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REPRESENTATIONS OF A FREE GROUP OF RANK TWO BY TIME-VARYING MEALY AUTOMATA

Adam Woryna

Institute of Mathematics
Silesian University of Technology
Kaszubska 23, 44–100 Gliwice, Poland

e-mail: Adam.Woryna@polsl.pl

Abstract

In the group theory various representations of free groups are used. A representation of a free group of rank two by the so-called time-varying Mealy automata over the changing alphabet is given. Two different constructions of such automata are presented.

Key words and phrases: changing alphabet, Mealy automaton, time-varying automaton, group generated by time-varying automaton, free group.

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1. Introduction

The theory of Mealy automata and groups generated by them is a part of geometric group theory which describes groups acting on a rooted tree. The study of the groups of automata was initiated in 70th years by several outstanding mathematicians (mainly by R. Grigorchuk). It has rapidly expanded in recent years and now plays an important role in algebra, theory of dynamical systems, spectral theory, ergodic theory and others (see [2]–[4], [7], [8]).

The Mealy automata considered so far had the same structure at every moment of a discrete time-scale (the so-called fixed Mealy automata).

A concept of a time-varying Mealy automata over a changing alphabet was introduced for the first time in paper [9]. In papers [10], [11], it is shown that time-varying Mealy automaton is a useful tool to describing groups acting on level homogeneous rooted trees which are not homogeneous.

One of key problems in the theory of groups of automata is the problem of embeddability of other known classes of groups into these groups. This problem was solved positively for free groups. The first example of such an automaton representation of a free group was suggested in [1]. However, the complete proof is still unpublished. An example with the complete proof was first presented in [5]. In paper [6] a representation of a free group of rank two by infinite unitriangular matrices is regarded as a group of transformations generated by fixed Mealy automata over a two-letter alphabet. In this paper we describe a representation of a free group of rank two by time-varying Mealy automata. Two different constructions will be presented.

2. Time-varying Mealy automata and groups generated by them

Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ be a set of nonnegative integers. A *changing alphabet* is an infinite sequence

$$X = (X_t)_{t\in\mathbb{N}_0},$$

where X_t are nonempty, finite sets (sets of letters). A *word* over the changing alphabet X is a finite sequence $x_0x_1\ldots x_l$, where $x_i \in X_i$ for $i = 0, 1, \ldots, l$. We denote by X^* the set of all words (including the empty word $\emptyset$). By $|w|$ we denote the length of the word $w \in X^*$. The set of words of the length t we denote by $X^{(t)}$. For any $t \in \mathbb{N}_0$ we also consider the set $X_{(t)}$ of finite sequences in which the i-th letter ($i = 1, 2, \ldots$) belongs to the set X_{t+i-1}. In particular $X_{(0)} = X^*$.

Definition 2.1 . A *time-varying Mealy automaton* is a quintuple

$$A = (Q, X, Y, \varphi, \psi),$$

where:

1. $Q = (Q_t)_{t\in\mathbb{N}_0}$ is a sequence of sets of *inside states*,
2. $X = (X_t)_{t\in\mathbb{N}_0}$ is a changing *input alphabet*,

3. $Y = (Y_t)_{t\in\mathbb{N}_0}$ is a changing *output alphabet*,
4. $\varphi = (\varphi_t)_{t\in\mathbb{N}_0}$ is a sequence of *transitions functions* of the form

$$\varphi_t \colon Q_t \times X_t \to Q_{t+1},$$

5. $\psi = (\psi_t)_{t\in\mathbb{N}_0}$ is a sequence of *output functions* of the form

$$\psi_t \colon Q_t \times X_t \to Y_t.$$

We say that an automaton A is *finite* if the set $S = \bigcup_{t\in\mathbb{N}_0} Q_t$ of all its inside states is finite. If $|S| = n$, we say that A is an *n-state automaton.*

It is convenient to present a time-varying Mealy automaton as a labelled, directed, locally finite graph with vertices corresponding to the inside states of the automaton. For every $t \in \mathbb{N}_0$ and every letter $x \in X_t$ an arrow labelled by x starts from every state $q \in Q_t$ and is going to the state $\varphi_t(q,x)$. Each vertex $q \in Q_t$ is labelled by the corresponding *state function*

$$\sigma_{t,q} \colon X_t \to Y_t, \quad \sigma_{t,q}(x) = \psi_t(q,x). \tag{1}$$

To make the graph of the automaton clear, the sets of vertices V_t and $V_{t'}$ corresponding to the sets Q_t and $Q_{t'}$ respectively, are disjoint whenever $t \neq t'$ (in particular, different vertices may correspond to the same inside state). Moreover, we will substitute a large number of arrows connecting two fixed states and having the same direction for a one multi-arrow labelled by suitable letters and if the labelling of such a multi-arrow is obvious we will omit this labelling.

