v]$ holds in V . When M is the whole monoid $Hyp(\tau)$, an M -solid variety is called a solid variety. Two hypersubstitutions σ_1, σ_2 are said to be V -equivalent if for every operation symbol f_i of type τ , $\sigma_1(f_i) \approx \sigma_2(f_i)$

is an identity in V . In this case we write $\sigma_1 \sim_V \sigma_2$. In [11] it was proved that if $\widehat{\sigma}_1[s] \approx \widehat{\sigma}_1[t]$ is an identity in V for given terms $s, t \in W_\tau(X)$ and $\sigma_1 \sim_V \sigma_2$ then $\widehat{\sigma}_2[s] \approx \widehat{\sigma}_2[t]$ is an identity in V . Therefore, at most one element from each equivalence class of \sim_V is needed to test the M -solidity.

The motivation of studying M -solid varieties comes from following result of Denecke and Reichel in [6]. For each monoid M of $Hyp(\tau)$, the collection of all M -solid varieties of type τ forms a complete lattice, which is a complete sublattice of the lattice $\mathcal{L}(\tau)$ of all varieties of type τ . This lattice $\mathcal{L}(\tau)$ is in general large and complicated, and difficult to study, and the M -solid sublattices give us a way to study at least some of its sublattices. Thus it may be useful to study the monoid $Hyp(\tau)$ and its submonoids M and the corresponding M -solid varieties, both in general and for specific type τ , and the intersection of the lattice of all M -solid varieties with a fixed variety of type τ . For specific types, much work has been done for type $\tau = (2)$, and in particular for varieties of semigroups. L. Polák ([10]) has given a characterization of the lattice of solid semigroup varieties, and various authors have studied M -solid semigroup varieties for various choices of M . Moreover, for type $\tau = (2, 2)$, in [3], all solid varieties of semirings are determined and, for type $\tau = (2, 1, 0)$, J. Koppitz ([9]) determined M -solid varieties of groups. More informations about hypersubstitutions, one can find in [8].

Our goal in this paper is a similar investigation for type (n) , for $n \geq 3$. Only a few solid varieties of type (n) have been known (see [1] and [7]). We will consider the concept of an n -semigroup, which is a natural extension of the concept of a semigroup. An n -semigroup is an algebra of type (n) , where the n -ary operation satisfies the $[i, j]$ -associative laws

$$\begin{aligned} x_1 \dots x_{i-1} (x_i \dots x_{i+n-1}) x_{i+n} \dots x_{2n-1} &\approx \\ x_1 \dots x_{j-1} (x_j \dots x_{j+n-1}) x_{j+n} \dots x_{2n-1}, &\text{ for } 1 \leq i < j \leq n. \end{aligned}$$

Each n -group is an n -semigroup (see Dörnte [2]). Each semigroup $(S; \cdot)$ induce an n -semigroup in the following way: Let $f_n : S^n \rightarrow S$ be defined by $f_n(a_1, a_2, \dots, a_n) := a_1 \cdot a_2 \cdot \dots \cdot a_n$ (we use the binary operation \cdot of the given semigroup). Since \cdot is associative, f_n satisfies the $[i, j]$ -associative laws for $1 \leq i < j \leq n$, i.e., $(S; f_n)$ is an n -semigroup. Clearly, in the case $n = 2$ we have the $[1, 2]$ -associative law $(x_1 x_2) x_3 \approx x_1 (x_2 x_3)$. So the notion of a 2-semigroup is the same as the notion of a semigroup.

We also introduce the monoids $NPer(n)$ and $Pre(n)$ and give a characterization of all $NPer(n)$ -solid as well as all $Pre(n)$ -solid varieties of semigroups.

2. HYPERSUBSTITUTIONS OF TYPE (n)

In this section we present some background information about hypersubstitutions and varieties of type (n) , and introduce the special monoids we shall be studying. We assume throughout a fixed type (n) , with $n \geq 3$, so we have one n -ary operation symbol which we shall denote by f . For Σ any set of identities of type (n) , we will denote by $Mod(\Sigma)$ the variety determined by the set Σ and by IdV we denote the set of all identities which hold in a given variety V . Because of the $[i, j]$ -associative laws, $1 \leq i < j \leq n$, a term over a variety of n -semigroups can be regarded as a word of the length $(n-1)r+1$ for a suitable natural number r . By $l(t)$ we denote the length of a given term $t \in W_{(n)}(X)$ and $var(t)$ means the set of variables occurring in t . By $cv(t)$ we mean the cardinality of $var(t)$. For example, if $t = f(x_1, \dots, x_1)$ then $l(t) = n$, $var(t) = \{x_1\}$, and $cv(t) = 1$. An identity $u \approx v$ is said to be normal if $u = v$ or both terms u and v are different from a variable. Since any hypersubstitution σ in $Hyp(n)$ is completely determined by what it does to f , we will denote by σ_t the hypersubstitution which maps f to the term t . For convenience, we list here some sets of terms and varieties of type (n) that we shall discuss later:

$W_{(n)}^{np}(X_n)$ be the set of all $t \in W_{(n)}(X_n)$ containing a subword s with $n = l(s) > cv(s)$;

$\widetilde{W}_{(n)}^{np}(X) := \{t \in W_{(n)}(X) \mid l(t) > cv(t)\}$;

$\widetilde{V}_n := Mod\{x_1 \dots x_{2n-1} \approx x_1 \dots x_{i-1} x_{i+1} x_{i+2} x_i x_{i+3} \dots x_{2n-1} \mid 1 \leq i \leq 2n-3\}$;

$\widetilde{W}_n := Mod\{t \approx x^n \mid t \in W_{(n)}(X_n), n = l(t) > cv(t)\}$;

$V_n := \widetilde{V}_n \cap \widetilde{W}_n$.

