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EMBEDDINGS OF CHAINS INTO CHAINS

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Abstract

Continuity of isotone mappings and embeddings of a chain G into another chain are studied. Especially, conditions are found under which the set of points of discontinuity of such a mapping is dense in G.

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1. Introduction

If G is a set, then $|G|$ denotes its cardinality; a set G will be called *nontrivial* if $|G| \geq 2$. Let G be a chain and $x, y \in G$; the symbol $x \prec y$ means that x, y are *consecutive elements* in G, i.e., $x < y$ and $x < z < y$ for no $z \in G$. In this case we will also say that x is an *up-isolated element* and y is *down-isolated element* in G. An element $x \in G$ is *isolated* if it is up-isolated or down-isolated.

Let G be a chain, H a (partially) ordered set. A mapping $f : G \to H$ is *isotone* if $x, y \in G, x < y$ implies $f(x) \leq f(y)$; it is an *embedding* if it is isotone and injective. A surjective embedding $f : G \to H$ is clearly an isomorphism of G onto H.

If G and H are disjoint chains (or, more generally, ordered sets), then $G \oplus H$ denotes its *ordinal sum.* Also, if G is a chain and $\{H_x : x \in G\}$ is a system of pairwise disjoint chains, then $\bigoplus_{x \in G} H_x$ denotes the ordinal sum of chains H_x over G. Moreover, $G \circ H$ is the *ordinal product* of chains G, H.

Recall that a *cut* of a chain G is a couple $[A, B]$ of its nonvoid subsets such that $G = A \oplus B$. The terms *jump* and *gap* will be used in the usual sense. Note that a chain G contains no jumps if it has no isolated elements and it contains no gaps if it is a conditionally complete lattice, i.e., any nonvoid up-bounded subset of G has the supremum in G and any nonvoid down-bounded subset of G has the infimum in G.

Let G be a chain and $a, b \in G, a \leq b$. We denote $\langle a, b\rangle$ the *closed interval* with end-points a, b, i.e., $\langle a, b\rangle = \{x \in G:\ a \leq x \leq b\}$. Similarly, if $a < b$, then $(a, b) = \{x \in G:\ a < x < b\}$ is the *open interval* with end-points a, b. Symbols $\langle a, b)$ and $(a, b\rangle$ have the obvious meaning. Also, $(a, \infty) = \{x \in G:\ x > a\}$, $(-\infty, a) = \{x \in G:\ x < a\}$ and $(-\infty, \infty) = G$ are open intervals. More generally, by an *interval* in a chain G we mean any subset $I \subseteq G$ having the property $a, b \in I, a < b \Rightarrow \langle a, b\rangle \subseteq I$.

Let $\mathcal{S}$ be any system of nonvoid subsets of a chain G. We define a relation $\prec$ on $\mathcal{S}$ by setting $A \prec B \Leftrightarrow a < b$ for all $a \in A$ and all $b \in B$. Trivially, this relation is a (partial) order on $\mathcal{S}$; we will call it a *natural order* on $\mathcal{S}$. If elements of $\mathcal{S}$ are pairwise disjoint intervals, then $\prec$ is a linear order on $\mathcal{S}$. Especially, if $A = \langle a_1, a_2\rangle$, $B = \langle b_1, b_2\rangle$ are closed intervals in G, then $A \prec B$ is equivalent to $a_2 < b_1$.

In following chapters we assume that two nontrivial chains G and L are given where L contains no gaps (and thus it is a conditionally complete lattice).

2. Isotone mappings and embeddings

Let $\varphi : G \to L$ be an isotone mapping. Put for any $x \in G$

$$a_\varphi(x) = \begin{cases} \sup\{\varphi(t):\ t \in G, t < x\}, & \text{if } x \text{ is not the least element in } G, \\ \varphi(x), & \text{if } x \text{ is the least element in } G; \end{cases}$$

$$b_\varphi(x) = \begin{cases} \inf\{\varphi(t):\ t \in G, t > x\}, & \text{if } x \text{ is not the greatest element in } G, \\ \varphi(x), & \text{if } x \text{ is the greatest element in } G. \end{cases}$$

In the sequel, symbols $a_\varphi(x)$, $b_\varphi(x)$ always have this meaning.

Lemma 2.1. *Let $\varphi : G \to L$ be an isotone mapping. Then $a_\varphi(x) \leq \varphi(x) \leq b_\varphi(x)$ for any $x \in G$.*

Proof. This is clear and follows from the definition. ∎

Further, the set of all closed intervals in L is denoted by $\mathcal{L}$; $(\mathcal{L}, \prec)$ is thus an ordered set. If $\varphi : G \to L$ is an isotone mapping, then we define a mapping $f[\varphi] : G \to \mathcal{L}$ by $f[\varphi](x) = \langle a_\varphi(x), b_\varphi(x)\rangle$.

Lemma 2.2. *Let $\varphi : G \to L$ be an isotone mapping. Then it holds:*

(i) *If $f[\varphi]$ is an embedding of $(G, <)$ into $(\mathcal{L}, \prec)$, then φ is an embedding of $(G, <)$ into $(L, <)$.*

(ii) *If G contains no jumps and if φ is an embedding of $(G, <)$ into $(L, <)$, then $f[\varphi]$ is an embedding of $(G, <)$ into $(\mathcal{L}, \prec)$.*

Proof.

(i) Suppose that $f[\varphi]$ is an embedding of $(G, <)$ into $(\mathcal{L}, \prec)$ and let $x < y$ for some $x, y \in G$, . Then $f[\varphi](x) \prec f[\varphi](y)$, i.e., $b_\varphi(x) < a_\varphi(y)$. As $\varphi(x) \leq b_\varphi(x)$, $\varphi(y) \geq a_\varphi(y)$, we have $\varphi(x) < \varphi(y)$ and φ is an embedding of $(G, <)$ into $(L, <)$.