Example 2.1 . Let $(m_t)_{t\in\mathbb{N}_0}$ be a sequence of positive integers. The Figure 1 presents a 2-state time-varying automaton $A = (Q, X, Y, \varphi, \psi)$ in which:

1. $Q_t = \{0, 1\}$,
2. $X_t = Y_t = \{0, 1, \dots, m_t - 1\}$,
2. $\varphi_t(q,x) = \mu(q)\cdot\mu(x)$,
3. $\psi_t(q,x) = x +_{m_t} q$,

where $+_{m_t}$ is an addition (mod m_t) and the function $\mu\colon \mathbb{R} \to \{0,1\}$ is defined as follows:

$$\mu(0) = 1 \quad \text{and} \quad \mu(x) = 0 \quad \text{for} \quad x \neq 0.$$

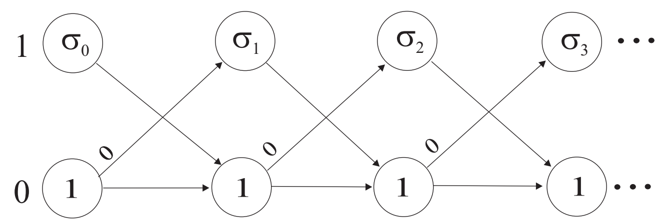

Figure 1. Example of a time-varying automaton.

In the Figure the state functions 1 and σ_t constitute respectively the identity function and the cyclical permutation $(0, 1, \ldots, m_t - 1)$ of X_t.

A time-varying automaton may be interpreted as a machine, which being at a moment $t \in \mathbb{N}_0$ in a state $q \in Q_t$ and reading on the input tape a letter $x \in X_t$, goes to the state $\varphi_t(q,x)$, types on the output tape the letter $\psi_t(q,x)$, moves both tapes to the next position and then proceeds further to the next moment $t+1$.

The automaton A with a fixed *initial state* $q \in Q_0$ is called the *initial automaton* and is denoted by A_q. The above interpretation defines a natural action of A_q on the words. Namely, the initial automaton A_q defines a function $f_q^A\colon X^* \to Y^*$ as follows:

$$f_q^A(x_0 x_1 \ldots x_l) = \psi_0(q_0, x_0)\psi_1(q_1, x_1)\ldots\psi_l(q_l, x_l),$$

where the sequence $q_0, q_1, \ldots, q_l$ of inside states is defined recursively:

$$(2) \qquad q_0 = q, \qquad q_i = \varphi_{i-1}(q_{i-1}, x_{i-1}) \quad \text{for} \quad i = 1, 2, \ldots, l.$$

The function f_q^A is called the *automaton function* defined by A_q. The image of a word $w = x_0x_1 \dots x_l$ under a map f_q^A can be easily found using the graph of the automaton. One must find a directed path starting in a vertex $q \in Q_0$ and with consecutive labels $x_0, x_1, \dots, x_l$. Such a path will be unique. If $\sigma_0, \sigma_1, \dots, \sigma_l$ are the labels of consecutive vertices in this path, then the word $f_q^A(w)$ is equal to $\sigma_0(x_0)\sigma_1(x_1)\dots\sigma_l(x_l)$.