It is easy to verify that there is no nontrivial solid variety of n -semigroups.

Theorem 1. *For each natural number $n \geq 3$ there is not nontrivial solid variety of n -semigroups.*

Proof. Let V be a solid variety of n -semigroups. Then $\widehat{\sigma}_{x_2}[(x_1 \dots x_n) x_{n+1} \dots x_{2n-1}] \approx \widehat{\sigma}_{x_2}[x_1 \dots x_{n-1} (x_n \dots x_{2n-1})] \in IdV$, i.e., $x_{n+1} \approx x_2 \in IdV$ and V is the trivial variety of type (n) . ■

A hypersubstitution σ is called a pre-hypersubstitution if $\sigma(f)$ is not a variable. The set $Pre(n)$ of all pre-hypersubstitutions forms a submonoid of the monoid $Hyp(n)$ of all hypersubstitutions of type (n) . A variety of n -semigroups is called presolid if it is M -solid for $M = Pre(n)$. Note that any solid variety is also presolid. By S_n we will denote the set of all bijections on the set $\{1, \dots, n\}$. For $\pi \in S_n$, the hypersubstitution σ with $\sigma(f) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$ will be denoted by σ_π . We will use the following notations of sets of hypersubstitutions:

$Pre(n) := Hyp(n) \setminus \{\sigma_{x_i} \mid 1 \leq i \leq n\}$ the set of all pre-hypersubstitutions;

$Per(n) := \{\sigma_\pi \mid \pi \in S_n\}$;

$Nper(n) := \{\sigma_t \mid t \in W_{(n)}^{np}(X_n)\} \cup \{\sigma_{id}\}$.

Proposition 2. *For $2 \leq n \in \mathbb{N}$, $Nper(n)$ forms a monoid.*

Proof. We have to check that $\sigma_1 \circ_h \sigma_2 \in Nper(n)$ for any $\sigma_1, \sigma_2 \in Nper(n)$. For this let $\sigma_1, \sigma_2 \in Nper(n)$. Then there are $r, t \in W_{(n)}^{np}(X_n)$ such that $\sigma_1(f) = r$ and $\sigma_2(f) = t$. In particular, r contains a subword s with $n = l(s) > cv(s)$. Further, $\hat{\sigma}_1[t]$ contains a subterm $S_n^n(r, x_{i_1}, \dots, x_{i_n})$. Since r contains a subword s with $n = l(s) > cv(s)$, the term $S_n^n(r, x_{i_1}, \dots, x_{i_n})$ contains a subword \tilde{s} with $n = l(\tilde{s}) > cv(\tilde{s})$. Consequently, $\hat{\sigma}_1[t]$ contains the subword \tilde{s} with $n = l(\tilde{s}) > cv(\tilde{s})$, i.e., $\sigma_1 \circ_h \sigma_2(f) = \hat{\sigma}_1[t] \in W_{(n)}^{np}(X_n)$ and thus $\sigma_1 \circ_h \sigma_2 \in Nper(n)$. ■

3. PRESOLID VARIETIES OF n -SEMIGROUPS

We begin the investigations of presolid varieties of n -semigroups by looking for a variety that contains all presolid varieties.

Proposition 3. *Let $3 \leq n \in \mathbb{N}$ and V be any $Pre(n)$ -solid variety of n -semigroups. Then $V \subseteq \tilde{V}_n$.*

Proof. Let $\pi \in S_n$ with $\pi(1) = 2$, $\pi(2) = 1$ and $\pi(k) = k$ for $3 \leq k \leq n$. If we apply σ_π to the $[1, n]$ -associative law we get $x_{n+1}x_2x_1x_3 \dots x_nx_{n+2} \dots x_{2n-1} \approx x_2x_1x_3 \dots x_{n+1}x_nx_{n+2}x_{n+3} \dots x_{2n-1} \in IdV$ since V is $Pre(n)$ -solid. By suitable substitution we get $x_1 \dots x_{2n-1} \approx x_2 \dots x_nx_1x_{n+1} \dots x_{2n-1} \in IdV$. If $n \geq 4$ then the application of σ_π to the $[3, 4]$ -associative law gives $x_2x_1x_4x_3x_5 \dots x_{2n-1} \approx x_2x_1x_3x_5x_4x_6 \dots x_{2n-1} \in IdV$.