(ii) Let G contain no jumps and let $\varphi : G \to L$ be an embedding. Take any $x, y \in G, x < y$; then there exist $x_0, y_0 \in G$ with $x < x_0 < y_0 < y$. From this $\varphi(x) < \varphi(x_0) < \varphi(y_0) < \varphi(y)$. From definition of $b_\varphi(x), a_\varphi(y)$, it follows $b_\varphi(x) \leq \varphi(x_0), a_\varphi(y) \geq \varphi(y_0)$. Thus, $b_\varphi(x) < a_\varphi(y)$ implies $f[\varphi](x) = \langle a_\varphi(x), b_\varphi(x)\rangle \prec \langle a_\varphi(y), b_\varphi(y)\rangle = f[\varphi](y)$ and $f[\varphi]$ is an embedding of $(G, <)$ into $(\mathcal{L}, \prec)$.

∎

Therefore, we have

Corollary 2.3. *Let $\varphi : G \to L$ be an isotone mapping and let G contain no jumps. Then the following statements are equivalent:*

(i) *φ is an embedding of $(G, <)$ into $(L, <)$;*

(ii) *$f[\varphi]$ is an embedding of $(G, <)$ into $(\mathcal{L}, \prec)$.*

∎

Example 2.4. Denote $G = (0, 1) \subseteq \mathbb{R}$ with the natural ordering of reals and $L = G \circ \langle 0, 1\rangle$. Let $\varphi : G \to L$ be mapping such that $\varphi(x) = [x, \frac{1}{2}]$ for $x \in G$. Then φ is an embedding of G into L; as G contains no jumps, $f[\varphi]$ is an embedding of G into $(\mathcal{L}, \prec)$. Clearly, we have $a_\varphi(x) = [x, 0]$, $b_\varphi(x) = [x, 1]$ for any $x \in G$; thus, $f[\varphi](x) = \langle [x, 0], [x, 1]\rangle$.

We can also prove

Lemma 2.5. *Let $\varphi : G \to L$ be an embedding. Then G contains no jumps iff $f[\varphi]$ is an embedding of $(G, <)$ into $(\mathcal{L}, \prec)$.*

Proof. If G contains no jumps, then $f[\varphi]$ is embedding by 2.2.(ii). On the other hand, suppose that G contains consecutive elements $x \prec\!\!\!\!- y$. Then $\varphi(x) < \varphi(y)$ in L and, trivially, $b_\varphi(x) = \varphi(y), a_\varphi(y) = \varphi(x)$. Thus, $a_\varphi(y) < b_\varphi(x)$ and the relation $f[\varphi](x) \prec f[\varphi](y)$ is not valid. Hence, $f[\varphi]$ is not an embedding of $(G, <)$ into $(\mathcal{L}, \prec)$. ∎

Theorem 2.6. *Let $\varphi : G \to L$ be an isotone mapping and let G contain no gaps. Then $\bigcup f[\varphi](G)$ is an interval in L.*

Proof. Take any $a, b \in \bigcup f[\varphi](G)$ and suppose the existence of an element $c \in L$ such that $a < c < b$, $c \notin \bigcup f[\varphi](G)$. Denote $A = \{t \in G : \ b_\varphi(t) < c\}$ and $B = \{t \in G : \ a_\varphi(t) > c\}$; we show that $[A, B]$ is a cut in G. As $a \in \bigcup f[\varphi](G)$, there exists $x \in G$ such that $a \in f[\varphi](x)$, i.e., $a_\varphi(x) \leq a \leq b_\varphi(x)$ and $a_\varphi(x) < c$. Then necessarily $b_\varphi(x) < c$; otherwise $c \in \langle a_\varphi(x), b_\varphi(x)\rangle \subseteq \bigcup f[\varphi](G)$, a contradiction. From this $x \in A$ and $A \neq \emptyset$. Similarly, we can prove $B \neq \emptyset$. Let $x \in A$ and $y \in B$. Then $b_\varphi(x) < c$ and $a_\varphi(y) > c$ so that $b_\varphi(x) < a_\varphi(y)$. As $\varphi(x) \leq b_\varphi(x)$ and $\varphi(y) \geq a_\varphi(y)$, we have $\varphi(x) < \varphi(y)$. As φ is isotone, this implies $x < y$ and we have proved $A \prec B$. Let $x \in G$ be any element. Then $c \notin f[\varphi](x) = \langle a_\varphi(x), b_\varphi(x)\rangle$. Thus either $c > b_\varphi(x)$ or $c < a_\varphi(x)$, i.e., either $x \in A$ or $x \in B$. We have shown $A \cup B = G$ and $[A, B]$ is a cut in G. As G contains no gaps, there exists $\max(A)$ or $\min(B)$ in G. Suppose the existence of $\max(A) = z$. Then $b_\varphi(z) < c$ by definition of A; at the same time $c \leq \inf(\{a_\varphi(t) : \ t \in B\}) = \inf(\{a_\varphi(t) : \ t \in G, t > z\}) \leq \inf(\{\varphi(t) : \ t \in G, t > z\}) = b_\varphi(z)$, a contradiction. If B has the minimum in G, then the conclusion is similar. ∎

J. Novák proved in [5] that if H is a chain without jumps and gaps and if $\mathcal{H}$ is a system of its nonvoid pairwise disjoint closed intervals such that $\bigcup \mathcal{H} = H$, then $(\mathcal{H}, \prec)$ is a chain without jumps and gaps. We prove a more general assertion.

