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T-VARIETIES AND CLONES OF T-TERMS

Klaus Denecke

University of Potsdam, Institute of Mathematics
Am Neuen Palais, 14415 Potsdam, Germany

e-mail: kdenecke@rz.uni-potsdam.de

and

Prakit Jampachon*

KhonKaen University, Department of Mathematics
KhonKaen, 40002 Thailand

e-mail: prajam@.kku.ac.th

Abstract

The aim of this paper is to describe how varieties of algebras of type τ can be classified by using the form of the terms which build the (defining) identities of the variety. There are several possibilities to do so. In [3], [19], [15] normal identities were considered, i.e. identities which have the form $x \approx x$ or $s \approx t$, where s and t contain at least one operation symbol. This was generalized in [14] to k-normal identities and in [4] to P-compatible identities. More generally, we select a subset T of $W_\tau(X)$, the set of all terms of type τ, and consider identities from $T \times T$. Since any variety can be described by one heterogenous algebra, its clone, we are also interested in the corresponding clone-like structure. Identities of the clone of a variety V correspond to M-hyperidentities for certain monoids M of hypersubstitutions. Therefore we will also investigate these monoids and the corresponding M-hyperidentities.

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1. Introduction

Let $\tau = (n_i)_{i\in I}$ be a type of algebras, indexed by a set I, with operation symbols f_i of arity n_i. Let $X = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite set of variables, and for each $n \geq 1$ let $X_n = \{x_1, x_2, \ldots, x_n\}$. We denote by $W_\tau(X)$ and $W_\tau(X_n)$ the sets of all terms, and of all n-ary terms of type τ, respectively. These two sets are the universes of two absolutely free algebras,

$$\mathcal{F}_\tau(X) \;:=\; \Big(W_\tau(X); (\overline{f_i})_{i\in I}\Big)$$

and

$$\mathcal{F}_\tau(X_n) \;:=\; \Big(W_\tau(X_n); (\overline{f_i})_{i\in I}\Big),$$

respectively. The operations $\overline{f_i}$ are defined by setting

$$\overline{f_i}(t_1, \ldots, t_{n_i}) := f_i(t_1, \ldots, t_{n_i}).$$

The algebras $\mathcal{F}_\tau(X)$ and $\mathcal{F}_\tau(X_n)$, respectively are examples of algebras of type τ, i.e. pairs $\mathcal{A} = (A; (f_i^A)_{i\in I})$, consisting of a carrier set (universe) A and a sequence $(f_i^A)_{i\in I}$ of operations defined on A, where f_i^A is n_i-ary and where $\tau = (n_i)_{i\in I}$ is the sequence of the arities of the f_i^A's. Let $Alg(\tau)$ be the class of all algberas of type τ. Another operation on sets of terms is the composition or superposition of terms which plays an important role in universal algebra, clone theory and theoretical computer science. For each pair of natural numbers m and n greater than zero, the superposition operation S^n_m maps one n-ary term and n m-ary terms to an m-ary term, so that

$$S^n_m : W_\tau(X_n) \times W_\tau(X_m)^n \;\to W_\tau(X_m).$$

The operation S^n_m is defined inductively, by setting $S^n_m(x_j, t_1, \ldots, t_n) := t_j$ for any variable $x_j \in X_n$, and

$$S^n_m(f_r(s_1, \ldots, s_{n_r}), t_1, \ldots, t_n)$$

$$:= f_r(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_r}, t_1, \ldots, t_n)).$$

Using these operations, we form the heterogeneous or multi-based algebra

$$clone(\tau) \ := \ (\, (W_\tau(X_n))_{n>0}; \ (S^n_m)_{n,m>0}, (x_i)_{i\leq n, n>0} \,).$$

