Discussiones Mathematicae
General Algebra and Applications 24 (2004) 225–249

GENERALIZED INFLATIONS AND NULL EXTENSIONS

Qiang Wang

School of Mathematics and Statistics
Carleton University
Ottawa, Ont., Canada K1S–5B6

e-mail: wang@math.carleton.ca

and

Shelly L. Wismath*

Department of Mathematics and C.S.
University of Lethbridge
Lethbridge, Ab., Canada T1K–3M4

e-mail: wismaths@cs.uleth.ca

Abstract

An inflation of an algebra is formed by adding a set of new elements to each element in the original or base algebra, with the stipulation that in forming products each new element behaves exactly like the element in the base algebra to which it is attached. Clarke and Monzo have defined the generalized inflation of a semigroup, in which a set of new elements is again added to each base element, but where the new elements are allowed to act like different elements of the base, depending on the context in which they are used. Such generalized inflations of semigroups are closely related to both inflations and null extensions. Clarke and Monzo proved that for a semigroup base algebra which is a union of groups, any semigroup null extension must be a generalized inflation, so that the concepts of null extension and generalized inflation coincide in the case of unions of groups. As a consequence, the collection of all associative generalized inflations formed from algebras in a variety of unions of groups also forms a variety.

In this paper we define the concept of a generalized inflation for any type of algebra. In particular, we allow for generalized inflations

*Research supported by NSERC of Canada.

of semigroups which are no longer semigroups themselves. After some general results about such generalized inflations, we characterize for several varieties of bands which null extensions of algebras in the variety are generalized inflations, and which of these are associative. These characterizations are used to produce examples which answer, in our more general setting, several of the open questions posed by Clarke and Monzo.

Keywords: inflation, generalized inflation, null extension, variety of semigroups, bands.

2000 Mathematics Subject Classification: 08A05, 20M07, 20M10.

1. Introduction and definitions

An inflation of an algebra $\mathcal{B}$ is a new algebra formed from $\mathcal{B}$ by adding a set of new elements to each element b in $\mathcal{B}$, with the stipulation that new elements attached to b always act like b in forming products in the new algebra. Inflations have been extensively studied, particularly for semigroups; see for instance [4], [6], [11], and [12]. Recently several variations of inflations have also been introduced. The concept of a k-inflation was developed by Bogdanović and Milić ([1]) for semigroups and extended to arbitrary type by Milić ([10]), and defined in a slightly different way by Denecke and Wismath for k-normalizations ([7]). In [5] Clarke and Monzo studied what they called generalized inflations of semigroups, in which a set of new elements is added to each base element but the new elements are also allowed to act like different elements of the base depending on the context in which they are used. Such generalized inflations of semigroups are closely related to both inflations and null extensions. In the semigroup case, a null extension of a semigroup $\mathcal{B}$ is a semigroup whose square is $\mathcal{B}$. Any semigroup which is a generalized inflation of a semigroup $\mathcal{B}$ is also a null extension. Clarke and Monzo proved in [5] that any semigroup which is a null extension of a semigroup which is a union of groups must be a generalized inflation, so that the two concepts of null extension and generalized inflation coincide in the case of unions of groups.

In this paper we define the concept of a generalized inflation for any type of algebra. In particular, we allow for generalized inflations of semigroups which no longer need be associative. After introducing our basic definitions in the remainder of this section, we study the interconnections between inflations, null extensions and generalized inflations in this more general setting in Section 2. Then in Sections 3 to 6 we characterize, for the semigroup varieties of left zero bands, rectangular bands, normal bands and semilattices, which null extensions of algebras in the variety are generalized inflations, and of these, which are associative. These characterizations provide us with examples to answer, in our more general setting, several open problems posed by Clarke and Monzo.

Throughout this paper, we let $\tau = (n_i)_{i \in I}$ be any type of algebras, with f_i an n_i-ary operation symbol for each i in some index set I.