Lemma 2.7. *Let $\mathcal{L}_0$ be a system of nonvoid pairwise disjoint intervals in L such that $\bigcup \mathcal{L}_0$ is an interval in L. Then the chain $(\mathcal{L}_0, \prec)$ contains no gaps.*

Proof. Denote $\bigcup \mathcal{L}_0 = I$; thus I is an interval in L. Suppose that $(\mathcal{L}_0, \prec)$ contains a cut $[\mathcal{A}, \mathcal{B}]$ which is a gap. Put $A = \bigcup \mathcal{A}$ and $B = \bigcup \mathcal{B}$. Then $[A, B]$ is a cut in I. As $(\mathcal{A}, \prec)$ does not contain the greatest element, $(A, <)$ does not contain the greatest element. Similarly, $(B, <)$ does not contain the least element. The cut $[A, B]$ in I is thus a gap which is impossible. ∎

Theorem 2.8. *Let G contain no jumps and let $\varphi : G \to L$ be an embedding. Then the following statements are equivalent:*

(i) *G contains no gaps;*

(ii) *In L there exists a system $\mathcal{L}_G$ of pairwise disjoint closed intervals such that $\bigcup \mathcal{L}_G$ is an interval in L and that $(G, <)$ and $(\mathcal{L}_G, \prec)$ are isomorphic.*

Proof. (i) implies (ii) by Lemma 2.2 and Theorem 2.6. On the other hand, (ii) implies (i) by Lemma 2.7. ∎

3. Continuity interval topology

Recall that the *interval topology* on a chain H is a topology the base of which is the system of all open intervals. In the following text, when speaking about a continuity of a mapping $\varphi : G \to L$, we mean the continuity of φ with respect to interval topology on G and on L.

Lemma 3.1. *Let $\varphi : G \to L$ be an isotone mapping and let $x \in G$. If $a_\varphi(x) = b_\varphi(x)$, then φ is continuous at x.*

Proof. Let $a_\varphi(x) = b_\varphi(x)$ so that $a_\varphi(x) = b_\varphi(x) = \varphi(x)$ and let U be any neighborhood of the element $\varphi(x)$ in L. Then there exists an open interval $I \subseteq U$ such that $\varphi(x) \in I$. Let, at first, $I = (a, b)$, where $a, b \in L$, $a < \varphi(x) < b$. Suppose that x is not an end-element in G. Then there exists $t_1 \in G$ with $t_1 < x$ such that $\varphi(t_1) > a$ (otherwise it would be $a_\varphi(x) \leq a$) and $t_2 \in G$ with $t_2 > x$ such that $\varphi(t_2) < b$. If we put $V = (t_1, t_2)$, then V is a neighborhood of x and $\varphi(V) \subseteq U$. If x is the least element in G, then it is not the greatest and we find $t_2 \in G$ with $t_2 > x$ such that $\varphi(t_2) < b$. Then $V = (-\infty, t_2) = \langle x, t_2)$ is a neighborhood of x with $\varphi(V) \subseteq U$. The case when x is the greatest element in G is similar.

Now, suppose $I = (a, \infty)$ so that $\varphi(x) > a$. If x is not the least element in G, then there exists $t_1 \in G$ with $t_1 < x$ such that $\varphi(t_1) > a$. We put $V = (t_1, \infty)$ and we have $\varphi(V) \subseteq U$. If x is the least element in G, it suffices to put $V = (-\infty, \infty)$. If $I = (-\infty, b)$, then the considerations are similar and case $I = (-\infty, \infty)$ is trivial. ■

Lemma 3.2. *Let $\varphi : G \to L$ be an isotone mapping, let $x \in G$ and let φ be continuous at x. If x is not up-isolated, then $b_\varphi(x) = \varphi(x)$; if x is not down-isolated, then $a_\varphi(x) = \varphi(x)$.*

Proof. Let x be not up-isolated. If x is the greatest element in G or $\varphi(x)$ is the greatest element in L, then, trivially, $b_\varphi(x) = \varphi(x)$. Suppose that x is not the greatest in G and $\varphi(x)$ is not the greatest in L. Choose arbitrarily $c \in L, c > \varphi(x)$. Then $U = (-\infty, c)$ is a neighborhood of $\varphi(x)$ in L; thus there exists a neighborhood V of x in G such that $\varphi(V) \subseteq U$. As x is not up-isolated, there exists $t \in V$ such that $x < t$. From this $\varphi(x) \leq \varphi(t)$ and $\varphi(t) \in U$, i.e., $\varphi(t) < c$. From the definition of $b_\varphi(x)$, we have $b_\varphi(x) \leq \varphi(t) < c$. Hence, $b_\varphi(x) < c$ for any $c \in L$, $c > \varphi(x)$ implies $b_\varphi(x) \leq \varphi(x)$. From this $b_\varphi(x) = \varphi(x)$. The assertion for $a_\varphi(x)$ can be proved similarly. ■

Taking into account Lemmas 3.1 and 3.2 we have:

Corollary 3.3. *Let $\varphi : G \to L$ be an isotone mapping and let $x \in G$ be not an isolated element. Then φ is continuous at x iff $a_\varphi(x) = b_\varphi(x)$.* ■

From the preceding Lemmas we get

Theorem 3.4. *Let G contain no jumps and let $\varphi : G \to L$ be an isotone mapping. Then φ is continuous precisely at those elements $x \in G$ for which $a_\varphi(x) = b_\varphi(x)$.* ■

Now we will investigate the problem of existence of an embedding $\varphi : G \to L$ with the prescribed set of points of discontinuity.

Lemma 3.5. *Let G contain no jumps and let $H \subseteq G$. Then the following statements are equivalent:*

(i) *there exists an embedding $\varphi : G \to L$ which is discontinuous at each element $x \in H$;*

(ii) *L contains a system $\mathcal{L}_G$ of pairwise disjoint closed intervals such that there exists an isomorphism $f : (G, <) \to (\mathcal{L}_G, \prec)$ with the property $x \in H \Rightarrow |f(x)| > 1$.*

Proof. (i) implies (ii) by Lemma 2.5. and Theorem 3.4. Suppose that (ii) holds and let $f(x) = \langle a(x), b(x) \rangle$ for $x \in G$. By the assumption, $a(x) < b(x)$ for $x \in H$. Let us define a mapping $\varphi : G \to L$ like this:

$$\varphi(x) = a(x) \quad \text{if } x \text{ is not the greatest element of } G,$$
$$\varphi(x) = b(x) \quad \text{if } x \text{ is the greatest element of } G.$$