It is well-known and easy to check that this algebra satisfies the clone axioms

$$\text{(C1)} \quad \overline{S^p_m}\left(\tilde{Z}, \overline{S^n_m}\left(\tilde{Y}_1, \tilde{X}_1, \ldots, \tilde{X}_n\right), \ldots, \overline{S^n_m}\left(\tilde{Y}_p, \tilde{X}_1, \ldots, \tilde{X}_n\right)\right)$$

$$\approx \overline{S^n_m}\left(\overline{S^p_n}\left(\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_p\right), \tilde{X}_1, \ldots, \tilde{X}_n\right), \text{for} \quad m, n, p = 1, 2, 3, \ldots,$$

$$\text{(C2)} \quad \overline{S^n_m}\left(\lambda_j, \tilde{X}_1, \ldots, \tilde{X}_n\right) \ \approx \ \tilde{X}_j, \quad \text{for } 1 \leq j \leq n \ \text{ and } \ m, n = 1, 2, 3, \ldots,$$

$$\text{(C3)} \quad \overline{S^m_m}\left(\tilde{X}_j, \lambda_1, \ldots, \lambda_m\right) \ \approx \ \tilde{X}_j, \quad \text{for } 1 \leq j \leq m \ \text{ and } \ m = 1, 2, 3, \ldots,$$

where $\overline{S^p_m}$ and $\overline{S^n_m}$ are operation symbols corresponding to the operations S^p_m and S^n_m of *clone*(τ), where $\lambda_1, \ldots, \lambda_m$ are nullary operation symbols and where $\tilde{Z}$, $\tilde{Y}_1, \ldots, \tilde{Y}_p$, $\tilde{X}_1, \ldots, \tilde{X}_m$ are variables. The algebra *clone*(τ) is also called the clone of terms of type τ.

Since later on we have to consider subalgebras and congruences of heterogeneous algebras, we recall these concepts. A subalgebra of *clone*(τ) consists of a sequence $(T^{(n)})_{n>0}$, where $T^{(n)} \subseteq W_\tau(X_n)$ for all $n > 0$ which is closed under all operations of *clone*(τ). A congruence on *clone*(τ) is a sequence $(\theta_n)_{n>0}$ of binary relations, where $\theta_n \subseteq W_\tau(X_n) \times W_\tau(X_n)$, which is preserved by all operations from *clone*(τ). For more background on heterogeneous algebras see [16], [1].

Since the set $W_\tau(X_n)$ of all n-ary terms of type τ is closed under the superposition operation $S^n := S^n_n$, there is a homogeneous analogue of this structure. The algebra $(W_\tau(X_n); S^n, x_1, \ldots, x_n)$ is an algebra of type $(n+1, 0, \ldots, 0)$, which still satisfies the clone axioms above for the case that $p = m = n$. Such an algebra is called a *unitary Menger algebra of rank* n (see [22]).

Let *n-clone*$(\tau) := (W_\tau(X_n); S^n)$ be the reduct of the unitary Menger algebra $(W_\tau(X_n); S^n, x_1, \ldots, x_n)$ of rank n. The algebra *n-clone*(τ) is called a *Menger algebra of rank n.*

If we consider the sequence $(W_\tau(X_n))_{n>0}$ together with the sequence of operations $(S^n_m)_{m,n>0}$, we obtain a heterogeneous algebra $((W_\tau(X_n))_{n>0}; (S^n_m)_{m,n>0})$ which we denote by $Menger(\tau)$. This heterogeneous algebra is called a *Menger system* (see [22]).

2. T- Identities

In [9] the authors studied the algebra $(W^{nf}_\tau(X_n); S^n)$, the algebra of n-full terms of type τ and in [6], [14] the algebras of strongly full terms and of k-normal n-ary terms are studied. All of them are subalgebras of $(W_\tau(X_n); S^n)$. Now we generalize these results to an arbitrary subalgebra $\mathcal{T} = (\underline{T}; (S^n_m)_{m,n>0})$ with $\underline{T} := (T^{(n)})_{n>0}$ and $T := \cup_{n>0} T^{(n)}$ of the heterogeneous algebra $Menger(\tau)$. For any variety V of type τ, we define $Id^T_n V := \{s \approx t \in IdV \mid s, t \in T^{(n)}\}$, that is $Id^T_n V := IdV \cap T^{(n)} \times T^{(n)}$ for every $n > 0$, where IdV is the set of all identities of V, and then $\underline{Id^T V} := (Id^T_n V)_{n>0}$ is called the sequence of all T-identities of V. Then $Id^T V = \cup_{n>0} Id^T_n V$. We will also use the notation $\underline{IdV} := (Id_n V)_{n>0}$, where IdV is the union of the sets $Id_n V$. *i.e.* $IdV = \cup_{n>0} Id_n V$.