Definition 1.1. Let $\mathcal{A} = (A; (f_i)_{i \in I})$ be an algebra of type τ. We define the image of $\mathcal{A}$ as the set $Im(\mathcal{A})$ consisting of all elements of the form $f_i^A(a_1, \ldots, a_{n_i})$, for some $i \in I$ and some $a_1, \ldots, a_{n_i} \in A$. An algebra $\mathcal{A}$ is called a *null extension of an algebra* $\mathcal{B}$ if $Im(\mathcal{A}) \subseteq B$.

Let $\mathcal{B}$ be an algebra of type τ. Both inflations and generalized inflations of $\mathcal{B}$ are defined by attaching a set of new elements to each base element $b \in B$. For each $b \in B$, let S_b be a set containing b, with $S_b \cap S_c = \emptyset$ for $b \neq c$, so that the sets S_b are pairwise disjoint. Let $A = \bigcup_{b \in B} S_b$. For each $x \in A$, there is a unique element $b \in B$ such that $x \in S_b$; we denote this element by $\hat{x}$, and call it the *base element* of x. The function $\hat{}\ : A \to B$ thus defined will be called the *'hat' function*.

Definition 1.2. Let $\mathcal{A} = (A; (f_i^A)_{i \in I})$ be an algebra with $A = \bigcup_{b \in B} S_b$ and operations f_i^A defined by $f_i^A(a_1, \ldots, a_{n_i}) = f_i^B(\hat{a}_1, \ldots, \hat{a}_{n_i})$. Then $\mathcal{A}$ is called an *inflation of the algebra* $\mathcal{B}$, and $\mathcal{B}$ is called the *base of the inflation*.

A generalized inflation of $\mathcal{B}$ also uses a universe set $A = \bigcup_{b \in B} S_b$. The operations on A will be defined using some *selector* or *role model functions* Γ_i. That is, for each operation symbol f_i of the type, we define an n_i-ary function $\Gamma_i : A^{n_i} \to B$, with the property that $\Gamma_i(x_1, \ldots, x_{n_i}) = x_1$ if $x_1 \in B$. Then we define the operations f_i^A for our new algebra $\mathcal{A}$ by

$$\begin{aligned} &f_i^A(a_1, \ldots, a_{n_i}) \\ &= f_i^B(\Gamma_i(a_1, \hat{a}_2, \ldots, \hat{a}_{n_i}), \Gamma_i(a_2, \hat{a}_1, \hat{a}_3, \ldots, \hat{a}_{n_i}), \ldots, \Gamma_i(a_{n_i}, \hat{a}_2, \ldots, \hat{a}_{n_i - 1})). \end{aligned}$$

Definition 1.3. The algebra $\mathcal{A} = (A; (f_i^A)_{i\in I})$ constructed in this way is called a *generalized inflation algebra* of $\mathcal{B}$, with $\mathcal{B}$ as the base of the generalized inflation. We will write $\mathcal{A} = (\mathcal{B}, \Gamma, \hat{\ })$ for the generalized inflation of $\mathcal{B}$ formed using the functions Γ and $\hat{\ }$.

Many of the results and examples we present in later sections deal with type (2) and semigroups. In this case we have a single binary operation symbol, usually denoted by f, and a single selector function $\Gamma : A \times A \to B$, with multiplication in the new algebra defined by $f^A(x, y) = f^B(\Gamma(x, \hat{y}), \Gamma(y, \hat{x}))$. Note that in fact it is only necessary to define Γ on the set $A \times B$. We think of the function Γ as selecting for each element x its role model when used with a base element b or $\hat{y}$ in B. In an inflation, each new element has exactly one role model in all contexts, and that role model is its base element; in a generalized inflation a new element can have many different role models in different contexts.

When $\mathcal{A}$ and $\mathcal{B}$ are both semigroups, our definition of a generalized inflation is exactly the process described by Clarke and Monzo in [5]. Here however we consider arbitrary algebras, even within type (2). In this sense our definition is broader than that of Clarke and Monzo: if $\mathcal{A}$ is a semigroup and a generalized inflation of $\mathcal{B}$, then $\mathcal{B}$ is a semigroup, but a generalized inflation of a semigroup $\mathcal{B}$ need not be a semigroup. The following example illustrates this.