In the set of words over a changing alphabet, we consider for any $k \in \mathbb{N}_0$ the equivalence relation $\sim_k$ as follows:

$$w \sim_k v \text{ iff } w \text{ and } v \text{ have a common prefix of the length } k.$$

Let X and Y be changing alphabets and let f be a function of the form $f\colon X^* \to Y^*$. If f preserves the relation $\sim_k$ for any k, then we say that f *preserves beginnings of the words.* If $|f(w)| = |w|$ for any $w \in X^*$, then we say that f *preserves lengths of the words.*

Theorem 2.1 ([9])**.** *The function $f\colon X^* \to Y^*$ is an automaton function iff it preserves beginnings and lengths of the words.*

Definition 2.2. Let $w \in X^*$ be a word of the length $n = |w|$. The function $f_w\colon X_{(n)} \to Y_{(n)}$ defined by the equality

$$f(wv) = f(w)f_w(v)$$

is called a *remainder* of f on the word w or simply a *w-remainder* of f.

Definition 2.3. Let $A = (Q, X, Y, \varphi, \psi)$ be a time-varying Mealy automaton. For any $t_0 \in \mathbb{N}_0$ the automaton $A|^{t_0} = (Q', X', Y', \varphi', \psi')$ defined as follows

$$Q'_t = Q_{t_0+t}, \quad X'_t = X_{t_0+t}, \quad Y'_t = Y_{t_0+t}, \quad \varphi'_t = \varphi_{t_0+t}, \quad \psi'_t = \psi_{t_0+t},$$

is called a *t_0-remainder* of A.

If f is generated by the initial automaton A_q and the word $w = x_0x_1 \dots x_l$, then the w-remainder f_w is an automaton function generated by the automaton B_{q_l}, where $B = A|^l$ and the initial state q_l is obtained from (2).

Definition 2.4 . An automaton A in which input and output alphabets coincide and every its state function $\sigma_{t,q}\colon X_t \to X_t$ is a permutation of X_t is called a *permutational automaton*.

If A is a permutational automaton, then for every $q \in Q_0$ the transformation f_q^A defines a permutation of X^*.

The set $SA(X)$ of automaton functions defined by all initial automata over a common input and output alphabet X forms a monoid with the identity function as the neutral element. The subset $GA(X)$ of functions generated by permutational automata is a group of invertible elements in $SA(X)$. The group $GA(X)$ is an example of residually finite groups (see [10]).

Definition 2.5 . Let $A = (Q, X, X, \varphi, \psi)$ be a time-varying permutational automaton. The group of the form

$$G(A) = \langle f_q^A \colon q \in Q_0 \rangle$$

is called the *group generated by automaton A*.

For any permutational automaton A, the group $G(A)$ is residually finite, as a subgroup of $GA(X)$. It turns out that groups of this form include the class of finitely generated residually finite groups.

Theorem 2.2 ([10])**.** *For any n-generated residually finite group G, there is an n-state time-varying automaton A such that $G \cong G(A)$.*

3. The embedding into the permutational wreath product

Let $X = (X_t)_{t \in \mathbb{N}_0}$ be a changing alphabet and let G be any subgroup of $GA(X)$. For any $i \in \mathbb{N}_0$ we consider the group

$$G_i = \left\langle g_w \colon\ g \in G,\ w \in X^{(i)} \right\rangle$$

which is a group generated by w-remainders of functions from G on all words $w \in X^{(i)}$. In particular $G_0 = G$.

Example 3.1 . If $G = G(A)$, then G_i is in general a subgroup of $G(A|^i)$. If we additionally assume that A is *accessible*, that is every state of A may be obtained from the recurrence (2) for some initial state $q \in Q_0$ and some word $w = x_0 x_1 \dots x_l$, then the equality $G_i = G(A|^i)$ holds for every $i \in \mathbb{N}_0$.

Proposition 3.1 . *For any $f, g \in SA(X)$ and any word $w \in X^*$, we have*

$$(f \circ g)_w = f_w \circ g_{f(w)}. \tag{3}$$

If $g \in GA(X)$, then

$$\left(g^{-1}\right)_w = \left(g_{g^{-1}(w)}\right)^{-1}. \tag{4}$$

Proof. For any $u \in X_{(|w|)}$ we have $(f \circ g)(wu) = (f \circ g)(w)(f \circ g)_w(u)$. On the other hand

$$\begin{aligned} (f \circ g)(wu) &= g(f(wu)) = g(f(w)f_w(u)) = \\ &= g(f(w))g_{f(w)}(f_w(u)) = (f \circ g)(w)(f_w \circ g_{f(w)})(u), \end{aligned}$$

what gives (3) from the previous equality. The formula (4) follows by substitution of f for g^{-1} in (3). ∎

Let us arrange the letters of X_i in the sequence: $x_0, x_1, \dots, x_{m-1}$.