Now we recall the following well-known facts:

Lemma 2.1. *For any variety V of type τ, $\underline{IdV}$ is a congruence on the algebra $Menger(\tau)$.*

Proof. This follows from the fact that IdV is a fully invariant congruence on the absolutely free algebra $\mathcal{F}_\tau(X)$. ■

Now we consider subalgebras $\mathcal{T}$ of the algebra $Menger(\tau)$ and will prove that the set of all T-identities of a variety V is a congruence on $\mathcal{T}$. We can use that $\underline{IdV} = (Id_n V)_{n>0}$ is a congruence on $Menger(\tau)$ and the well-known fact that for a congruence θ on an algebra $\mathcal{B}$, and for a subalgebra $\mathcal{A} \subseteq \mathcal{B}$, the relation $\theta_A := \theta \cap (A \times A)$ is a congruence on $\mathcal{A}$. Then we have

Theorem 2.2. *For a subalgebra $\mathcal{T} = ((T^{(n)})_{n>0}; (S^n_m)_{m,n>0})$ of the heterogeneous algebra $Menger(\tau)$ and for a variety V of type τ, the sequence $\underline{Id^T V}$ is a congruence on $\mathcal{T}$.* ■

Because of the previous theorem, we can define the quotient algebra $\mathcal{T}/\underline{Id^T V}$ which we denote by $clone_T(V)$.

If $\mathcal{A}$ is an algebra of type τ and if $s \approx t$ is an equation consisting of terms of type τ, then $\mathcal{A} \models s \approx t$ means that $s \approx t$ is satisfied as an identity in $\mathcal{A}$. Let T be a subset of $W_\tau(X)$ and let R_T be the relation between Alg(τ), the set of all algebras of type τ, and T^2, which is defined by

$$R_T := \{(\mathcal{A}, s \approx t) \mid \mathcal{A} \in Alg(\tau), s, t \in T \quad (\mathcal{A} \models s \approx t)\}.$$

This relation induces a Galois connection (Mod^T, Id^T) between $Alg(\tau)$ and T^2 where the operations Mod^T, Id^T are defined as follows: For $K \subseteq Alg(\tau)$ and for $\Sigma \subseteq T^2$,

$$Id^T(K) = \{s \approx t \in T^2 \mid \forall \mathcal{A} \in K(\mathcal{A} \models s \approx t)\} \quad \text{and}$$

$$Mod^T(\Sigma) = \{\mathcal{A} \in Alg(\tau) \mid \forall s \approx t \in \Sigma(\mathcal{A} \models s \approx t)\}.$$

Clearly, the operator Mod^T is the restriction of the usual operator Mod to T^2. From the properties of a Galois connection we obtain that the products $Mod^T Id^T$ and $Id^T Mod^T$ are closure operators on the power set of $Alg(\tau)$ and of T^2, respectively.

Now we consider the variety $T(V) := Mod^T Id^T V$ for a given variety V and $Id^T V := IdV \cap T^2$. It is clear that if $IdV \subset T^2$, then $T(V) = V$. Since T^2 must not be an equational theory in general, the converse is not true.

Proposition 2.3. *Let T be a subset of $W_\tau(X)$ and let $\mathcal{L}(\tau)$ be the lattice of all varieties of type τ. Then the operator $C_T : \mathcal{L}(\tau) \to \mathcal{L}(\tau)$ defined by $C_T(V) = T(V)$ is a closure operator.*

Proof. From $Id^T V \subseteq IdV$ there follows $V = Mod\ Id\ V \subseteq Mod^T Id^T V = T(V) = C_T(V)$. Using the fact that $Mod^T Id^T$ is a closure operator, we obtain $C_T(C_T(V)) = T(T(V)) = Mod^T Id^T (Mod^T Id^T V) = Mod^T Id^T V = T(V) = C_T(V)$. Finally, from $V_1 \subseteq V_2$, we have $C_T(V_1) = T(V_1) = Mod^T Id^T V_1 = Mod^T(IdV_1 \cap T^2) \subseteq Mod^T(IdV_2 \cap T^2) = T(V_2) = C_T(V_2)$. Altogether, we obtain that C_T is a closure operator. ∎

The set of all fixed points of C_T forms a sublattice of the lattice $\mathcal{L}(\tau)$, in fact it is a complete lattice (see [12]). Now we are interested in the variety $Mod(T \times T)$.