Clearly, φ is an embedding of $(G, <)$ into $(L, <)$. Let $x \in H$ so that $a(x) < b(x)$. If x is not the greatest element in G, then $U = (-\infty, b(x))$ is a neighborhood of $a(x) = \varphi(x)$ in L. Let V be any neighborhood of x in G. As G contains no jumps, V contains an element $t > x$. Then $f(x) \prec f(t)$, i.e., $b(x) < a(t)$ and $\varphi(t) \geq a(t) > b(x)$ implies $\varphi(t) \notin U$. Thus, $\varphi(V) \subseteq U$ for no neighborhood V of x and φ is discontinuous at x. Now suppose that x is the greatest element in G. Put $U = (a(x), \infty)$ Then U is a neighborhood of $b(x) = \varphi(x)$ in L. If V is any neighborhood of x in G, then V contains an element $t < x$. Therefore, $f(t) \prec f(x)$, i.e., $b(t) < a(x)$ implies $\varphi(t) \leq b(t) < a(x)$ and $\varphi(t) \notin U$. Thus, again, $\varphi(V) \subseteq U$ for no neighborhood V of x; and φ is discontinuous at x. ■

Lemma 3.6. *Let G contain no jumps and let $H \subseteq G$. Assume that L contains a system $\mathcal{L}_G$ of pairwise disjoint closed intervals such that $\bigcup \mathcal{L}_G$ is an interval in L and that there exists an isomorphism $f : (G, <) \to (\mathcal{L}_G, \prec)$ with the property $x \in H \Leftrightarrow |f(x)| > 1$. Then there exists an embedding $\varphi : G \to L$ such that H is the set of points of discontinuity of the mapping φ.*

Proof. Let $f(x) = \langle a(x), b(x) \rangle$ so that $a(x) < b(x)$ for $x \in H$ and $a(x) = b(x)$ for $x \in G \setminus H$. Let us define a mapping $\varphi : G \to L$ in the same way as in the proof of Lemma 3.5, i.e., $\varphi(x) = b(x)$ if x is the greatest element in G and $\varphi(x) = a(x)$ in the other case. Then φ is an embedding of G into L and, as it has been proved in Lemma 3.5, φ is discontinuous at any element $x \in H$. Let $x \in G \setminus H$ such that $f(x) = \{a(x)\}$ and let U be any neighborhood of the element $\varphi(x) = a(x)$. Suppose at first that x is not end-element in G. Then $\varphi(x)$ is not end-element in L and hence, there exist $a, b \in L$ such that $a < \varphi(x) < b$ and $(a, b) \subseteq U$. Suppose that $b(t) < a$ for any $t \in G, t < x$. As $f(x) = \{\varphi(x)\} = \{a(x)\}$ and for $t \in G, t > x$ implies $f(t) \succ f(x)$, i.e., $a(t) > \varphi(x)$, we have $a \notin \langle a(t), b(t)\rangle = f(t)$ for all $t \in G$ and a $\notin \bigcup f(G) = \bigcup \mathcal{L}_G$. This contradicts the fact that $\bigcup \mathcal{L}_G$ is an interval in L. Thus it must exist an element $t_0 \in G, t_0 < x$ such that $b(t_0) \geq a$. Take an element $t_1 \in G$ such that $t_0 < t_1 < x$. Then $f(t_0) \prec f(t_1) \prec f(x)$, i.e., $b(t_0) < a(t_1) \leq b(t_1) < \varphi(x)$ and $\varphi(t_1) = a(t_1) > a$. In the similar way we find an element $t_2 \in G$ with $t_2 > x$ such that $b(t_2) < b$. Then $V = (t_1, t_2)$ is a neighborhood of x such that $\varphi(V) \subseteq (a, b) \subseteq U$ and φ is continuous at x.

Now let x be the greatest element in G. Then it is not the least element and, hence, $\varphi(x)$ is not the least element in L. Thus, there exists $a \in L$ with $a < \varphi(x)$ such that $(a, \varphi(x)\rangle \subseteq U$. In a similar way as above, we find an element $t_1 \in G$ with $t_1 < x$ such that $\varphi(t_1) > a$. Then $V = (t_1, \infty) = (t_1, x\rangle$ is a neighborhood of x with $\varphi(V) \subseteq U$ and φ is continuous at x. If x is the least element in G, then the proof is similar. ∎

From the preceding two Lemmas, we now obtain the conclusion:

Theorem 3.7. *Let G contain no jumps and no gaps and let $H \subseteq G$. Then the following statements are equivalent:*

(i) *there exists an embedding $\varphi : G \to L$ such that H is the set of points of discontinuity of the mapping φ;*

(ii) *L contains a system $\mathcal{L}_G$ of pairwise disjoint closed intervals such that $\bigcup \mathcal{L}_G$ is an interval in L and that there exists an isomorphism $f : (G, <) \to (\mathcal{L}_G, \prec)$ with the property $x \in H \Leftrightarrow |f(x)| > 1$.*

Proof. If (i) is valid, then (ii) holds by Theorems 2.8 and 3.4. On the other hand, (ii) implies (i) by Lemma 3.6.

∎

Especially, by setting $H = G$, we get

Theorem 3.8. *Let G contain no jumps and no gaps. Then the following statements are equivalent:*

(i) *there exists an embedding $\varphi : G \to L$ which is discontinuous at each point of G;*

(ii) *L contains a system $\mathcal{L}_G$ of pairwise disjoint nontrivial closed intervals such that $\bigcup \mathcal{L}_G$ is an interval in L and that $(G, <)$ and $(\mathcal{L}_G, \prec)$ are isomorphic.* ∎

Example 3.9. Let $G = (0, 1) \subseteq \mathbb{R}$ be the open interval with usual ordering and $L = G \circ \langle 0, 1\rangle$. Then $\mathcal{L}_G = \{\langle [x, 0], [x, 1]\rangle : \ x \in G\}$ is an ordered naturally system of pairwise disjoint nontrivial closed intervals in L whose union is L and which is isomorphic with G. Thus, according to Theorem 3.8, exists an embedding of G into L which is discontinuous at each point of G. By the considerations in proofs of Lemmas 3.5 and 3.6, the mapping $\varphi(x) = [x, 0]$, (where $x \in G$) is such an embedding. Clearly, the embedding constructed in Example 2.4 has also this property.