Both identities together provide $x_1 \dots x_{2n-1} \approx x_1 \dots x_{i-1} x_{i+1} x_{i+2} x_i x_{i+3} \dots x_{2n-1} \in IdV$ for $1 \leq i \leq n-2$. Let $\rho \in S_n$ with $\rho(2n-1) = 2n-2$, $\rho(2n-2) = 2n-1$ and $\rho(k) = k$ for $1 \leq k \leq 2n-3$. Dually, then the application of σ_ρ to the $[1, n]$ -associative law as well as to the $[n-3, n-2]$ -associative law (if $n \geq 4$) provides identities from which we can derive $x_1 \dots x_{2n-1} \approx x_1 \dots x_{i-1} x_{i+1} x_{i+2} x_i x_{i+3} \dots x_{2n-1} \in IdV$ for $n \leq i \leq 2n-3$. Finally, we have

$$\begin{aligned}
& x_1 \dots x_{2n-1} \\
& \approx x_1 \dots x_{n-1} x_{n+1} x_{n+2} x_n x_{n+3} \dots x_{2n-1} \\
& \approx x_1 \dots x_{n+1} x_{n-2} x_{n-1} x_{n+2} x_n x_{n+3} \dots x_{2n-1} \\
& \approx x_1 \dots x_{n+1} x_{n-2} x_n x_{n-1} x_{n+2} x_{n+3} \dots x_{2n-1} \\
& \approx x_1 \dots x_{n-2} x_n x_{n+1} x_{n-1} x_{n+2} x_{n+3} \dots x_{2n-1}, \text{ i.e.,} \\
& x_1 \dots x_{2n-1} \approx x_1 \dots x_{n-2} x_n x_{n+1} x_{n-1} x_{n+2} x_{n+3} \dots x_{2n-1} \in IdV.
\end{aligned}$$

Altogether we have $x_1 \dots x_{2n-1} \approx x_1 \dots x_{i-1} x_{i+1} x_{i+2} x_i x_{i+3} \dots x_{2n-1} \in IdV$ for $1 \leq i \leq 2n-3$. ■

Now we will determine identities satisfying by presolid varieties.

Lemma 4. *Let $4 \leq n \in 2\mathbb{N}$ and V be any $Pre(n)$ -solid variety of n -semigroups. Then $x_1 \dots x_{2n-1} \approx x_{\pi(1)} \dots x_{\pi(2n-1)}$ for all $\pi \in S_{2n-1}$.*

Proof. Let $\pi \in S_{2n-1}$ with $\pi(1) = 2$, $\pi(2) = 1$ and $\pi(k) = k$ for $3 \leq k \leq 2n-1$. If we apply σ_π to the $[1, n]$ -associative law we get $x_{n+1} x_2 x_1 x_3 \dots x_n x_{n+2} \dots x_{2n-1} \approx x_2 x_1 x_3 \dots x_{n+1} x_n x_{n+2} \dots x_{2n-1} \in IdV$ since V is $Pre(n)$ -solid and by suitable substitution we obtain

$$(1) \quad x_1 \dots x_{2n-1} \approx x_2 \dots x_n x_1 x_{n+1} \dots x_{2n-1} \in IdV.$$

By Proposition 3 we have $V \subseteq \tilde{V}_n$. Using the identities of \tilde{V}_n we get $x_2 \dots x_n x_1 x_{n+1} \dots x_{2n-1} \approx x_2 x_1 x_3 \dots x_{2n-1} \in IdV$ (since n is a even number). Together with (1) we obtain $x_1 \dots x_{2n-1} \approx x_2 x_1 x_3 \dots x_{2n-1} \in IdV$. It is easy to see that one can derive $x_1 \dots x_{2n-1} \approx x_{\pi(1)} \dots x_{\pi(2n-1)}$ for all $\pi \in S_{2n-1}$ from $x_1 \dots x_{2n-1} \approx x_2 x_1 x_3 \dots x_{2n-1}$ and the identities of \tilde{V}_n . ■

Lemma 5. *Let $3 \leq n \in \mathbb{N}$, $2n - 1 \leq p \in (n - 1)\mathbb{N} + 1$ and V be a variety of n -semigroups with $V \subseteq \widetilde{V}_n$. Then for each $\pi \in S_p$ holds*

$$x_{\pi(1)} \dots x_{\pi(p)} \approx x_1 \dots x_p \in IdV \text{ or}$$

$$x_{\pi(1)} \dots x_{\pi(p)} \approx x_2 x_1 x_3 \dots x_p \in IdV.$$

Proof. Let $\pi \in S_p$. We consider the term $x_{\pi(1)} \dots x_{\pi(p)}$ and move step by step x_p, x_{p-1}, \dots, x_3 to the $p^{th}, (p-1)^{th}, \dots, 3^{th}$ position using the identities of \widetilde{V}_n . Then we have on the first both positions $x_1 x_2$ or $x_2 x_1$. This shows $x_{\pi(1)} \dots x_{\pi(p)} \approx x_1 \dots x_p \in IdV$ or $x_{\pi(1)} \dots x_{\pi(p)} \approx x_2 x_1 x_3 \dots x_p \in IdV$. ■

It is easy to check that $Nper(n) \subseteq Pre(n)$. So, any presolid variety has to be $Nper(n)$ -solid. Next we find the lattice of all $Nper(n)$ -solid varieties of n -semigroups.