Definition 2.4. Let T be a subset of $W_\tau(X)$. An algebra $\mathcal{A}$ of type τ is called T-quasi constant algebra if there exists a term $t_0 \in T$ such that $t^{\mathcal{A}} = t_0^{\mathcal{A}}$ for all $t \in T$.

Let TQ be the class of all T-quasi constant algebras of type τ. This definition generalizes the concept of a constant algebra introduced in [3], and that of a quasi-constant algebra introduced in [9].

Proposition 2.5. *Let T be a subset of $W_\tau(X)$. Then $Mod(T \times T) = TQ$.*

Proof. Let $\mathcal{A} \in TQ$. Then there exists a term $t_0 \in T$ such that $t^{\mathcal{A}} = t_0^{\mathcal{A}}$ for all $t \in T$. Let t_1, t_2 be arbitrary terms in T. Then $t_1^{\mathcal{A}} = t_0^{\mathcal{A}} = t_2^{\mathcal{A}}$. This means $\mathcal{A} \models t_1 \approx t_2$ and hence $\mathcal{A} \in Mod(T \times T)$. Conversely, let $\mathcal{A} \in Mod(T \times T)$. Then $t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$ for all $t_1, t_2 \in T$. Therefore, $\mathcal{A} \in TQ$. ∎

Corollary 2.6. *For any variety V, we have* $T(V) = V \vee TQ$.

Proof. $T(V) = Mod^T Id^T V = Mod(IdV \cap T \times T) = ModIdV \vee Mod(T \times T) = V \vee TQ$. ∎

We notice that a similar approach is contained in [13].

3. T-hypersubstitutions and T-hyperidentities

To study the identities in the algebra $\mathcal{T} = ((T^{(n)})_{n>0}; (S^n_m)_{m,n>0})$, we need the concepts of T-hypersubstitutions and T-hyperidentities.

A hypersubstitution σ of type τ is a mapping which assigns to each operation symbol f_i of type τ an n_i-ary term $\sigma(f_i)$ of type τ. Any hypersubstitution σ induces a mapping $\hat{\sigma}$ on the set $W_\tau(X)$ of all terms of type τ, given by the following inductive definition:

(i) $\hat{\sigma}[x_j] := x_j$, if $x_j \in X$ is a variable,

(ii) $\hat{\sigma}[f_i(t_1,\ldots,t_{n_i})] := S_n^{n_i}(\sigma(f_i),\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_{n_i}])$,

for compound terms $f_i(t_1,\ldots,t_{n_i})$.

Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ. A binary operation $\circ_h$ can be defined on this set, by $\sigma_1 \circ_h \sigma_2 = \hat{\sigma}_1 \circ \sigma_2$, where $\circ$ is the usual composition of mappings. It is well-known that $(Hyp(\tau);\circ_h,\sigma_{id})$ is a monoid, where σ_{id} is the identity hypersubstitution which is defined by $\sigma_{id}(f_i) = f_i(x_1,\ldots,x_{n_i})$ for all $i \in I$.

Let $\mathcal{T} = ((T^{(n)})_{n>0};(S_m^n)_{m,n>0})$ be a subalgebra of the algebra $Menger(\tau)$. We define $Hyp^T(\tau)$, the set of all T-hypersubstitutions as follows;

$$Hyp^T(\tau) = \{\sigma \in Hyp(\tau) \mid \ \forall\, i \in I \ \ \sigma(f_i) \in T \text{ and } \forall\, t \in T \ \ \hat{\sigma}[t] \in T\}.$$

Then we get:

Proposition 3.1. *Let $\mathcal{T}$ be a subalgebra of the algebra $Menger(\tau)$. Then $(Hyp^T(\tau);\circ_h)$ is a subsemigroup of the semigroup $(Hyp(\tau);\circ_h)$. Moreover, $(Hyp^T(\tau)\cup\{\sigma_{id}\};\circ_h,\sigma_{id})$ is a submonoid of $(Hyp(\tau);\circ_h,\sigma_{id})$.* ■

Definition 3.2. Let $\mathcal{T}$ be a subalgebra of $Menger(\tau)$ and let V be a variety of type τ and Id^TV be the set of all identities of V consisting of terms from T, i.e., $Id^TV = IdV \cap T^2$. Then $s \approx t \in Id^TV$ is called a *T-hyperidentity* in V if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^TV$ for all $\sigma \in Hyp^T(\tau)$. If every identity in Id^TV is satisfied as a T-hyperidentity, then the variety V is called *$Hyp^T(\tau)$-solid*; for short, we will write *T-solid*.

