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NOTE ON ALGEBRAIC INTERIOR SYSTEMS

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Abstract

We get an interrelation between an algebraic closure system and its conjugated interior system. We introduce the concept of algebraic interior system and we get its representation.

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Let A be a non-void set. Denote by $ExpA$ the power set of A, i.e., the system of all subsets of A. A subsystem $\mathcal{S} \subseteq ExpA$ is called an *interior system over* A if $\mathcal{S}$ is closed under arbitrary unions, i.e., if for each $\mathcal{N} \subseteq \mathcal{S}$ also $\bigcup \mathcal{N} \subseteq \mathcal{S}$. Of course, we have $\emptyset \in \mathcal{S}$ for every interior system because it is a union of an empty family of subsets of $\mathcal{S}$. A dual concept is the so-called *closure system over* A; it is a subsystem $\mathcal{T} \subseteq ExpA$ which is closed under arbitrary intersections, i.e., for each subsystem $\mathcal{M} \subseteq \mathcal{T}$ we have $\bigcap \mathcal{M} \subseteq \mathcal{T}$.

A mapping $I : ExpA \to ExpA$ is called an *interior operator* on A if for every $X, Y \subseteq A$ we have:

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(i) $I(X) \subseteq X$,

(ii) $X \subseteq Y$ implies $I(X) \subseteq I(Y)$,

(iii) $I(I(X)) = I(X)$.

Dually, a mapping $C : ExpA \to ExpA$ is called a *closure operator* on A if for every $X, Y \subseteq A$ we have:

(i)′ $X \subseteq C(X)$,

(ii)′ $X \subseteq Y$ implies $C(X) \subseteq C(Y)$,

(iii)′ $C(C(X)) = C(X)$.

The connection between an interior system and an interior operator is established by the following well-known

Lemma 1. *Let $\mathcal{S}$ be an interior system and I an interior operator on A.*

(a) *Define $I_{\mathcal{S}}(X) = \bigcup\{B \in \mathcal{S} : B \subseteq X\}$. Then $I_{\mathcal{S}}$ is an interior operator on A.*

(b) *Define $\mathcal{S}_I = \{I(X) : X \in ExpA\}$. Then $\mathcal{S}_I$ is an interior system over A.*

For the proof, see e.g., [3].

We will call $I_{\mathcal{S}}$ or $\mathcal{S}_I$ the **assigned interior operator** or **system** to the interior system $\mathcal{S}$ or the interior operator I.

Of course, analogous results are well known also for closure systems and closure operators. Moreover, there are connections between closure operators (systems) and interior operators (systems) as follows:

Lemma 2. *Let C be a closure operator and I an interior operator on a set A. Define*

$$I_C(X) = A \setminus C(A \setminus X) \quad \text{and} \quad C_I(X) = A \setminus I(A \setminus X).$$

Then I_C or C_I is an interior or a closure operator on A, respectively.

The proof is evident.

The interior operator I_C will be called to be *conjugated* to the closure operator C; C_I is a *conjugated* closure operator to the interior operator I.

Of course, if $\mathcal{S}$ or $\mathcal{T}$ be an interior or a closure system over A, respectively, then

$$\mathcal{S}_T = \{A \setminus B; B \in \mathcal{T}\}$$

is an interior operator and

$$\mathcal{T}_S = \{A \setminus B; B \in \mathcal{S}\}$$

is a closure system over A. This follows immediatelly by Lemma 2. The system $\mathcal{S}_T$ is called dual to $\mathcal{T}$ and $\mathcal{T}_S$ is called dual to $\mathcal{S}$.

A closure system $\mathcal{T}$ is called *algebraic* (see e.g., [2]) if it is closed under unions of chains, i.e., for every chain $\{B_\lambda; \lambda \in \Lambda\}$ (that is $B_{\lambda_1} \subseteq B_{\lambda_2}$ for $\lambda_1 \leq \lambda_2$) of members of $\mathcal{T}$ also $\bigcup\{B_\lambda; \lambda \in \Lambda\} \in \mathcal{S}$. It was proved by G. Birkhoff and O. Frink [1] that for every algebraic closure system $\mathcal{T}$ over A there exists an algebra $\mathcal{A} = (A, F)$ on this support A such that $\mathcal{T} = Sub\mathcal{A}$ (the system are all subalgebras of $\mathcal{A}$). Moreover, for the assigned closure operator C_T and $X \subseteq A$ we have $C_T(X) = [X]$, the subalgebra of $\mathcal{A}$ generated by X. We are going to relate the conjugale interior system.

Theorem 1. *Let $\mathcal{S}$ be an interior system over A and I_S its assigned interior operator. Then $\mathcal{S}$ is conjugated to an algebraic closure system if and only if*

$$(*) \qquad I_S(A \setminus X) = \bigcap\{I_S(A \setminus Y); Y \subseteq X \ \ and \ \ Y \ \ finite\}.$$

Proof. Let $\mathcal{S}$ be an interior system over A conjugated to a closure system $\mathcal{T}_S$. By Lemma 2 we have $C_{I_S}(X) = A \setminus I_S(A \setminus X)$, i.e., $A \setminus \bigcap\{I_S(A \setminus Y); Y \subseteq X \text{ and } Y \text{ finite}\} = \bigcup\{A \setminus I_S(A \setminus Y); Y \subseteq X \text{ and } Y \text{ finite}\} = \bigcup\{C_{T_S}(Y); Y \subseteq X \text{ and } Y \text{ finite}\}$.