Lemma 6. *Let $3 \leq n \in \mathbb{N}$ and V be any variety of n -semigroups with $V \subseteq \widetilde{V}_n$. Then for each $t \in \widetilde{W}_{(n)}^{np}(X)$ holds $t \approx z^n \in IdV$.*

Proof. Let $t \in \widetilde{W}_{(n)}^{np}(X)$. Then there is a variable $w \in X$ that occurs at least two times in t . If $l(t) = n$ then $l(t) > cv(t)$ and $t \approx x^n \in IdV$ since $V \subseteq \widetilde{W}_n$. Suppose now that $l(t) > n$. Using the identities of \widetilde{V}_n we can move w on the first and the second position, respectively, i.e., $t \approx ww u_3 \dots u_{l(t)}$ with $u_3, \dots, u_{l(t)} \in X$. Since $x_1 x_1 x_3 \dots x_n \approx z^n \in IdV$ we have $ww u_3 \dots u_{n-1} (u_n \dots u_{l(t)}) \approx z^n \in IdV$, i.e., $t \approx z^n \in IdV$. ■

Lemma 7. *Let $3 \leq n \in \mathbb{N}$ and V be any nontrivial variety of n -semigroups with $V \subseteq \widetilde{W}_n$. Then only normal identities hold in V .*

Proof. Assume that a non-normal identity $u \approx v$ holds in V . Then $u \neq v$ and one of the terms u and v is a variable. Without loss of generality let u be a variable. Since V is a nontrivial variety the term v ($\neq u$) is not a variable. Then by substitution we get $y \approx y^{l(v)} \in IdV$ where $l(v) > 1$. Clearly, $l(v) = r(n-1) + 1$ for some natural number $r \geq 1$. From $xy^{n-1} \approx z^n \in IdV$ it follows $y^{r(n-1)+1} \approx z^n \in IdV$, i.e., $y^{l(v)} \approx z^n \in IdV$. But $y \approx y^{l(v)}$ and $y^{l(v)} \approx z^n$ provide $x \approx y$, and V is the trivial variety, a contradiction. ■

Proposition 8. *Let $3 \leq n \in \mathbb{N}$. A nontrivial variety V of n -semigroups is $Nper(n)$ -solid iff $V \subseteq \widetilde{W}_n$.*

Proof. Assume that V is $Nper(n)$ -solid. We have $t_1 := x_1x_2^{n-1} \in W_{(n)}^{np}(X_n)$, i.e., $\sigma_{t_1} \in Nper(n)$ and its application to the $[1, 3]$ -associative law gives

$$(1) \quad x_1x_2^{n-1}x_{n+1}^{n-1} \approx x_1x_2^{n-1} \in IdV.$$

Further, we have $t_2 := x_2x_3^{n-1} \in W_{(n)}^{np}(X_n)$, i.e., $\sigma_{t_2} \in Nper(n)$ and its application to the $[1, 2]$ -associative law gives

$$(2) \quad x_{n+1}x_{n+2}^{n-1} \approx x_3x_4^{n-1}x_{n+2}^{n-1} \in IdV.$$

Then one obtains $xy^{n-1} \stackrel{(1)}{\approx} xy^{n-1}z^{n-1} \stackrel{(2)}{\approx} wz^{n-1} \in IdV$, i.e., we have $xy^{n-1} \approx z^n \in IdV$. Dually, we can show that $x^{n-1}y \approx z^n \in IdV$. Let now $t \in W_{(n)}(X_n)$ with $n = l(t) > cv(t)$. Then there are $u_1, \dots, u_n \in X$ such that $t = u_1 \dots u_n$. Since $l(t) > cv(t)$ there are $i, j \in \{1, \dots, n\}$ with $i < j$ such that $u_i = u_j$. Then the term $s := x_1 \dots x_{j-1}x_ix_{j+1} \dots x_n$ belongs to $W_{(n)}^{np}(X_n)$, i.e., $\sigma_s \in Nper(n)$. Without loss of generality let $i \neq 1$. Then the application of σ_s to the $[1, j]$ -associative law gives $x_1 \dots x_{j-1}x_ix_{j+1} \dots x_n x_{n+1} \dots x_{n+j-2}x_{n+i-1}x_{n+j} \dots x_{2n-1} \approx x_1 \dots x_{j-1}x_ix_{n+j} \dots x_{2n-1}$. Then $x_{n+1} \notin \{x_1, \dots, x_{j-1}, x_i, x_{n+j}, \dots, x_{2n-1}\}$ since $1 < i < j \neq 1$. So, we substitute x_{n+1} by x_{n+1}^n and get $x_1 \dots x_{j-1}x_ix_{n+j} \dots x_{2n-1} \approx x_1 \dots x_{j-1}x_ix_{j+1} \dots x_n x_{n+1}^n \dots x_{n+j-2}x_{n+i-1}x_{n+j} \dots x_{2n-1}$. It is easy to check that one can derive $x_1 \dots x_{j-1}x_ix_{j+1} \dots x_n x_{n+1}^n \dots x_{n+j-2}x_{n+i-1}x_{n+j} \dots x_{2n-1} \approx z^n$ using $xy^{n-1} \approx x^{n-1}y \approx z^n \in IdV$, i.e., one gets $x_1 \dots x_{j-1}x_ix_{n+j} \dots x_{2n-1} \approx z^n \in IdV$. Consequently, if we substitute x_i by u_i for $1 \leq i \leq n$ we get $u_1 \dots u_n \approx z^n \in IdV$, i.e., $t \approx z^n \in IdV$. Altogether, $V \subseteq \widetilde{W}_n$.