4. T-hypersubstitutions and endomorphisms of $\mathcal{T}$

There is a close connection between extensions of T-hypersubstitutions and endomorphisms of $\mathcal{T}$. This connection will be used later on to describe identities in the quotient algebra $\mathcal{T}/\underline{Id^TV}$.

For a hypersubstitution σ in $Hyp(\tau)$, it is well-known that the induced mapping $\hat{\sigma}$, regarded as a sequence $\underline{\hat{\sigma}} := (\hat{\sigma}^{(n)})_{n>0}$ with

$$\hat{\sigma}^{(n)} : W_\tau(X_n) \to W_\tau(X_n)$$

is an endomorphism on $clone(\tau)$. If we apply $\hat{\sigma}$ on a subalgebra $\mathcal{T}$, instead of $\hat{\sigma}/_T$ we will simply write $\hat{\sigma}$. Consequently, we have:

Theorem 4.1. *For any hypersubstitution $\sigma \in Hyp^T(\tau)$, the sequence $\underline{\hat{\sigma}}$ is an endomorphism on the algebra $\mathcal{T}$.*

As a consequence of Theorem 4.1, the set $Im(\hat{\sigma}) := \{\hat{\sigma}[t] \mid t \in W_\tau(X)\}$ is the universe of a subalgebra of $\mathcal{T}$.

Since it is not clear that every subalgebra $\mathcal{T}$ has an independent generating set, we assume in addition that the algebra $\mathcal{T}$ has an independent generating set $\underline{G} := (G^{(n)})_{n>0}$. That is, $\mathcal{T}$ is free with respect to itself, freely generated by the set $\underline{G}$. Then any substitution $\underline{\eta} := (\eta^{(n)})_{n>0}$ from $\underline{G}$ into $\underline{T}$ with $\eta^{(n)} : G^{(n)} \to T^{(n)}$ for all $n > 0$ can be uniquely extended to an endomorphism $\underline{\overline{\eta}} := (\overline{\eta}^{(n)} : T^{(n)} \to T^{(n)})_{n>0}$ of $\mathcal{T}$. Such mappings are called *$\underline{T}$-substitutions with respect to $\underline{G}$*. Let $Subst_{\underline{G}}(\underline{T})$ be the set of all such $\underline{T}$-substitutions with respect to $\underline{G}$. Together with a binary composition $\odot$ defined by $\underline{\eta_1} \odot \underline{\eta_2} := (\eta_1^{(n)} \odot \eta_2^{(n)})_{n>0} := (\overline{\eta_1}^{(n)} \circ \eta_2^{(n)})_{n>0}$, where $\circ$ is the usual composition of functions, $(Subst_{\underline{G}}(\underline{T}); \odot)$ is a semigroup. In fact, together with the identity mapping $\underline{id_G}$ it is a monoid. Let $End(\mathcal{T})$ be the monoid of all endomorphisms of the algebra $\mathcal{T}$. In the next theorems, we describe the connection between the monoids $Subst_{\underline{G}}(\underline{T})$, $End(\mathcal{T})$ and $Hyp^T(\tau) \cup \{\sigma_{id}\}$.

Theorem 4.2. *Let $\mathcal{T}$ be a subalgebra of $Menger(\tau)$ and $\underline{G}$ be an independent generating system of $\mathcal{T}$. Then the monoids $Subst_{\underline{G}}(\underline{T})$ and $End(\mathcal{T})$ are isomorphic.*

Proof. We define a heterogeneous mapping $\underline{\psi} : Subst_{\underline{G}}(\underline{T}) \to End(\mathcal{T})$ by $\underline{\psi}(\underline{\eta}) = \underline{\overline{\eta}}$ for $\underline{\eta} \in Subst_{\underline{G}}(\underline{T})$. Clearly, $\underline{\psi}$ is well-defined since $\underline{\overline{\eta}}$ is uniquely determined by $\underline{\eta}$. For any $\underline{\eta_1}, \underline{\eta_2} \in Subst_{\underline{G}}(\underline{T})$, we have