4. CASE $G = L$

In this part we will investigate the following problem: Can there exist a chain G and an embedding of G into G which is discontinuous at each element of G? If G contains no jumps and no gaps, then it is possible (by Theorem 3.8) only in case when G contains an ordered naturally system of pairwise disjoint nontrivial closed intervals which is isomorphic with G.

Theorem 4.1. *Let G be a chain without gaps and containing both end-elements. If $\mathcal{G}$ is any system of nontrivial subsets of G, then $(\mathcal{G}, \prec)$ is not isomorphic to $(G, <)$.*

Proof. From assumptions it follows that G is a complete lattice; denote 0 its least element. Suppose that there exists some system $\mathcal{G}$ of its nontrivial subsets and an isomorphism $f : (G, <) \to (\mathcal{G}, \prec)$. Denote $A = \{x \in G : \ \{x\} \prec f(x)\}$. Of course, $A \subseteq G$. Now we show $A \neq \emptyset$. If $\{0\} \prec f(0)$, then $0 \in A$; in the opposite case an element $x \in f(0), x > 0$ exists and then $x \in A$ for $f(0) \prec f(x)$. Denote $\sup(A) = a$ and choose two elements $u, v \in f(a), u < v$. Suppose $a \leq u$. Then $a < v$, hence, $f(a) \prec f(v)$ implies $\{v\} \prec f(v)$. This means $v \in A$ which contradicts the fact $\sup(A) = a$ and $v > a$. Thus $a > u$ so that $a \notin A$ and it is $f(u) \prec f(a)$, especially $f(u) \prec \{u\}$. From this $u \notin A$. When we should have an $x \in A$ such that $u < x < a$, then $\{x\} \prec f(x) \prec f(a)$ and $x < u$, a contradiction. Thus no $x \in A$ with $u < x < a$ exists and $A \cap \langle u, a\rangle = \emptyset$. But this contradicts the fact $\sup(A) = a$. ∎

As a special case, if G is a chain without gaps and with end-elements, then it contains no system $\mathcal{G}$ of pairwise disjoint nontrivial closed intervals such that $(\mathcal{G}, \prec)$ is isomorphic to $(G, <)$. This, together with Theorem 3.8, gives:

Corollary 4.2. *Let G be a chain without jumps and gaps containing end-elements. Then there is no embedding of G into G which is discontinuous at each point of G.* ■

If G does not contain some end-element, then Theorem 4.1 does not hold. Indeed, it suffices to put G equal with the set of all positive integers and $f(n) = \{2n-1, 2n\}$.

M. Novotný in [6] constructed an example of a chain G without jumps and gaps containing a system $\mathcal{G}$ of pairwise disjoint nontrivial closed intervals such that $(\mathcal{G}, \prec)$ is isomorphic to $(G, <)$. The following Theorem gives a general construction of chains with this property.

Theorem 4.3. *Let G be a chain, H be a nontrivial chain. Put $G_1 = G, G_{n+1} = G_n \circ H$ for $n \geq 1$, $K = \bigoplus_{n \in \mathbb{N}} G_n$. Then K contains a system $\mathcal{K}$ of pairwise disjoint nontrivial closed intervals such that $(K, <)$ is isomorphic to $(\mathcal{K}, \prec)$.*

Proof. Choose two fixed elements $a, b \in H$ such that $a < b$. Let $x \in K$. Then there exists (a unique) $n \in \mathbb{N}$ such that $x \in G_n$. We put $f(x) = \langle [x,a], [x,b] \rangle \subseteq G_{n+1}$. Let us denote $\mathcal{K} = \{f(x) : \ x \in K\}$. We show that $f : (K, <) \to (\mathcal{K}, \prec)$ is an isomorphism. Let $x, y \in K, x < y$. If there exists $n \in \mathbb{N}$ such that $x, y \in G_n$, then $[x,b] < [y,a]$ in G_{n+1} and $f(x) = \langle [x,a], [x,b] \rangle \prec \langle [y,a], [y,b] \rangle = f(y)$. If $x \in G_m, y \in G_n$, where $m \neq n$, then $m < n$ and $[x,b] \in G_{m+1}, [y,a] \in G_{n+1}$. This implies $[x,b] < [y,a]$ again in K and $f(x) \prec f(y)$ in $\mathcal{K}$ as well. ■

Note that if H contains end-elements 0,1 and if we put $a = 0, b = 1$ in the above construction, then $\bigcup \mathcal{K}$ is an interval in K.

A chain having the property from Theorem 4.3 can be constructed such that it contains no jumps and no gaps. This follows from two lemmas which proofs are left to reader.

Lemma 4.4. *Let G, H be chains. If G and H contain no jumps, then $G \circ H$ contains no jumps; if G and H contain no gaps and H contains both end-elements, then $G \circ H$ contains no gaps.* ■

Lemma 4.5. *Let G and H be chains. If G and H contain no jumps and G has not the greatest element or H has not the least element, then $G \oplus H$ contains no jumps. If G and H contain no gaps and G has the greatest element or H has the least element, then $G \oplus H$ contains no gaps.* ∎

We also have:

Corollary 4.6. *There exists a chain K without jumps and gaps which contains a system $\mathcal{K}$ of pairwise disjoint nontrivial closed intervals such that $\cup\mathcal{K}$ is an interval in K and that $(K, <)$ is isomorphic with $(\mathcal{K}, \prec)$.*

Proof. Let $G = G_1$ be a chain without jumps and gaps and with the least element and without the greatest element, and let H be a nontrivial chain without jumps and gaps containing end-elements 0,1. Put $G_{n+1} = G_n \circ H$ for $n \in \mathbb{N}$ and $K = \bigoplus_{n\in\mathbb{N}} G_n$. Then, by Theorem 4.3, K is a chain which contains a system $\mathcal{K}$ with desired properties. Further, if we put $a = 0, b = 1$ in the construction in the proof of Theorem 4.3, then $\bigcup\mathcal{K}$ is an interval in K. Further, by Lemma 4.4, $G_2 = G_1 \circ H$ contains no jumps and no gaps and, clearly, it contains the least element and does not contain the greatest element. By Lemma 4.5, $G_1 \oplus G_2$ contains no jumps and no gaps and it does not contain the greatest element. By induction, we prove that, for any $n \in \mathbb{N}$, $G_1 \oplus G_2 \oplus \cdots \oplus G_n$ contains no jumps and no gaps and no greatest element and G_{n+1} contains no jumps and no gaps and it contains the least element and does not contain the greatest element. Then $G_1 \oplus G_2 \oplus \cdots \oplus G_n \oplus G_{n+1}$ contains no jumps and no gaps and no greatest element. From this it follows that $K = \bigoplus_{n\in\mathbb{N}} G_n$ contains no jumps and no gaps. ∎

Note that the chain K constructed above has not the greatest element but it contains the least element.