Suppose now that $V \subseteq \widetilde{W}_n$. Let $t \in W_{(n)}^{np}(X_n)$. Then t contains a subterm s with $n = l(s) > cv(s)$ and there are words u and v (the empty word λ is also possible for u as well as for v) such that $t = usv$. Since $s \approx z^n \in IdV$ we have $t \approx uz^n v \in IdV$. The repeated application of $xy^{n-1} \approx x^{n-1}y \approx z^n \in IdV$ to $uz^n v$ gives finally $uz^n v \approx z^n$, i.e., $t \approx z^n \in IdV$. This shows that any $\sigma \in Nper(n)$ is V -equivalent to $\sigma_{x_1^n}$.

Let $u \approx v \in IdV$. If $u = v$ then clearly $\widehat{\sigma}_{x_1^n}[u] \approx \widehat{\sigma}_{x_1^n}[v] \in IdV$. If $u \neq v$ and $u \approx v$ is a normal identity of V then there are natural numbers $r, s \geq 1$ such that $\widehat{\sigma}_{x_1^n}[u] \approx u_1^{n^r}$ and $\widehat{\sigma}_{x_1^n}[v] \approx v_1^{n^s}$ where u_1 (v_1) is the first letter in u (in v). From $xy^{n-1} \approx z^n \in IdV$ it follows $x^n \approx z^n \in IdV$ and thus $u_1^{n^r} \approx v_1^{n^s} \in IdV$, i.e., $\widehat{\sigma}_{x_1^n}[u] \approx \widehat{\sigma}_{x_1^n}[v] \in IdV$. Since only normal identities are satisfied in V by Lemma 7 we can conclude that V is $Nper(n)$ -solid. ■

After the following lemma we are able to characterize all presolid varieties of n -semigroups.

Lemma 9. *Let $3 \leq n \in 2\mathbb{N} + 1$, V be a variety of n -semigroups with $V \subseteq \tilde{V}_n$, and $\sigma \in \text{Per}(n)$. Then there holds*

$$\hat{\sigma}[x_1 \dots x_i (x_{i+1} \dots x_{i+n}) x_{i+n+1} \dots x_{2n-1}] \approx x_1 \dots x_{2n-1} \in \text{Id}V$$

for $0 \leq i \leq n - 1$.

Proof. Let $\pi \in S_n$. Without loss of generality let $i = 0$. Then

(1) $x_{\pi(1)} \dots x_{\pi(n)} x_{n+1} \dots x_{2n-1} \approx x_1 \dots x_{2n-1} \in \text{Id}V$ or

(2) $x_{\pi(1)} \dots x_{\pi(n)} x_{n+1} \dots x_{2n-1} \approx x_2 x_1 x_3 \dots x_{2n-1} \in \text{Id}V$ by Lemma 5. We put $y_1 := x_1 \dots x_n$ in case (1) ($y_1 := x_2 x_1 x_3 \dots x_n$ in case (2)) and $y_j := x_{n+j-1}$ for $2 \leq j \leq n$. Using the identities of \tilde{V}_n it is easy to check that $y_{\pi(1)} \dots y_{\pi(n)} \approx x_1 \dots x_{2n-1} \in \text{Id}V$ in case (1) and $y_{\pi(1)} \dots y_{\pi(n)} \approx x_{n+1} x_2 x_1 x_3 \dots x_n x_{n+2} \dots x_{2n-1} \in \text{Id}V$ in case (2), respectively. Further, we have $x_{n+1} x_2 x_1 x_3 \dots x_n x_{n+2} \dots x_{2n-1} \approx x_1 x_{n+1} x_2 x_3 \dots x_n x_{n+2} \dots x_{2n-1} \approx x_1 x_2 x_3 \dots x_n x_{n+1} x_{n+2} \dots x_{2n-1} \in \text{Id}V$ (since n is an odd number). This shows that $\hat{\sigma}_\pi[(x_1 \dots x_n) x_{n+1} \dots x_{2n-1}] \approx S_{2n-1}^n(\sigma_\pi(f), S_{2n-1}^n(\sigma_\pi(f), x_1, \dots, x_n), x_{n+1}, \dots, x_{2n-1}) \approx x_1 \dots x_{2n-1} \in \text{Id}V$. ■

Theorem 10. *Let $n \geq 3$ be a natural number and V be a nontrivial variety of n -semigroups. Then V is $\text{Pre}(n)$ -solid iff the following statements hold:*

- (i) $V \subseteq V_n$;
- (ii) If $x_{\pi(1)} \dots x_{\pi(n)} \approx x_1 \dots x_n \in \text{Id}V$ for some $\pi \in S_n$ then $x_{\pi \circ s(1)} \dots x_{\pi \circ s(n)} \approx x_{s(1)} \dots x_{s(n)} \in \text{Id}V$ for all $s \in S_n$;
- (iii) If $n \in 2\mathbb{N}$ then $x_1 \dots x_{2n-1} \approx x_{\pi(1)} \dots x_{\pi(2n-1)}$ for all $\pi \in S_{2n-1}$.