Example 1.4. Consider type (2), with one binary operation symbol f. The example given by Clarke and Monzo in [5] is the following: $B = \{e, g, h\}$, with the multiplication of a left zero semigroup, and in the generalized inflation one new element $x \in S_e$ is added, with $\Gamma(x, e) = \Gamma(x, h) = \Gamma(x, x) = e$, and $\Gamma(x, g) = h$. In this case the new algebra $\mathcal{A}$ is still a semigroup.

But now consider the same base set $B = \{e, g, h\}$, again a left zero semigroup, with one new element $p \in S_e$ added. We set $\Gamma(p, e) = e$, $\Gamma(p, p) = e$, $\Gamma(p, g) = g$, $\Gamma(p, h) = h$. The following calculation shows that the generalized inflation defined by Γ is no longer associative:

$$\begin{aligned} f^A(f^A(p,p),h) &= f^A(f^B(\Gamma(p,\hat{p}),\Gamma(p,\hat{p})),h) \\ &= f^A(f^B(\Gamma(p,e),\Gamma(p,e)),h) = f^A(f^B(e,e),h) \\ &= f^A(e,h) = f^B(\Gamma(e,\hat{h}),\Gamma(h,\hat{e})) = f^B(e,h) = e; \end{aligned}$$

but

$$\begin{aligned} f^A(p, f^A(p,h)) &= f^A(p, f^B(\Gamma(p,\hat{h}), \Gamma(h,\hat{p}))) \\ &= f^A(p, f^B(\Gamma(p,h), \Gamma(h,e))) = f^A(p, f^B(h,h)) \\ &= F^A(p,h) = f^B(\Gamma(p,\hat{h}), \Gamma(h,\hat{p})) = f^B(h,h) = h. \end{aligned}$$

We begin with some basic observations about the concepts of inflation, generalized inflation and null extension.

Lemma 1.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of type τ. Then:*

(i) *Any inflation of $\mathcal{B}$ is a generalized inflation of $\mathcal{B}$;*

(ii) *Any generalized inflation of $\mathcal{B}$ is a null extension of $\mathcal{B}$;*

(iii) *Any algebra is an inflation and a generalized inflation of itself;*

(iv) *If $\mathcal{A}$ is a generalized inflation of $\mathcal{B}$, then $\mathcal{B}$ is a subalgebra of $\mathcal{A}$;*

(v) *Any inflation $\mathcal{C}$ of a generalized inflation $\mathcal{A}$ of $\mathcal{B}$ is a generalized inflation of $\mathcal{B}$;*

Proof. (i): Let $\mathcal{A}$ be an inflation of $\mathcal{B}$. Then each element $x \in A$ is assigned a base element $\hat{x} \in B$ by the inflation. For each $i \in I$, we define the role model function Γ_i by $\Gamma_i(x_1, x_2, \ldots, x_{n_i}) = \hat{x}_1$. Then the generalized inflation operation f_i^A induced by Γ_i on A satisfies $f_i^A(a_1, \ldots, a_{n_i}) = f_i^B(\hat{a}_1, \ldots, \hat{a}_{n_i})$, and is the same as the inflation operation on A. This shows that the inflation may be viewed as a generalized inflation of $\mathcal{B}$.

(ii): It follows from the definition of the operations in a generalized inflation $\mathcal{A}$ of an algebra $\mathcal{B}$, that $Im(\mathcal{A}) \subseteq B$.

(iii): If $\mathcal{A}$ is an inflation or generalized inflation of $\mathcal{B}$ in which no new elements are added to B, so that $S_b = \{b\}$ for all $b \in B$, then $\mathcal{A} = \mathcal{B}$.

(iv): This follows from the fact that $f_i^A|_B = f_i^B$.