$$\underline{\psi}(\underline{\eta_1} \odot \underline{\eta_2}) = \overline{\underline{\eta_1} \odot \underline{\eta_2}} = \overline{\underline{\overline{\eta}_1} \circ \underline{\eta_2}} = \underline{\overline{\eta}_1} \circ \underline{\overline{\eta}_2} = \underline{\psi}(\underline{\eta_1}) \circ \underline{\psi}(\underline{\eta_2}),$$

since $(\underline{\overline{\eta}_1} \circ \underline{\eta_2})|_{\underline{G}} = \underline{\overline{\eta}_1} \circ \underline{\overline{\eta}_2}|_{\underline{G}}$ and using the uniqueness of $\overline{\underline{\overline{\eta}_1} \circ \underline{\eta_2}}$. Therefore, $\underline{\psi}$ is a homomorphism. For injectivity, let $\underline{\eta_1}, \underline{\eta_2} \in Subst_{\underline{G}}(\underline{T})$ such that $\underline{\psi}(\underline{\eta_1}) = \underline{\psi}(\underline{\eta_2})$. Then $\underline{\overline{\eta}_1} = \underline{\overline{\eta}_2}$ and so $\underline{\overline{\eta}_1}|_{\underline{G}} = \underline{\overline{\eta}_2}|_{\underline{G}}$. This means $\underline{\eta_1} = \underline{\eta_2}$. Thus, $\underline{\psi}$ is injective. Clearly, $\underline{\psi}$ is surjective since for any endomorphism $\underline{\eta}$ on $\mathcal{T}$ we have $\underline{\eta}|_{\underline{G}}$ is a $\underline{T}$-substitution. This proves the theorem. ∎

By Theorem 4.1, we may consider the set $\{\underline{\hat{\sigma}}/\mathcal{T} \mid \sigma \in Hyp^T(\tau)\} \cup \{\underline{\hat{\sigma}}_{id}/\mathcal{T}\}$. Clearly, this set forms a submonoid of the monoid $(End(\mathcal{T}); \circ_h, id_T)$ of all endomorphisms of $\mathcal{T}$.

5. T-hyperidentities and identities in $\mathcal{T}$

We recall that for a subalgebra $\mathcal{T} = ((T^{(n)})_{n>0}; (S^n_m)_{m,n>0})$ of the heterogenous algebra $Menger(\tau) = ((W_\tau(X_n))_{n>0}; (S^n_m)_{m,n>0})$ and for any variety V of type τ, by Theorem 2.2, the set of all T-identities of V, $\underline{Id^T V}$, is a congruence on the algebra $\mathcal{T}$. This allows us to define $clone_T(V) = \mathcal{T}/\underline{Id^T V}$.

Now we will use the following "translation mechanism" between elements of $\mathcal{T}$ and elements of a subalgebra of the absolutely free algebra $\mathcal{F}_\tau(X)$. The components of the sequence from the generating system $\underline{G}$ form a set G of terms of type τ and the elements of $\underline{T}$ produced by application of the operation $S^{n_i}_m$ from $\underline{G}$ correspond to elements of $W_\tau(X)$ which arise by application of the operations $\overline{f}_i$ to elements from G. This gives a one-to-one mapping φ (see [18]) between terms of type τ and so-called operator terms formulated in the language of the heterogeneous algebra $Menger(\tau)$.

To consider identities in $clone_\tau(V)$, we need to build up the free heterogeneous algebra in a variety defined by (C_1) generated by the new variable system $\underline{G}^*$ which has the same cardinality as the system $\underline{G}$. This free heterogeneous algebra is denoted by $\mathcal{F}_{\tau^*}(G^*)$. This gives a one-to-one mapping φ from the system $\underline{G}$ onto the system $\underline{G}^*$. The extension of this mapping assigns to arbitrary elements from T the corresponding terms over the heterogeneous algebra $\mathcal{T}$.