Proof. Suppose that V is $\text{Pre}(n)$ -solid. Then $V \subseteq \tilde{V}_n$ by Proposition 3. Further, V is $N\text{per}(n)$ -solid since $N\text{per}(n) \subseteq \text{Pre}(n)$. Then by Proposition 8 we get $V \subseteq \tilde{W}_n$. Therefore, $V \subseteq \tilde{V}_n \cap \tilde{W}_n = V_n$ and it holds (i). Suppose that $x_{\pi(1)} \dots x_{\pi(n)} \approx x_1 \dots x_n \in \text{Id}V$ for some $\pi \in S_n$. Further let $\rho \in S_n$. Then $\sigma_\rho \in \text{Pre}(n)$. Since V is $\text{Pre}(n)$ -solid we have $\hat{\sigma}_\rho[x_1 \dots x_n] \approx \hat{\sigma}_\rho[x_{\pi(1)} \dots x_{\pi(n)}] \in \text{Id}V$, i.e., $x_{\pi \circ \rho(1)} \dots x_{\pi \circ \rho(n)} \approx x_{\rho(1)} \dots x_{\rho(n)} \in \text{Id}V$. This shows (ii). Finally, (iii) it follows from Lemma 4.

Suppose that (i)–(iii) are satisfied. Let $\sigma_t \in \text{Pre}(n)$. If $\sigma_t \notin \text{Per}(n)$ then $t \in \widetilde{W}_{(n)}^{np}(X)$ and $t \approx z^n \in \text{IdV}$ by Lemma 6, i.e., σ_t is V -equivalent to $\sigma_{x_1^n}$, where $\sigma_{x_1^n} \in \text{Nper}(n)$. But (i) implies that V is $\text{Nper}(n)$ -solid by Proposition 8. Thus $\widehat{\sigma}_{x_1^n}[u] \approx \widehat{\sigma}_{x_1^n}[v] \in \text{IdV}$ for all $u \approx v \in \text{IdV}$, i.e., $\widehat{\sigma}_t[u] \approx \widehat{\sigma}_t[v] \in \text{IdV}$ for all $u \approx v \in \text{IdV}$. Let now $\sigma_t \in \text{Per}(n)$ and $u \approx v \in \text{IdV}$. If $\text{var}(u) \neq \text{var}(v)$ then without loss of generality there is a $w \in \text{var}(u) \setminus \text{var}(v)$. We substitute w by x^n and get $\tilde{u} \approx v \in \text{IdV}$ from $u \approx v \in \text{IdV}$ where x^n is a subterm of \tilde{u} , i.e., $\tilde{u} \in \widetilde{W}_{(n)}^{np}(X)$. Then by Lemma 6 we have $\tilde{u} \approx x^n \in \text{IdV}$, i.e., $u \approx v \approx x^n \in \text{IdV}$. If $l(u) > cv(u)$ or $l(v) > cv(v)$ then $u \in \widetilde{W}_{(n)}^{np}(X)$ or $v \in \widetilde{W}_{(n)}^{np}(X)$ and thus $u \approx v \approx x^n \in \text{IdV}$ by Lemma 6. Consequently, if $\text{var}(u) \neq \text{var}(v)$ or $l(u) > cv(u)$ or $l(v) > cv(v)$ then $u \approx v \approx x^n \in \text{IdV}$. If, in particular, $l(u) = cv(u)$ then $u = u_1 \dots u_{l(u)}$ with $u_1, \dots, u_{l(u)} \in X$ and there is a $\pi \in S_{l(u)}$ such that $\widehat{\sigma}_t[u] \approx u_{\pi(1)} \dots u_{\pi(l(u))}$. But from $u \approx x^n \in \text{IdV}$ we get by the substitution $u_i \mapsto u_{\pi(i)}$ for $1 \leq i \leq l(u)$ that $u_{\pi(1)} \dots u_{\pi(l(u))} \approx x^n \in \text{IdV}$, i.e., $\widehat{\sigma}_t[u] \approx x^n \in \text{IdV}$. If, in particular, $l(v) = cv(v)$ then we get $\widehat{\sigma}_t[v] \approx x^n \in \text{IdV}$ in the same matter. If $l(u) > cv(u)$ ($l(v) > cv(v)$) then $u \in \widetilde{W}_{(n)}^{np}(X)$ ($v \in \widetilde{W}_{(n)}^{np}(X)$) and it is easy to check that $\widehat{\sigma}_t[u] \in \widetilde{W}_{(n)}^{np}(X)$ ($\widehat{\sigma}_t[v] \in \widetilde{W}_{(n)}^{np}(X)$), too. Then $\widehat{\sigma}_t[u] \approx x^n \in \text{IdV}$ ($\widehat{\sigma}_t[v] \approx x^n \in \text{IdV}$) by Lemma 6. Consequently, $\widehat{\sigma}_t[u] \approx x^n \approx \widehat{\sigma}_t[v] \in \text{IdV}$. The remaining case is $\text{var}(u) = \text{var}(v)$ and $l(u) = cv(u)$ and $l(v) = cv(v)$. We put $s := l(u)$ and $\{u_1, \dots, u_s\} = \text{var}(u) = \text{var}(v)$. Because of Lemma 9 (if $n \in 2\mathbb{N} + 1$) and of (iii) (if $n \in 2\mathbb{N}$), respectively, we have $\widehat{\sigma}_t[x_1 \dots x_{i-1}(x_i \dots x_{i+n-1})x_{i+n} \dots x_{2n-1}] \approx \widehat{\sigma}_t[x_1 \dots x_{j-1}(x_j \dots x_{j+n-1})x_{j+n} \dots x_{2n-1}] \in \text{IdV}$ for $1 \leq i < j \leq n$. Therefore we can assume that