Proposition 3.2 . *The group G_i embeds into the permutational wreath product $G_{i+1} \wr_{X_i} S(X_i)$ by the mapping*

$$\Psi \colon g \mapsto (g_{x_0}, g_{x_1}, \dots, g_{x_{m-1}})\sigma_g,$$

where $\sigma_g \in S(X_i)$ is defined by the equality $\sigma_g(x) = g(x)$.

Proof. The mapping Ψ is one-to-one, what follows from the equalities $g(xu) = \sigma_g(x) g_x(u)$ for $x \in X_i$ and $u \in X_{(i+1)}$. By Proposition 3.1, we have:

$$\begin{aligned}
\Psi(f \circ g) &= ((f \circ g)_{x_0}, \ldots, (f \circ g)_{x_{m-1}})\sigma_{f \circ g} = \\
&= (f_{x_0} \circ g_{\sigma_f(x_0)}, \ldots, f_{x_{m-1}} \circ g_{\sigma_f(x_{m-1})})\, \sigma_f \circ \sigma_g = \\
&= (f_{x_0}, f_{x_1}, \ldots, f_{x_{m-1}})\sigma_f \;\; (g_{x_0}, g_{x_1}, \ldots, g_{x_{m-1}})\sigma_g = \\
&= \Psi(f)\Psi(g).
\end{aligned}$$

Hence Ψ is a homomorphism. ■

We will write

$$g = [g_{x_0}, g_{x_1}, \ldots, g_{x_{m-1}}]\sigma_g$$

without special comments.

4. Representations of a free group of rank two by time-varying Mealy automata

In this chapter we describe a representation of a free group of rank two by time-varying Mealy automata. Two different constructions of such automata will be presented.

The first construction gives a representation by a 2-state automaton. It uses the following reverse order relation $\prec$ among freely reduced group words in symbols a, b:

1. the empty word $\prec a \prec a^{-1} \prec b \prec b^{-1}$,
2. if $|w_1| < |w_2|$, then $w_1 \prec w_2$,
3. if $|w_1| = |w_2|$ and w_1, w_2 first differ (counting from the right side) in their k-th terms, then the order of these words depends on their k-th terms.

The crucial point of this construction constitute two permutations a, b of the set $\mathbb{N}$ with the following property: if w is a group word in a and b on the l-th position ($l = 1, 2, \ldots$) in the above ordering, then the permutation of $\mathbb{N}$ defined by w maps the number 1 into l. The permutations a, b may be defined by the following formulas:

$$a(n) = \begin{cases} 2, & \text{if} \quad n = 1, \\ n + 4 \cdot 3^k, & \text{if} \quad 2 \cdot 3^k \leq n < 3^{k+1}, \\ n - 2 \cdot 3^k, & \text{if} \quad 3^{k+1} \leq n < 4 \cdot 3^k, \\ n + 3^{k+1}, & \text{if} \quad 4 \cdot 3^k \leq n < 2 \cdot 3^{k+1}; \end{cases}$$

$$b(n) = \begin{cases} 4, & \text{if} \quad n = 1, \\ n + 10 \cdot 3^k, & \text{if} \quad 2 \cdot 3^k \leq n < 5 \cdot 3^k, \\ n - [13 \cdot 3^{k-1}], & \text{if} \quad 5 \cdot 3^k \leq n < 17 \cdot 3^{k-1}, \\ n - 4 \cdot 3^k, & \text{if} \quad 17 \cdot 3^{k-1} \leq n < 2 \cdot 3^{k+1}. \end{cases}$$

Let $A = (Q, X, X, \varphi, \psi)$ be a time-varying automaton in which (exceptionally, all the components below are indexed from $t = 1$):