Further, by Theorem 3.8 and Corollary 4.6, we conclude:

Corollary 4.7. *There exists a chain K without jumps and gaps and an embedding $\varphi : K \to K$ which is discontinuous at all points of K.*

Example 4.8. Let $G = G_1 = \langle 0, 1) \subseteq \mathbb{R}$, $H = \langle 0, 1\rangle \subseteq \mathbb{R}$, both with the usual ordering, $G_{n+1} = G_n \circ H$ for $n \in \mathbb{N}$ and $K = \bigoplus_{n\in\mathbb{N}} G_n$. By Corollary 4.6 and its proof, K is a chain without jumps and gaps (containing the least element and not containing the greatest element) which contains a system $\mathcal{K}$ of pairwise disjoint nontrivial closed intervals such that $\bigcup\mathcal{K}$ is an interval in K. Therefore that $(K,<)$ and $(\mathcal{K},\prec)$ are isomorphic. Thus, by Theorem 3.8, it exists an embedding $\varphi : K \to K$ which is discontinuous at all elements of K. An example of such an embedding is the following one: if $x \in K$, then there exists (a unique) $n \in \mathbb{N}$ such that $x \in G_n$; finaly, we put $\varphi(x) = [x, \frac{1}{2}] \in G_{n+1}$.

5. Density of the set of points of discontinuity

In this part we again suppose that there are given two nontrivial chains G, L and L contains no gaps. Symbols $a_\varphi(x)$, $b_\varphi(x)$, for a given isotone mapping $\varphi : G \to L$ have the same meaning as in parts 2 and 3.

Recall that a subset H of a chain G is called *dense* in G (in the Hausdorff sense, see [3], p. 89) if for any elements $x, y \in G$, $x < y$ there exist $x_0, y_0 \in H$ such that $x \leq x_0 < y_0 \leq y$. Trivially, if H is dense in G and $x, y \in G, x \prec y$, then $x, y \in H$. If G contains no jumps, then $H \subseteq G$ is dense in G in the above sense iff it is topologically dense in G at interval topology.

Lemma 5.1. *Let $\varphi : G \to L$ be an isotone mapping, let $H \subseteq G$ be dense in G and let $x \in G$. Then the following hold:*

(i) *If x is not the least element in G, then*
$a_\varphi(x) = \sup(\{\varphi(t) :\ t \in H, t < x\})$.

(ii) *If x is not the greatest element in G, then*
$b_\varphi(x) = \inf(\{\varphi(t) :\ t \in H, t > x\})$.

Proof. Ad (i): Let x be not the least element in G. Of course,

$$\sup(\{\varphi(t) :\ t \in H, t < x\}) \leq a_\varphi(x).$$

If there exists an $x_0 \in G$ such that $x_0 \prec x$, then, clearly, $a_\varphi(x) = \varphi(x_0)$; but $x_0 \in H$ so that $\sup(\{\varphi(t) :\ t \in H, t < x\}) = \varphi(x_0)$. In the opposite case, for any $t \in G, t < x$, there exists $t_0 \in H$ with $t < t_0 < x$ implying $\varphi(t) \leq \varphi(t_0)$. From this $a_\varphi(x) = \sup(\{\varphi(t) :\ t \in G, t < x\}) \leq \sup(\{\varphi(t) :\ t \in H, t < x\})$ and we have $a_\varphi(x) = \sup(\{\varphi(t) :\ t \in H, t < x\})$.

(ii) can be proved by dual considerations. ■

Lemma 5.2. *Let $\varphi : G \to L$ be an isotone mapping and let G contain no jumps. Let $H \subseteq G$ be dense in G and let $x \in G$. Then the following hold:*

(i) *If x is not the least in G, then* $a_\varphi(x) = \sup(\{b_\varphi(t) : t \in H, t < x\})$.

(ii) *If x is not the greatest in G, then* $b_\varphi(x) = \inf(\{a_\varphi(t) : t \in H, t > x\})$.

Proof. Ad (i): Suppose that x is not the least element in G. By Lemma 5.1, $a_\varphi(x) = \sup(\{\varphi(t) : t \in H, t < x\})$. As $\varphi(t) \leq b_\varphi(t)$ for all $t \in G$, we have $a_\varphi(x) \leq \sup(\{b_\varphi(t) : t \in H, t < x\})$. On the other hand, if $t \in H$ and $t < x$, then there exists $t_0 \in H$ with $t < t_0 < x$ implying $b_\varphi(t) \leq \varphi(t_0)$. From this $\sup(\{b_\varphi(t) : t \in H, t < x\}) \leq \sup(\{\varphi(t) : t \in H, t < x\}) = a_\varphi(x)$ and the assertion follows.