$$u = (\dots (u_1 \dots u_n)u_{n+1} \dots u_{2n-1}) \dots u_{s-1}u_s)$$

$$v = (\dots (u_{\pi(1)} \dots u_{\pi(n)})u_{\pi(n+1)} \dots u_{\pi(2n-1)}) \dots u_{\pi(s-1)}u_{\pi(s)})$$

for some permutation $\pi \in S_s$. Further there is a $\rho \in S_n$ such that $\sigma_t = \sigma_\rho$. If $s = 1$ we have obviously $\widehat{\sigma}_\rho[u] \approx \widehat{\sigma}_\rho[v] \in \text{IdV}$. If $s = n$ then $\widehat{\sigma}_\rho[u] \approx u_{\rho(1)} \dots u_{\rho(n)}$ and $\widehat{\sigma}_\rho[v] \approx u_{\pi \circ \rho(1)} \dots u_{\pi \circ \rho(n)}$. By (ii) from $x_{\pi(1)} \dots x_{\pi(n)} \approx x_1 \dots x_n \in \text{IdV}$ it follows $x_{\pi \circ \rho(1)} \dots x_{\pi \circ \rho(n)} \approx x_{\rho(1)} \dots x_{\rho(n)} \in \text{IdV}$, i.e., $\widehat{\sigma}_\rho[u] \approx \widehat{\sigma}_\rho[v] \in \text{IdV}$. Let now $s > n$. Then there is a $\phi \in S_s$ such that $\widehat{\sigma}_t[u] \approx u_{\phi(1)} \dots u_{\phi(s)}$ and $\widehat{\sigma}_t[v] \approx u_{\pi \circ \phi(1)} \dots u_{\pi \circ \phi(s)}$.

By Lemma 5 we have $\hat{\sigma}_t[u] \approx u_1 \dots u_s$ or $\hat{\sigma}_t[u] \approx u_2 u_1 u_3 \dots u_s =: \tilde{u}$. If $\hat{\sigma}_t[u] \approx u$, i.e., $x_{\phi(1)} \dots x_{\phi(s)} \approx u_1 \dots u_s \in IdV$ then by the substitution $u_i \mapsto u_{\pi(i)}$ for $1 \leq i \leq s$ we get $u_{\pi \circ \phi(1)} \dots u_{\pi \circ \phi(s)} \approx u_{\pi(1)} \dots u_{\pi(s)} \in IdV$, i.e., $\hat{\sigma}_t[v] \approx v$, and from $u \approx v \in IdV$ it follows $\hat{\sigma}_t[u] \approx \hat{\sigma}_t[v] \in IdV$. If $\hat{\sigma}_t[u] \approx \tilde{u}$, i.e., $u_{\phi(1)} \dots u_{\phi(s)} \approx u_2 u_1 u_3 \dots u_s$ then by the same substitution we get $u_{\pi \circ \phi(1)} \dots u_{\pi \circ \phi(s)} \approx u_{\pi(2)} u_{\pi(1)} u_{\pi(3)} \dots u_{\pi(s)} =: \tilde{v}$, i.e., $\hat{\sigma}_t[v] \approx \tilde{v} \in IdV$. Moreover, from Lemma 5 we get

$$u_{\pi(2)} u_{\pi(1)} u_{\pi(3)} \dots u_{\pi(s)} \approx u_1 \dots u_s \quad \text{or}$$

$$u_{\pi(2)} u_{\pi(1)} u_{\pi(3)} \dots u_{\pi(s)} \approx u_2 u_1 u_3 \dots u_s$$

as well as

$$u_{\pi^{-1}(2)} u_{\pi^{-1}(1)} u_{\pi^{-1}(3)} \dots u_{\pi^{-1}(s)} \approx u_1 \dots u_s \quad \text{or}$$

$$u_{\pi^{-1}(2)} u_{\pi^{-1}(1)} u_{\pi^{-1}(3)} \dots u_{\pi^{-1}(s)} \approx u_2 u_1 u_3 \dots u_s.$$

i.e.,

$$u_2 u_1 u_3 \dots u_s \approx u_{\pi(1)} \dots u_{\pi(s)} \quad \text{or}$$

$$u_2 u_1 u_3 \dots u_s \approx u_{\pi(2)} u_{\pi(1)} u_{\pi(3)} \dots u_{\pi(s)}.$$

This shows $\tilde{v} \approx u$ or $\tilde{v} \approx \tilde{u}$ as well as $\tilde{u} \approx v$ or $\tilde{u} \approx \tilde{v}$. This implies $\tilde{v} \approx \tilde{u}$ or both $\tilde{v} \approx u$ and $\tilde{u} \approx v$ hold in V . Since $u \approx v \in IdV$ we have altogether $\tilde{v} \approx \tilde{u} \in IdV$ and thus $\hat{\sigma}_t[u] \approx \hat{\sigma}_t[v] \in IdV$ because of $\hat{\sigma}_t[u] \approx \tilde{u} \in IdV$ and $\hat{\sigma}_t[v] \approx \tilde{v} \in IdV$. \blacksquare

Let us apply Theorem 10 for the case $n = 3$. We obtain the following characterization of all presolid varieties of 3-semigroups.