(v): This was proved in [5] for the case that $\mathcal{C}$ is a semigroup inflation of a semigroup generalized inflation of $\mathcal{B}$; the same proof may be generalized to cover arbitrary type. We let $\mathcal{A} = (\mathcal{B}, \Gamma, \hat{\ })$, and let $\gamma : C \to A$ by $\gamma : x \mapsto \overline{x}$ be a function which represents $\mathcal{C}$ as an inflation of $\mathcal{A}$. For each $i \in I$, define $\Gamma' : C^{n_i} \to B$ by $\Gamma'(x_1, \ldots, x_{n_i}) = \Gamma(\overline{x_1}, \ldots, \overline{x_{n_i}})$. Then $\mathcal{C}$ is easily seen to be a generalized inflation of $\mathcal{B}$, using Γ'. ∎

2. Class operators and identities

Let $Alg(\tau)$ be the class of all algebras of type τ. We define the following class operators on $Alg(\tau)$: for V any class of algebras of type τ, let

$$Inf(V) = \{\mathcal{A} \in Alg(\tau) \mid \mathcal{A} \text{ is an inflation of some } \mathcal{B} \text{ in } V\}.$$

$$GInf(V) = \{\mathcal{A} \in Alg(\tau) \mid \mathcal{A} \text{ is a generalized inflation of some } \mathcal{B} \text{ in } V\}.$$

$$NExt(V) = \{\mathcal{A} \in Alg(\tau) \mid \mathcal{A} \text{ is a null extension of some } \mathcal{B} \text{ in } V\}.$$

These three operators are clearly monotone and extensive. It is straightforward to show that Inf is also idempotent, i.e. any inflation of an inflation of an algebra $\mathcal{B}$ is also an inflation of $\mathcal{B}$. Thus Inf is a closure operator on $Alg(\tau)$. But a null extension of a null extension of an algebra is not necessarily a null extension, and we shall give an example in the next section to show that a generalized inflation of a generalized inflation is not always a generalized inflation. This answers, in our more generalized setting, a question posed by Clarke and Monzo in [5].

It follows from Lemma 1.5 that for any variety V, we have:

$$V \subseteq Inf(V) \subseteq GInf(V) \subseteq NExt(V).$$

Another question posed by Clarke and Monzo is whether the class $GInf(V)$ is a variety when V is a variety. They showed that when V is a variety of unions of groups, the class of semigroups in $GInf(V)$ forms a variety of semigroups, and asked whether this property holds for all varieties of semigroups. In our more general setting, we ask whether the classes $Inf(V)$, $GInf(V)$ and $NExt(V)$ are varieties when V is a variety of type τ.

To answer this question for $NExt(V)$, we must consider identities of type τ. An identity of type τ is any equation $u \approx v$, where u and v are terms of type τ. Such terms are built up from the operation symbols f_i of the type and a standard set of variables, $X = \{x_1, x_2, x_3, \ldots\}$. For any variety V of type τ, we denote by IdV the set of all identities satisfied in V; and for any set Σ of identities of type τ, we denote by $Mod\Sigma$ the variety of all algebras which satisfy all the identities in Σ.

In order to define a new operation on identities, we introduce the following notation. For each natural number j, we consider a set of new variables $\{x_{j1}, x_{j2}, x_{j3}, \ldots\}$. We set $T_j = \{f_i(x_{j1}, x_{j2}, \ldots, x_{jn_i}) \mid i \in I\}$. Then for any identity $u \approx v$ of type τ, we let $(u \approx v)^*$ be the set of all identities formed from $u \approx v$ by consistent replacement of each variable x_j in u or v

by any element of T_j. For Σ a set of identities, we denote by Σ^* the set of all identities $(u \approx v)^*$ for $u \approx v \in \Sigma$.

Note that in the type (2) setting, this operation merely replaces each variable x_j in an identity by the "doubled" variable $x_{j1}x_{j2}$. In particular, the associative identity $x_1(x_2x_3) \approx (x_1x_2)x_3$ becomes the double-associative identity $(x_{11}x_{12})[(x_{21}x_{22})(x_{31}x_{32})] \approx [(x_{11}x_{12})(x_{21}x_{22})](x_{31}x_{32})$. This was the construction used by Clarke and Monzo to prove the type (2), semigroup version of the following result.