Theorem 5.1. *Let $\mathcal{T}$ be a subalgebra of the algebra $Menger(\tau)$ which has an independent generating system $\underline{G}$, let V be a variety of type τ and let $s \approx t \in Id^T V$. If $\varphi(s) \approx \varphi(t)$ is an identity in $clone_T(V)$, then it is a T-hyperidentity in V. (That is $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity in V for all $\sigma \in Hyp^T(\tau)$.)*

Proof. Let $\varphi(s) \approx \varphi(t)$ be an identity in $clone_T(V)$. Then for every valuation ν we have $\overline{\nu}(\varphi(s)) = \overline{\nu}(\varphi(t))$. The composition

$$\underline{nat\ Id^T V \circ \hat{\sigma} \circ \varphi^{-1}}$$

is the extension of a valuation mapping into $clone_\tau(V)$, and so we have

$$\varphi(s) \approx \varphi(t) \in Id(clone_T(V)) \Rightarrow \overline{\underline{(natId^T V \circ \hat{\sigma} \circ \varphi^{-1})}}\varphi(s)$$

$$= \overline{\underline{(natId^T V \circ \hat{\sigma} \circ \varphi^{-1})}}\varphi(t)$$

$$\Rightarrow \underline{natId^T V \circ \hat{\sigma}}(s) = \underline{natId^T V \circ \hat{\sigma}}(t)$$

$$\Rightarrow \left[\hat{\sigma}^{(n)}[s]\right]_{Id_n^T V} = \left[\hat{\sigma}^{(n)}[t]\right]_{Id_n^T V} \quad \text{for every } n > 0$$

$$\Rightarrow \hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^T V$$

for every $\sigma \in Hyp^T(\tau)$. Therefore $s \approx t$ is satisfied as a T-hyperidentity in V. ■

Definition 5.2. Let T be a nonempty subset of the universe of the algebra $Menger(\tau)$. We call T a *j-ideal* of $Menger(\tau)$ if there is an integer j with $1 \leq j \leq n+1$ and for any terms $t_1, t_2, \ldots, t_{n+1}$, such that $t_j \in T$, imply $S_m^n(t_1, \ldots, t_j, \ldots, t_{n+1}) \in T$. A set T is called an *ideal* if it is a j-ideal for all $1 \leq j \leq n+1$ and for all n.

It is clear that every j-ideal is a subalgebra of $Menger(\tau)$.

Let $\tau = (1)$ and f be a unary operation symbol. We consider the algebra $(W_{(1)}(X_1); S_1^1)$. It is easy to see that the set $N^k := \{t \in W_{(1)}(X_1) \mid op(t) \geq k\}$ is an ideal of the algebra $(W_{(1)}(X_1); S_1^1)$. ($op(t)$ is the number of occurrences of the operation symbol f in the term t.)

Theorem 5.3. *Let T be an ideal of the algebra $Menger(\tau)$ which has an independent generating system G. Then $T \times T \cup \Delta_{W_\tau(X)}$ is a fully invariant congruence on the absolutely free algebra $\mathcal{F}_\tau(X)$.*

Proof. It is clear that $T \times T \cup \Delta_{W_\tau(X)} := \rho$ is an equivalence relation on $W_\tau(X)$. To prove the compatibility, we let $(s_1, t_1), \ldots, (s_{n_i}, t_{n_i}) \in \rho$. If there exists $s_i \in T$ for some $1 \leq i \leq n_i$, then t_i is also in T and we obtain

$\overline{f}(s_1, \dots, s_{n_i}) = S_n^{n_i}(f(x_1, \dots, x_{n_i}), s_1, \dots, s_{n_i}) \in T$ and similarly, we also have $\overline{f}(t_1, \dots, t_{n_i}) = S_n^{n_i}(f(x_1, \dots, x_{n_i}), t_1, \dots, t_{n_i}) \in T$, since T is an ideal of $Menger(\tau)$. If $s_i \notin T$ for all $1 \le i \le n_i$, then $s_i = t_i$ for all i, and therefore $\overline{f}(s_1, \dots, s_{n_i}) = \overline{f}(t_1, \dots, t_{n_i})$. Hence $(\overline{f}(s_1, \dots, s_{n_i}), \overline{f}(t_1, \dots, t_{n_i})) \in \rho$.