Hence, I_S satisfies $(*)$ if and only if C_{T_S} satisfies

$C_{T_S}(X) = \bigcup\{C_{T_S}(Y); Y \subseteq X \text{ and } Y \text{ finite}\}$.

By Lemma 6.3 in [2], $\mathcal{T}_S$ is an algebraic closure system. ∎

Example. Let $\mathcal{A} = (A, F)$ be an algebra. Denote by $Sub_0\mathcal{A} = \{\emptyset\} \cup Sub\mathcal{A}$. For every $X \subseteq A$ define $I(X) = \bigcup\{B \in Sub_0\mathcal{A}; B \subseteq X\}$. It is an easy exercise to verify that I is an interior operator on A.

The foregoing example motivated us to introduce the following concept.

Let $\mathcal{T}$ be a closure system over a set A and $\mathcal{T}_0 = \{\emptyset\} \cup \mathcal{T}$. The interior system $\mathcal{S}_I$ assigned to the interior operator

$$I(X) = \bigcup\{B \in \mathcal{T}_0; B \subseteq X\}$$

will be called *algebraic*.

Let $\mathcal{S}$ be an interior system over A. A subsystem $\mathcal{B} \subseteq \mathcal{S}$ is called a *base* of $\mathcal{S}$ if for each $D \in \mathcal{S}$ there exist $B_\lambda \in \mathcal{B}(\lambda \in \Lambda)$ such that

$$D = \bigcup\{B_\lambda; \lambda \in \Lambda\}.$$

Hence, an interior system $\mathcal{S}$ is algebraic if it has a base which is a closure system.

Theorem 2. *An interior system $\mathcal{S}$ over A is algebraic if and only if there exists an algebra $\mathcal{A} = (A, F)$ such that $Sub_0\mathcal{A}$ is a base of $\mathcal{S}$.*

Proof. We need only to find an algebra $\mathcal{A} = (A, F)$ such that $Sub_0\mathcal{A}$ is a base of $\mathcal{S}$.

Let $\mathcal{S}$ be an algebraic interior system over A and the closure system $\mathcal{C}$ over A is its base. Denote by $\mathcal{B}$ the system of all $\bigcup$-irreducible members of $\mathcal{C}$. Of course, $\mathcal{B}$ is also a base of $\mathcal{S}$. We use the Birkhoff-Frink construction of operations $f_{\bar{a}}$ (see [1]). Let $Y \subseteq A$ be a finite subset, say $Y = \{a_0, \cdots, a_{n-1}\}$. Let C be the closure operator on A assigned to the closure system $\mathcal{C}$ and let $a \in C(Y)$. Denote by $\bar{a} = \langle a_0, \cdots, a_{n-1}\rangle$ and define

$$f_{\bar{a}}(b_0, \cdots, b_{n-1}) = \begin{cases} a & \text{if } \langle b_0, \cdots, b_{n-1}\rangle = \bar{a} \\ b_0 & \text{otherwise.} \end{cases}$$

If $B \in \mathcal{B}$ and $B \neq \emptyset$ then $B \in Sub_0\mathcal{A}$, where $\mathcal{A} = (A, F)$ is an algebra on the support A whose all operations are $f_{\bar{a}}$ for all possible choices of $\bar{a}$ and arbitrary integers n. If $B \in \mathcal{B}$ and $B \neq \emptyset$ then it is routine way (see [1]) to show that B is closed under every $f_{\bar{a}}$ and hence $B \in Sub_0\mathcal{A}$. Thus for each subset $X \subseteq A$, $I_S(X)$ is a union of elementals of $Sub_0\mathcal{A}$.

On the other hand, suppose $B \in Sub_0\mathcal{A}$. If $B = \emptyset$ then $B \in \mathcal{S}$. If $B \neq \emptyset$ then B is a subalgebra of $\mathcal{A}$. Suppose $H \subseteq B$, H finite, e.g. $H = \{a_0, \cdots, a_{n-1}\}$. If $a \in C(H)$ then $a = f_{\bar{a}}(a_0, \cdots, a_{n-1})$ for $\bar{a} = \langle a_0, \cdots, a_{n-1}\rangle$ and hence $a \in B$ thus $C(H) \subseteq B$. Howeover, $B = \bigcup\{C(H);\ H \subseteq B$ and H finite$\} \in \mathcal{S}$ since $\mathcal{S}$ is an interior system and C is its base. We have shown that $Sub_0\mathcal{A}$ is a base of $\mathcal{S}$. ■

Corollary 1. *For any algebraic interior system $\mathcal{S}$ there exists an algebraic closure system $\mathcal{C}$ which is a base of $\mathcal{S}$.*

Corollary 2. *For any algebraic interior system $\mathcal{S}$ over A there exists an algebra $\mathcal{A} = (A, F)$ such that for each $X \subseteq A$ either $I_S(X) = \emptyset$ or $I_S(X)$ is a union of subalgebras of $\mathcal{A}$.*

REFERENCES

[1] G. Birkoff and O. Frink, *Representations of lattices by sets*, Trans. Amer. Math. Soc. **64** (1948), 299–316.

[2] G. Grătzer, *Universal Algebra*, Springer-Verlag (2nd ed.), New York-Heidelberg-Berlin 1979.

[3] K. Kuratowski, Topologie I, PWN, Warszawa 1958.

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