Theorem 2.1. *Let V be any variety of type τ. Then:*

(i) *$NExt(V) = Mod((IdV)^*)$, for any variety V.*

(ii) *$NExt(V)$ is a variety.*

(iii) *If $V = Mod\Sigma$, so that Σ is a basis for the set IdV of all identities of V, then $NExt(V) = Mod(\Sigma^*)$, and Σ^* is a basis for the identities of $NExt(V)$.*

Proof. (i): It follows from the definition of a null extension that any algebra $\mathcal{A}$ which is a null extension of an algebra $\mathcal{B}$ in V satisfies all the identities in IdV^*, so that $NExt(V) \subseteq Mod((IdV)^*)$. Conversely, let $\mathcal{A}$ be an algebra satisfying the identities in IdV^*. Then the algebra $\mathcal{B} = Im(\mathcal{A})$ satisfies all the identities of V, by the construction of IdV^*, and $\mathcal{A}$ is a null extension of $\mathcal{B} \in V$.

(ii): This is a consequence of (i), since any equational class is a variety.

(iii): It will suffice to show that for any identity $u \approx v$ which can be deduced from Σ, according to the usual five rules of deduction, the corresponding identity $(u \approx v)^*$ can be deduced from Σ^*. This can be shown by induction on the length of a deduction of $u \approx v$ from Σ. It is clear that if $u \approx v$ follows from any set of identities by application of the first three rules, reflexivity, symmetry or transitivity, then $(u \approx v)^*$ follows by the same rules from the corresponding starred identities. We verify that the analogous claim holds for the remaining two rules of deduction. If f_i is an n_i-ary operation symbol and $u_j \approx v_j$ holds for $1 \leq j \leq n_i$, then $f_i(u_1, \ldots, u_{n_i})^* = f_i(u_1^*, \ldots, u_{n_i}^*) \approx f_i(v_1^*, \ldots, v_{n_i}^*) = f_i(v_1, \ldots, v_{n_i})^*$ is a consequence of $u_j^* \approx v_j^*$ for $1 \leq j \leq n_i$. For the replacement rule, suppose that $u' \approx v'$ is obtained from $u \approx v$ by replacing each occurrence of a variable x_k by a term t. Then $(u' \approx v')^*$ can be deduced from $(u \approx v)^*$, by a suitable replacement of the variables in the terms in T_k used in place of x_k in the starred version. ∎

The properties of the class operator Inf are studied in [3], where it is shown that $Inf(V)$ is a variety when V is a variety. It is straightforward to verify that any finite product of generalized inflations of algebras from a variety V is a generalized inflation of the product of the algebras. This shows that the class $GInf(V)$ is closed under the formation of finite products. However, we shall give an example in the next section to show that $GInf(V)$ is not always closed under the formation of subalgebras, and hence is not in general a variety. We do not know if $GInf(V)$ is closed under homomorphic images.

3. Generalized inflations of left zero bands

In the remainder of this paper we investigate null extensions and generalized inflations of (varieties of) idempotent semigroups, or bands. Our type is thus (2), with one binary operation symbol f. For convenience we shall often write semigroup identities in the usual juxtaposition notation, writing xy instead of $f^A(x,y)$, and omitting brackets where allowed. Note however that although the base algebras of the extensions will always be semigroups, the extensions themselves need not be. Note also that for type (2), when $\mathcal{A} = (\mathcal{B}, \Gamma, \hat{\ })$ is a generalized inflation of $\mathcal{B}$, the binary operation on A is given by the formula $xy = \Gamma(x, \hat{y})\Gamma(y, \hat{x})$.

We provide here a list of the varieties of semigroups to which we shall refer:

Sem $= Mod(x(yz) \approx (xy)z)$, the variety of all semigroups;

TR $= Mod(x \approx y)$, the variety of all trivial semigroups;

Z $= Mod(xy \approx zw)$, the variety of all zero semigroups;

LZ $= Mod(x \approx xz)$, the variety of all left zero band semigroups;

RZ $= Mod(x \approx yx)$, the variety of all right zero band semigroups;

RB $= Mod(x \approx x^2, x(yz) \approx (xy)z \approx xz)$, the variety of all rectangular band semigroups;

NB $= Mod(x \approx x^2, x(yz) \approx (xy)z,\ xyzw \approx xzyw)$, the variety of all normal band semigroups;

SL $= Mod(x \approx x^2, x(yz) \approx (xy)z, xy \approx yx)$, the variety of all commutative band semigroups or semilattices;

B $= Mod(x(yz) \approx (xy)z, x \approx x^2)$, the variety of all idempotent semigroups, or bands.