1. $Q_t = \{0, 1\}$,
2. $X_t = \{1, 2, \ldots, t\}$,
3. $\varphi_t(q, x) = q$,
4. $\psi_t(0, x) = a_t(x)$, $\psi_t(1, x) = b_t(x)$,

where a_t, b_t are defined as follows:

$$a_t(x) = \begin{cases} a(x), & \text{if} \quad x \in X_t \cap a^{-1}(X_t), \\ \bar{a}_t(x), & \text{if} \quad x \in X_t \smallsetminus a^{-1}(X_t); \end{cases}$$

$$b_t(x) = \begin{cases} b(x), & \text{if} \quad x \in X_t \cap b^{-1}(X_t), \\ \bar{b}_t(x), & \text{if} \quad x \in X_t \smallsetminus b^{-1}(X_t); \end{cases}$$

and the mappings

$$\bar{a}_t\colon X_t \setminus a^{-1}(X_t) \to X_t \setminus a(X_t), \qquad \bar{b}_t(x)\colon X_t \setminus b^{-1}(X_t) \to X_t \setminus b(X_t)$$

are any bijections. It is not hard to see that $a_t, b_t \in S(X_t)$. In particular, the automaton A is permutational. The graph of this automaton is presented in Figure 2.

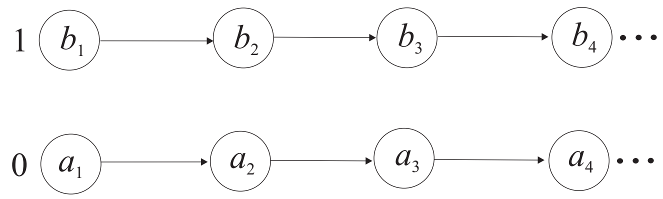

Figure 2. The automaton A which generates a free group of rank two.

Theorem 4.1 . *The group $G(A)$ generated by the functions f_0^A and f_1^A is a free group of rank two which is freely generated by these functions.*

Proof. The generators f_0^A and f_1^A map any word $x^* = x_1x_2\ldots x_l \in X^*$ into

$$f_0^A(x^*) = a_1(x_1)a_2(x_2)\ldots a_l(x_l), \;\; f_1^A(x^*) = b_1(x_1)b_2(x_2)\ldots b_l(x_l).$$

For every $n \in \mathbb{N}$ we have:

$$a_t(x) = a(x), \quad b_t(x) = b(x) \quad \text{for} \quad x = 1, 2, \ldots, n,$$

where $t = \max\{a(1), \ldots, a(n), b(1), \ldots, b(n)\}$. Thus, if w is a nonempty, freely reduced group word in f_0^A, f_1^A and the element $g \in G(A)$ is represented by w, then

$$g(\underbrace{11\ldots1}_{t}) \neq \underbrace{11\ldots1}_{t}$$

for t large enough. Indeed, if a group word which derives from w by substitution of all f_0^A for a and of all f_1^A for b is on the l-th position (in the above ordering), then $l \neq 1$ and the last letter of $g(\underbrace{11\dots1}_{t}) \in X^*$ is equal to l for t large enough. ■

In the second constructions we consider the automaton $B = (Q, X, X, \varphi, \psi)$ defined as follows:

1. $Q_t = X_t = \{0, 1, \dots, t+1\}$,
2. $\varphi_t(q, x) = \begin{cases} 1 + \mu(t + 1 - x), & \text{for} \quad q = 1, \\ 1 + q - \mu(q), & \text{for} \quad q \neq 1; \end{cases}$
3. $\psi_t(q, x) = \begin{cases} x, & \text{for} \quad q = 1, \\ \alpha_t^{\mu(q)-q}(x), & \text{for} \quad q \neq 1; \end{cases}$

where $\alpha_t = (0)(1, 2, \dots, t+1)$ is a cyclical permutation of X_t. The automaton B is permutational and its graph is presented in Figure 3.

The construction of the automaton B is quite different from the automaton A. In particular, the automaton B is not finite. On the other hand, the labelling of its inside states is quite straight. Namely, every state function $\sigma_{t,q}$ is a power of a cyclical permutation α_t.