(ii) can be proved similarly. ■

Lemma 5.3. *Let $\varphi : G \to L$ be an isotone mapping, let G contain no jumps and no gaps, and let $H \subseteq G$ be dense in G. If φ is continuous at any point $x \in G \smallsetminus H$, then $cl(\bigcup f[\varphi](H))$ is an interval in L.*

Proof. Suppose that $a, b \in cl(\bigcup f[\varphi](H)), a < b$ and that $c \in L, a < c < b$. If it exists $x \in H$ such that $c \in \langle a_\varphi(x), b_\varphi(x)\rangle = f[\varphi](x)$, then clearly $c \in cl(\bigcup f[\varphi](H))$. Thus, we can suppose that no $x \in H$ with this property exists. This means that either $b_\varphi(x) < c$ or $a_\varphi(x) > c$ for any $x \in H$. Denote $A = \{x \in H : b_\varphi(x) < c\}, B = \{x \in H : a_\varphi(x) > c\}$. Then $[A, B]$ is a cut in H. As H is dense in G and G contains no jumps and no gaps, there exists exactly one element $z \in G$ such that $z = \sup(A) = \inf(B)$. Assume at first $z \in A$, i.e., $z = \max(A)$. Then $b_\varphi(z) < c$. By Lemma 5.1, $b_\varphi(z) = \inf(\{\varphi(t) : t \in H, t > z\}) = \inf(\varphi(B))$. Thus it must exist $t_0 \in B$ with $\varphi(t_0) < c$. By definition of the set B, we have $\varphi(t_0) \geq a_\varphi(t_0) > c$, a contradiction. Similarly, we verify that $z \notin B$ and hence $z \in G \smallsetminus H$. By assumption φ is continuous at z and, by Theorem 3.4, $a_\varphi(z) = b_\varphi(z) = \varphi(z)$. From Lemma 5.2, there follows $\varphi(z) = \sup(\{b_\varphi(t) : t \in H, t < z\}) = \sup(\{b_\varphi(t) : t \in A\})$. As $b_\varphi(t) < c$ for all $t \in A$, we have $\sup(\{b_\varphi(t) : t \in A\}) \leq c$, i.e., $\varphi(z) \leq c$. Similarly, the fact $\varphi(z) = \inf(\{a_\varphi(t) : t \in H, t > z\})$ implies $\varphi(z) \geq c$ and we have $\varphi(z) = c$.

Let U be any neighborhood of the element $c \in L$. Then there exist $c_1, c_2 \in L$ such that $c_1 < c < c_2$ and $(c_1, c_2) \subseteq U$. As $c = \varphi(z) = \sup(\{b_\varphi(t) : t \in H, t < z\})$, there exists $t_0 \in H$, such that $t_0 < z$ and $c_1 < b_\varphi(t_0) \leq c$. This implies $\langle a_\varphi(t_0), b_\varphi(t_0)\rangle \cap U \neq \emptyset$. Thus, $U \cap (\bigcup f[\varphi](H)) \neq \emptyset$ for any neighborhood U of c and $c \in cl(\bigcup f[\varphi](H))$. We have proved that $cl(\bigcup f[\varphi](H))$ is an interval in L. ■

Lemma 5.4. *Let G, L contain no jumps and L contain end-elements. Let $H \subseteq G$ be dense in G and let L contain a system $\mathcal{L}_H$ of pairwise disjoint nontrivial closed intervals such that $cl(\bigcup \mathcal{L}_H)$ is an interval in L and that $(H, <)$ is isomorphic to $(\mathcal{L}_H, \prec)$. Then there exists an isotone mapping $\varphi : G \to L$ such that H is the set of points of discontinuity of the mapping φ.*

Proof. The assumptions imply that L is a complete lattice. Let f be an isomorphism of $(H, <)$ onto $(\mathcal{L}_H, \prec)$ and let $f(x) = \langle a(x), b(x) \rangle$ for $x \in H$. Put, for any $x \in H$, $\varphi(x) = a(x)$ if x is not the greatest element in G, and $\varphi(x) = b(x)$ otherwise. Then φ is a mapping of H into L. We can extend φ to a mapping with domain G, namely, for $x \in G \setminus H$, we put $\varphi(x) = \sup(\{\varphi(t) : t \in H, t < x\})$ if x is not the least element in G and $\varphi(x) = \inf(\{\varphi(t) : t \in H, t > x\})$ otherwise. We show that $\varphi : G \to L$ is isotone. Let $x, y \in G, x < y$. If $x, y \in H$, then $f(x) \prec f(y)$, i.e., $b(x) < a(y)$ implies $\varphi(x) = a(x) < b(x) < a(y) \leq \varphi(y)$. Suppose $x \in H, y \in G \setminus H$. Then there exists $t_0 \in H$ such that $x < t_0 < y$. From this it follows $\varphi(x) < \varphi(t_0)$ and, from the definition of φ, we have $\varphi(t_0) \leq \varphi(y)$. Thus, $\varphi(x) < \varphi(y)$. Let $x \in G \setminus H, y \in H$. If x is the least in G, then $\varphi(x) \leq \varphi(y)$ follows from the definition of φ; in the opposite case $t \in H, t < x$ implies $t < y$, and thus, $\varphi(t) < \varphi(y)$ and $\varphi(x) = \sup(\{\varphi(t) : t \in H, t < x\}) \leq \varphi(y)$. At the end, let $x, y \in G \setminus H$. Then there exist $t_1, t_2 \in H$ with $x < t_1 < t_2 < y$ and, from the preceding facts, we have $\varphi(x) \leq \varphi(t_1) < \varphi(t_2) \leq \varphi(y)$.

Let $x \in H$. If x is not the greatest in G, then $U = (-\infty, b(x))$ is a neighborhood of $\varphi(x) = a(x)$ in L. Let V be any neighborhood of x in G. As G contains no jumps, it exists $t \in H$ such that $t \in V$ and $t > x$. Then $f(x) \prec f(t)$, i.e., $b(x) < a(t) \leq \varphi(t)$ and $\varphi(t) \notin U$. If x is the greatest in G, then $U = (a(x), \infty)$ is a neighborhood of $\varphi(x) = b(x)$ in L. But any neighborhood V of x contains an element $t \in H, t < x$. Then $b(t) < a(x)$ and $\varphi(t) = a(t) < b(t) < a(x)$ implies $\varphi(t) \notin U$ again. Thus φ is discontinuous at x.