Next we will show that ρ is closed under any endomorphism on $\mathcal{F}_\tau(X)$. Let $(s,t) \in \rho$ and let φ be any endomorphism on $\mathcal{F}_\tau(X)$. If $s = t$, then clearly $\varphi(s) = \varphi(t)$, and so $(\varphi(s), \varphi(t)) \in \rho$. In the case $s, t \in T$ it is easy to see that $\varphi(s) = S_m^n(s, s_1, s_2, \dots, s_n)$ and $\varphi(t) = S_m^n(t, s_1, s_2, \dots, s_n)$, where $s_i = \varphi(x_i)$ for all $1 \le i \le n$ and x_i are variables occurring in terms s and t. Since T is an ideal and $s, t \in T$, we have $S_m^n(s, s_1, s_2, \dots, s_n)$ and $S_m^n(t, s_1, s_2, \dots, s_n)$ belongs to T. Thus $(\varphi(s), \varphi(t)) \in \rho$. ∎

Let V be a variety V of type τ. If $IdV \cap T^2$ is closed under all endomorphisms of $Menger(\tau)$, then, by Theorem 4.1, it is closed under $\hat{\sigma}$ for any T-hypersubstitution σ, and therefore the variety $T(V) = Mod(IdV \cap T^2)$ is T-solid. Then we have

Proposition 5.4. *Let V be a variety of type τ. If $Id^T V$ is a fully invariant congruence on $Menger(\tau)$, then the variety $T(V)$ is T-solid.*

References

[1] G. Birkhoff and J. D. Lipson, *Heterogeneous algebra,* J. Combin. Theory **8** (1970), 115–133.

[2] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra,* Springer-Verlag, New York 1981.

[3] I. Chajda, *Normally presented varieties,* Algebra Universalis **34** (1995), 327–335.

[4] I. Chajda, K. Denecke and S. L. Wismath, *A characterization of P-compatible varieties,* Preprint 2004.

[5] W. Chromik, *Externally compatible identities of algebras,* Demonstratio Math. **23** (1990), 345–355.

[6] K. Denecke and L. Freiberg, *The algebra of full terms*, preprint 2003.

[7] K. Denecke and K. Hałkowska, *P-compatible hypersubstitutions and M_P-solid varieties,* Studia Logica **64** (2000), 355–363.

[8] K. Denecke, P. Jampachon, *Clones of N-full terms,* Algebra and Discrete Math. (2004), no. **4**, 1–11.

[9] K. Denecke, P. Jampachon, *N-Full varieties and clones of n-full terms,* Southeast Asian Bull. Math. **25** (2005), 1–14.

[10] K. Denecke, P. Jampachon and S. L. Wismath, *Clones of n-ary algebras,* J. Appl. Algebra Discrete Struct. **1** (2003), 141–158.

[11] K. Denecke, M. Reichel, *Monoids of hypersubstitutions and M-solid varieties*, Contributions to General Algebra **9** (1995), 117–126.

[12] K. Denecke and S. L. Wismath, *Universal Algebra and Applications in Theoretical Computer Science*, Chapman & Hall/CRC, Boca Raton, London, New York, Washington D.C. 2002.

[13] K. Denecke and S. L. Wismath, *Galois connections and complete sublattices*, p. 211–230 in: "*Galois Connections and Applications*", Kluwer Academic Publ., Dordrecht 2004.

[14] K. Denecke and S. L. Wismath, *A characterization of k-normal varieties*, Algebra Universalis **51** (2004), 395–409.

[15] E. Graczyńska, *On Normal and regular identities and hyperidentities*, p. 107–135 in: "*Universal and Applied Algebra*", World Scientific Publ. Co., Singapore 1989.

[16] P.J. Higgins, *Algebras with a scheme of operators,* Math. Nachr. **27** (1963), 115–132.

[17] A. Hiller, *P-Compatible Hypersubstitutionen und Hyperidentitäten,* Diplomarbeit, Potsdam 1996.

[18] H.-J. Hoehnke and J. Schreckenberger, *Partial Algebras and their Theories,* Manuscript 2005.

[19] I.I. Melnik, *Nilpotent shifts of varieties*, (Russian), Mat. Zametki **14** (1973), 703–712, (English translation in: Math. Notes **14** (1973), 962–966).

[20] J. Płonka, *On varieties of algebras defined by identities of special forms*, Houston Math. J. **14** (1988), 253–263.

[21] J. Płonka, *P-compatible identities and their applications to classical algebras,* Math. Slovaca **40** (1990), 21–30.

[22] B. Schein and V. S. Trochimenko, *Algebras of multiplace functions,* Semigroup Forum **17** (1979), 1–64.

[23] W. Taylor, *Hyperidentities and Hypervarieties*, Aequat. Math. **23** (1981), 111–127.

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