We consider for any $i \in \mathbb{N}_0$ the remainders a_i and b_i of the functions f_0^B and f_1^B respectively, on the word $00\dots0$ of the length i. Let $G_i = G(B|^i)$ be a group generated by an i-remainder of the automaton B. From the graph of B, we see that $G_i = \langle a_i, a_i^2, \dots, a_i^{i-1}, b_i \rangle = \langle a_i, b_i \rangle$. In particular, $G_0 = G(B)$. The embedding of G_i into the permutational wreath product $G_{i+1} \wr_{X_i} S(X_i)$ is induced by the following equations:

$$a_i = [a_{i+1}, a_{i+1}, \dots, a_{i+1}]\alpha_i, \quad b_i = [b_{i+1}, \dots, b_{i+1}, a_{i+1}^i]. \tag{5}$$

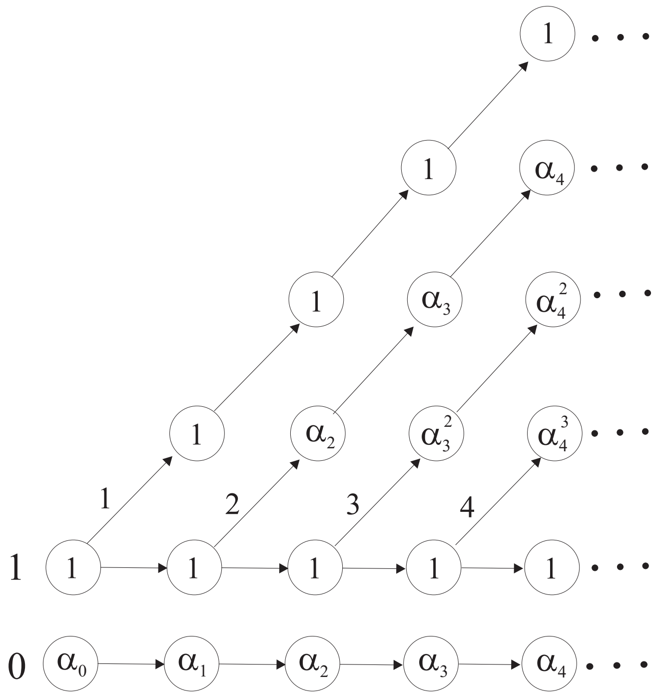

Figure 3. The automaton B which generates a free group of rank two.

Any element $g \in G_i$ is an automaton function over the changing alphabet $Y = (X_{i+t})_{t \in \mathbb{N}_0}$. It is represented by some freely reduced group word w in the symbols a_i, b_i:

$$g = w(a_i, b_i).$$

Let $x^* = x_0 x_1 \dots x_{l-1}$ be any word over Y and let

$$g_{x_0},\ g_{x_0 x_1},\ \dots,\ g_{x_0 x_1 \dots x_{l-1}}$$

be remainders of g on the consecutive beginnings of x^*. The remainder $g_{x_0 \dots x_j} \in G_{i+j+1}$ $(j = 0, 1, \dots l-1)$ is represented by some freely reduced group word $w_{x_0 \dots x_j}$ in the symbols a_{i+j+1}, b_{i+j+1}:

$$g_{x_0 \dots x_j} = w_{x_0 \dots x_j}(a_{i+j+1}, b_{i+j+1}).$$

Using the equations 5, we may derive $w_{x_0\ldots x_j}$ from $w_{x_0\ldots x_{j-1}}$ (from w if $j=0$) in the following way:

(i) if $x_j = 0$, then every syllable of the form a^s_{i+j} is substituted for a^s_{i+j+1} and every syllable of the form b^r_{i+j} is substituted for b^r_{i+j+1},

(ii) if $x_j \neq 0$, then every syllable of the form a^s_{i+j} is substituted for a^s_{i+j+1} and every syllable of the form b^r_{i+j} is substituted for b^r_{i+j+1} or - in case of $\alpha^s_{i+j}(x_j) = i+j+1$ - for $a^{r(i+j)}_{i+j+1}$, where s is the sum of all exponents on a_{i+j}-syllables on the left of b^r_{i+j}.

Thus $w_{x_0\ldots x_j}$ is a freely reduction of the word derived from $w_{x_0\ldots x_{j-1}}$ by the rules (i) and (ii). The word w_{x^*} is called an *x^*-remainder of w.* The rules (i) and (ii) define the action of G_i on the set Y^* as follows:

$$g(x_0x_1\ldots x_{l-1}) = \alpha_i^{S_0}(x_0)\alpha_{i+1}^{S_1}(x_1)\ldots\alpha_{i+l-1}^{S_{l-1}}(x_{l-1}), \tag{6}$$

where S_j is the sum of all exponents on a_{i+j}-syllables in $w_{x_0\ldots x_{j-1}}$.