Clarke's and Monzo's Theorem (Theorem 5 in [5]) says that any semigroup null extension of a base semigroup which is a union of groups must in fact be a generalized inflation. This means that for any variety V of unions of groups, $NExt(V) \cap Sem = GInf(V) \cap Sem$, and hence that $GInf(V) \cap Sem$ is a variety.

Our first observation is that $NExt(V)$ and $GInf(V)$ need not be equal, even if V is a variety of unions of groups, when we no longer require associativity. This is a consequence of the following basic fact about generalized inflations. If $\mathcal{A} = (\mathcal{B}, \Gamma, \hat{\ })$ is a generalized inflation of $\mathcal{B}$, then for every $x \in A \setminus B$, we have $xx = f^A(x, x) = f^B(\Gamma(x, \hat{x}), \Gamma(x, \hat{x})) = \Gamma(x, \hat{x})\Gamma(x, \hat{x})$. This means that any product xx in a generalized inflation must be a square in B. For example, let us take $\mathcal{B}$ to be the 2-element group $\{e, a\}$ with identity e. Let A have base set $\{e, a, x\}$. We can define a binary operation on A for which $xx = a$, such that $\mathcal{A}$ is a null extension of $\mathcal{B}$. But $\mathcal{A}$ cannot be a generalized inflation of $\mathcal{B}$, since $xx = a$ is not a square in B. This shows that not every (non-associative) null extension of a union of groups need be a generalized inflation.

Example 3.1. For $V = TR$, the trivial variety, it is easy to see that any generalized inflation or null extension of an algebra in V is a zero semigroup. Thus we have $Z = GInf(V) = NExt(V)$ in this case. In particular, $GInf(TR)$ is a variety.

Example 3.2. Let $V = Z$, the variety of zero semigroups. Any generalized inflation of a zero semigroup is still a zero semigroup: for any elements x, y, u, v in the generalized inflation, we have $xy = \Gamma(x, \hat{y})\Gamma(y, \hat{x}) = \Gamma(u, \hat{v})\Gamma(v, \hat{u}) = uv$. However, we can produce an example of an associative null extension of a zero semigroup which is not a generalized inflation: Take $A = \{b, c, x\}$, $B = \{b, c\} \in Z$, and set $B \times B = \{c\}$ but $xx = b$. This shows that $GInf(Z) \cap Sem \neq NExt(Z) \cap Sem$.

Next, we characterize generalized inflations of left zero bands from the variety LZ. Dual results may of course be shown for the variety RZ of right zero bands.

Lemma 3.3. *Let $\mathcal{A} = (\mathcal{B}, \Gamma, \hat{\ })$ be any generalized inflation of a left zero band $\mathcal{B}$. Then for all x, y, $z \in A$ and all $b \in B$, the following hold:*

(i) $(xy)z = xy$;

(ii) $xb = \Gamma(x, b)$ *and* $xy = x\hat{y} = \Gamma(x, \hat{y})$.

Proof. (i): Using the multiplication of the generalized inflation we have $(xy)z = (\Gamma(x,\hat{y})\Gamma(y,\hat{x}))\Gamma(z,\Gamma(x,\hat{y})\Gamma(y,\hat{x})) = \Gamma(x,\hat{y}) = \Gamma(x,\hat{y})\Gamma(y,\hat{x}) = xy$, for all x, $y \in A$.

(ii): We have $xb = \Gamma(x,b)\Gamma(b,x) = \Gamma(x,b)$ in LZ. Similarly, $xy = \Gamma(x,\hat{y})\Gamma(y,\hat{x}) = \Gamma(x,\hat{y}) = \Gamma(x,\hat{y})\hat{y} = x\hat{y}$. ■

It follows from part (ii) of this Lemma that if a null extension $\mathcal{A}$ of a left zero band $\mathcal{B}$ is to be a generalized inflation of $\mathcal{B}$, then there is a unique way to define the function $\Gamma : A \times B \to B$, namely that $\Gamma(x,b) = xb$. Thus, in order to determine whether a generalized inflation is possible, we must analyze how to choose the base elements $\hat{x}$ associated to new elements x. The sets H_x defined below are the sets of possible such "hats" for each x.