Corollary 11. *A nontrivial variety of 3-semigroups is Pre(3)-solid iff $V \subseteq Mod\{(xyz)wt \approx x(yzw)t \approx xy(zwt) \approx yzxwt \approx xzwyzt \approx xywtz, xyx \approx x^2y \approx xy^2 \approx z^3\} =: W$ and it holds the following condition:*

(*) *If $x_1 x_2 x_3 \approx x_{\pi(1)} x_{\pi(2)} x_{\pi(3)} \in IdV$ for some $\pi \in \{(12), (13), (23)\}$*

then $x_1 x_2 x_3 \approx x_{\rho(1)} x_{\rho(2)} x_{\rho(3)} \in IdV$ for all $\rho \in S_3$.

Proof. Suppose that V is $Pre(3)$ -solid. Then the conditions (i) and (ii) of Theorem 10 are satisfied. From (i) it follows that $xyzwt \approx yzxwt \approx xzwy t \approx xywtz \in IdV$ and $xyx \approx x^2y \approx xy^2 \approx z^3 \in IdV$. Hence $V \subseteq W$. Using (ii) we can verify condition (*): If $\pi = (13)$, i.e., $x_1x_2x_3 \approx x_3x_2x_1 \in IdV$ then $x_2x_1x_3 \approx x_2x_3x_1 \in IdV$ (for $s = (12)$). Both identities provide $x_1x_2x_3 \approx x_1x_3x_2 \approx x_2x_3x_1 \approx x_2x_1x_3 \approx x_2x_3x_1 \approx x_1x_3x_2 \in IdV$. If $\pi = (12)$, i.e., $x_1x_2x_3 \approx x_2x_1x_3 \in IdV$ then $x_1x_3x_2 \approx x_2x_3x_1 \in IdV$ (for $s = (23)$). If $\pi = (23)$, i.e., $x_1x_2x_3 \approx x_1x_3x_2 \in IdV$ then $x_2x_1x_3 \approx x_3x_1x_2 \in IdV$ (for $s = (12)$). In the latter two cases, we conclude in the same matter as before.

Suppose now that $V \subseteq W$ and (*) is satisfied. Since $V \subseteq W$, the condition (i) of Theorem 10 holds. We have now to show that also condition (ii) is satisfied. For this let $\pi \in S_3$. If $\pi \in \{(1), (12), (13), (23)\}$ then the condition is satisfied by (*). If $\pi = (123)$, i.e., $x_1x_2x_3 \approx x_2x_3x_1 \in IdV$ then we have to check that also $x_2x_1x_3 \approx x_3x_2x_1 \in IdV$, $x_3x_2x_1 \approx x_1x_3x_2 \in IdV$, $x_1x_3x_2 \approx x_2x_1x_3 \in IdV$, $x_2x_3x_1 \approx x_3x_1x_2 \in IdV$, and $x_3x_1x_2 \approx x_1x_2x_3 \in IdV$. Obviously, these five equations are consequences of the given identity $x_1x_2x_3 \approx x_2x_3x_1 \in IdV$. If $\pi = (132)$ the we conclude in the same matter. This shows (ii). Condition (iii) can be neglected since 3 is odd. Altogether, V is $Pre(3)$ -solid by Theorem 10. ■

REFERENCES

- [1] V. Budd, K. Denecke and S.L. Wismath, *Short-solid superassociative type (n) varieties*, East-West J. of Mathematics **3** (2) (2001), 129–145.
- [2] W. Dörnte, *Untersuchungen über einen verallgemeinerten Gruppenbegriff*, Math. Z. **29** (1928), 1–19.
- [3] K. Denecke and Hounnon, *All solid varieties of semirings*, Journal of Algebra **248** (2002), 107–117.
- [4] K. Denecke and J. Koppitz, *Pre-solid varieties of semigroups*, Archivum Mathematicum **31** (1995), 171–181.
- [5] K. Denecke and J. Koppitz, *Finite monoids of hypersubstitutions of type $\tau = (2)$* , Semigroup Forum **56** (1998), 265–275.
- [6] K. Denecke and M. Reichel, *Monoids of hypersubstitutions and M -solid varieties*, Contributions to General Algebra **9** (1995), 117–126.
- [7] K. Denecke, J. Koppitz and S.L. Wismath, *Solid varieties of arbitrary type*, Algebra Universalis **48** (2002), 357–378.

- [8] K. Denecke and S.L. Wismath, *Hyperidentities and clones*, Gordon and Breach Scientific Publisher, 2000.
- [9] J. Koppitz, *Hypersubstitutions and groups*, Novi Sad J. Math. **34** (2) (2004), 127–139.
- [10] L. Polák, *All solid varieties of semigroups*, Journal of Algebra **219** (1999), 421–436.
- [11] J. Płonka, *Proper and inner hypersubstitutions of varieties*, p. 106–115 in: “*Proceedings of the International Conference: ‘Summer School on General Algebra and Ordered Sets’, Olomouc 1994*”, Palacký University, Olomouc 1994.

Received 15 July 2005