Theorem 4.2. *The group $G(B)$ generated by the functions f_0^B and f_1^B is a free group of rank two which is freely generated by these functions.*

Proof. We show that for every $i \in \mathbb{N}_0$ the group G_i is freely generated by a_i, b_i. Let

$$w = a_i^{s_1}b_i^{r_1}\ldots a_i^{s_k}b_i^{r_k}$$

be any nonempty, freely reduced group word in a_i, b_i and let $N(w)$ be the number of b_i-syllables in w. We prove by induction on $N(w)$ that w does not define the neutral element in G_i. To this, we assume that for every $j \in \mathbb{N}_0$ any nonempty, freely reduced group word v in a_j, b_j with $N(v) < N(w)$ does not define the neutral element in G_j.

If $N(w) \leq 1$, then $w = a_i^{s_1}$ $(s_1 \neq 0)$ or $w = a_i^{s_1}b_i^{r_1}a_i^{s_2}$ $(r_1 \neq 0)$. In this case we easily check that none of the above words defines the neutral element in G_i. Let, now assume $N(w) > 1$. Then $k > 1$. Let us denote:

$$S = \sum_{j=1}^{k-1} s_j, \quad R = \max_{0<j<k}(|s_j| + |s_{j+1}|) + 1, \quad \text{and} \quad l = \max(0, R-i).$$

We consider the remainders w_{y^*} and w_{x^*} of w on sequences:

$$y^* = \underbrace{00\ldots0}_{l}, \quad x^* = \underbrace{00\ldots0}_{l}x,$$

where $x = \alpha_{i+l}^{-S}(i+l+1) \in X_{i+l}$. By the rule (i) we have

$$w_{y^*} = a_{i+l}^{s_1} b_{i+l}^{r_1} \ldots a_{i+l}^{s_k} b_{i+l}^{r_k}.$$

We may derive w_{x^*} from w_{y^*} by substitution of every its a_{i+l}-syllable for an appropriate a_{i+l+1}-syllable and every b_{i+l}-syllable for b_{i+l+1}-syllable or else a_{i+l+1}-syllable – according to the rules (i) and (ii). The substitution of any b_{i+l}-syllable for a_{i+l+1}-syllable we call simply as a_{i+l+1}-substitution.

There are no two consecutive syllables $b_{i+l}^{r_j}$, $b_{i+l}^{r_{j+1}}$ in w_{y^*} for which the a_{i+l+1}-substitutions hold. Otherwise $\alpha_{i+l}^{s_{j+1}}(i+l+1) = i+l+1$ and since, as $s_{j+1} \neq 0$, we have consequently

$$|s_{j+1}| \geq i+l+1 \geq i+(R-i)+1 > R > |s_{j+1}|.$$

If $v = a_{i+l}^{s_j} b_{i+l}^{r_j} a_{i+l}^{s_{j+1}}$ is any subword in w_{y^*} such that the a_{i+l+1}-substitution holds for $b_{i+l}^{r_j}$, then this subword will be substituted for $a_{i+l+1}^{s'}$ in w_{x^*}, where $s' = s_j + r_j(i+l) + s_{j+1}$. Since

$$|r_j(i+l)| \geq |i+l| \geq R \geq |s_j| + |s_{j+1}| + 1 > |s_j + s_{j+1}|,$$

we have $s' \neq 0$.

As a result of the above observation, we obtain that w_{x^*} is nonempty. Moreover, for the syllable $b_{i+l}^{r_{k-1}}$ in w_{y^*} the a_{i+l+1}-substitution holds. As a result we have

$$N(w_{x^*}) < N(w).$$

By inductive assumption, w_{x^*} does not define the neutral element in G_{i+l+1}. As a consequence, w does not define the neutral element in G_i. ∎

Remark 4.1 . The fact that the sequence $|X_0|, |X_1|, |X_2|, \ldots$ is unbounded is crucial in both first and the second construction. Moreover, if X is any changing alphabet with the above sequence unbounded, then the same arguments allow to construct in a similar way a time-varying automaton over X giving a representation of a free group of rank two.

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