Let $x \in G \setminus H$. Suppose at first that x is neither the least nor the greatest element of G. Then $\varphi(x)$ is not end-element of L (there exist $t_1, t_2 \in H$ such that $x < t_1 < t_2$ which implies $\varphi(x) \leq \varphi(t_1) < \varphi(t_2)$ and $\varphi(x)$ is not the greatest element; similarly it is not the least element). Choose any neighborhood U of $\varphi(x)$ in L. Then there exist $a, b \in L$ such that $a < \varphi(x) < b$ and $(a, b) \subseteq U$. The definition of $\varphi(x)$ implies existence of $t_1 \in H$ with $t_1 < x$ such that $\varphi(t_1) > a$. We want to show that there exists $t_2 \in H$ with $t_2 > x$ such that $\varphi(t_2) < b$. Suppose that not. Thus,

$\varphi(t) \geq b$ for all $t \in H$, $t > x$. Then $a(t) \geq b$ for all $t \in H$, $t > x$ and we show that $b(t) < \varphi(x)$ for all $t \in H$, $t < x$. Let $t \in H$ and $t < x$. Then there exists $t_0 \in H$ such that $t < t_0 < x$ and from this $f(t) \prec f(t_0)$, i.e., $b(t) < a(t_0)$. From the definition of $\varphi(x)$, there follows $a(t_0) \leq \varphi(x)$. We have shown $(\varphi(x), b) \cap \langle a(t), b(t)\rangle = \emptyset$ for all $t \in H$, i.e., $(\varphi(x), b) \cap (\cup \mathcal{L}_H) = \emptyset$. This means that no element of the interval $(\varphi(x), b)$ lies in $cl(\cup \mathcal{L}_H)$, which contradicts the assumption that $cl(\cup \mathcal{L}_H)$ is an interval in L. Thus, there exists $t_2 \in H$ with $t_2 > x$ such that $\varphi(t_2) < b$. Then $V = (t_1, t_2)$ is a neighborhood of x in G such that $\varphi(V) \subseteq U$ and φ is continuous at x.

Now suppose that x is the greatest element of G. Let U be any neighborhood of $\varphi(x)$ in L. Then there exists $a \in L$ with $a < \varphi(x)$ such that $(a, \varphi(x)\rangle \subseteq U$. From the definition of $\varphi(x)$, there follows the existence of $t_1 \in H$ with $t_1 < x$ such that $\varphi(t_1) > a$. Then $V = (t_1, \infty) = (t_1, x\rangle$ is a neighborhood of x such that $\varphi(V) \subseteq U$. If x is the least element of G and U is any neighborhood of $\varphi(x)$, then it exists $b \in L$, with $b > \varphi(x)$ such that $\langle \varphi(x), b) \subseteq U$. In a similar way, we can find an element $t_2 \in H$ such that $t_2 > x$ and $\varphi(t_2) < b$. Then $V = (-\infty, t_2) = \langle x, t_2)$ is a neighborhood of x for which $\varphi(V) \subseteq U$. Thus φ is continuous at x. ∎

In Lemmas 5.3 and 5.4 we can the phrase "isotone mapping" replace by the word "embedding" for the following simple assertion holds:

Lemma 5.5. *Let K_1, K_2 be chains, let K_1 contain no jumps and let $\varphi : K_1 \to K_2$ be an isotone mapping. If the set of all points off discontinuity of the mapping φ is dense in K_1, then φ is injective, i.e., it is an embedding of K_1 into K_2.*

Proof. Suppose that φ is isotone and not injective. Then there exist $x_1, x_2 \in K_1$ such that $x_1 < x_2$ and $\varphi(x_1) = \varphi(x_2)$. Therefore, φ is a constant mapping on interval $\langle x_1, x_2\rangle$ and, thus, continuous at each point of the interval (x_1, x_2). But then, the set of all points of discontinuity of mapping φ is not dense in K_1. ∎

From the preceding results, we can state the following theorem:

Theorem 5.6. *Let G contain no jumps and no gaps, L contain no jumps and have end-elements. Let $H \subseteq G$ be dense in G. Then the following statements are equivalent:*

(i) *There exists an isotone mapping $\varphi : G \to L$ such that H is the set of all points of discontinuity of φ.*

(ii) *L contains a system $\mathcal{L}_H$ of pairwise disjoint nontrivial closed intervals such that $cl(\cup\mathcal{L}_H)$ is an interval in L and, moreover, $(H,<)$ and $(\mathcal{L}_H,\prec)$ are isomorphic.*

Proof. If (i) holds, then (ii) is valid by Lemmas 5.3, 5.5 and Theorem 3.7. On the other hand (ii) implies (i) by Lemma 5.4. ■

By putting $G = L$, we get:

Corollary 5.7. *Let G be a chain without jumps and gaps containing end-elements, let $H \subseteq G$ be dense in G. Then the following statements are equivalent:*

(i) *there exists an isotone mapping $\varphi : G \to G$ such that H is the set of all points of discontinuity of φ;*

(ii) *G contains a system $\mathcal{G}_H$ of pairwise disjoint nontrivial closed intervals such that $cl(\cup\mathcal{G}_H)$ is an interval and, moreover, $(H,<)$ and $(\mathcal{G}_H,\prec)$ are isomorphic.* ■

Let G be a chain. In accordance with [4], we will say that

G has property (D) if there exists an isotone mapping of G into itself whose set of all points of discontinuity is dense in G.

If G contains no jumps, then "isotone mapping" can be replaced by "embedding".

From Corollary 5.7 we get directly:

Corollary 5.8. *Let G be a chain without jumps and gaps and with end-elements. Then G has property (D) if and only if there exists a subset $H \subseteq G$ dense in G and a system $\mathcal{G}_H$ of pairwise disjoint nontrivial closed intervals in G such that $cl(\cup\mathcal{G}_H)$ is an interval in G and, moreover, $(H,<)$ and $(\mathcal{G}_H,\prec)$ are isomorphic.*

Example 5.9. Interval $G = \langle 0,1\rangle \subseteq \mathbb{R}$ has property (D). In fact, the set H of dyadic rational numbers from interval $(0,1)$ is dense in G. If, further, $\mathcal{G}_H$ is the system of closures of complemented intervals of Cantor discontinuum in $\langle 0,1\rangle$, then $cl(\cup\mathcal{G}_H) = \langle 0,1\rangle$ and, as it is well-known, the chains $(H,<)$ and $(\mathcal{G}_H,\prec)$ are isomorphic. Our assertion follows from Corollary 5.8.

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