Theorem 3.4. *Let $\mathcal{A}$ be a null extension of $\mathcal{B} \in LZ$. For each $x \in A \setminus B$, let*

$$H_x = \{b \in B \mid wx = wb \text{ for all } w \in A \setminus B\}.$$

Then $\mathcal{A}$ is a generalized inflation of $\mathcal{B}$ iff for all $x \in A \setminus B$, the set H_x is non-empty.

Proof. If $\mathcal{A}$ is a generalized inflation of $\mathcal{B}$, then, by Lemma 3.3(ii), we see that $\hat{x} \in H_x$ for every new element $x \in A \setminus B$. This shows that each H_x is non-empty in a generalized inflation. Conversely, let $\mathcal{A}$ be a null extension of $\mathcal{B}$ for which all the sets H_x are non-empty. For each $x \in A \setminus B$, set $\hat{x}$ to be any element of H_x, and define $\Gamma : (A \setminus B) \times B \to B$ by $\Gamma(x,b) = xb$. Then the generalized inflation $(\mathcal{B}, \Gamma, \hat{\ })$ is precisely the algebra $\mathcal{A}$. ■

In [5], Clarke and Monzo give a construction for expressing any associative null extension of a semigroup which is a union of groups as a generalized inflation. Their method for choosing the 'hat' and Γ functions to use depends strongly on associativity. Our method from Theorem 3.4 can be used even for non-associative null extensions of left zero semigroups, and also gives all possible formulations of the null extension as a generalized inflation. The next example illustrates this process.

Example 3.5. Let $\mathcal{B}$ be a left zero band with universe $B = \{b, c, d\}$. Consider the null extension $\mathcal{A}$ of $\mathcal{B}$ whose universe is the set $\{b, c, d, x, y\}$, and whose binary operation is given by the following table.

$$\begin{array}{c|ccc|cc} \mathcal{A} & b & c & d & x & y \\ \hline b & b & b & b & b & b \\ c & c & c & c & c & c \\ d & d & d & d & d & d \\ \hline x & b & d & b & b & d \\ y & c & c & c & c & c \end{array}$$

Note that the definition of $p \in H_q$ for a new element q requires that the column in the table under p is the same as the column under q, so such sets can easily be computed from the table. In this example, we see that $H_x = \{b, d\}$ and $H_y = \{c\}$. Thus there are two generalized inflations corresponding to the algebra $\mathcal{A}$, one with $\hat{x} = b$ and one with $\hat{x} = d$.

Example 3.6. Now we consider as our base algebra the algebra $\mathcal{A}$ from the previous example. We define a generalized inflation $\mathcal{C} = (\mathcal{A}, \Gamma, \hat{\ })$ of $\mathcal{A}$ by adding a new element $q \in S_y$, so that $\hat{q} = y$, and setting $\Gamma(q, b) = \Gamma(q, c) = \Gamma(q, d) = d$ and $\Gamma(q, x) = \Gamma(q, y) = c$. Then $\mathcal{C}$ is a generalized inflation of a generalized inflation of the left zero band $\mathcal{B}$, with multiplication as shown below, and we can use Theorem 3.4 to determine whether $\mathcal{C}$ is also a generalized inflation of $\mathcal{B}$. However, H_q is empty in this case, since there is no element $p \in B$ whose column in the multiplication table is the same as the column for q. This means that there is no element in the base set B that can act as $\hat{q}$ in a generalized inflation. This example thus shows that a generalized inflation of a generalized inflation need not be a generalized inflation, answering in the more general non-associative setting a question of Clarke and Monzo for the semigroup situation. It also shows that the operator $GInf$ is not idempotent.

$$\begin{array}{c|ccccc|c} \mathcal{C} & b & c & d & x & y & q \\ \hline b & b & b & b & b & b & b \\ c & c & c & c & c & c & c \\ d & d & d & d & d & d & d \\ x & b & d & b & b & d & d \\ y & c & c & c & c & c & c \\ \hline q & d & d & d & c & c & c \end{array}$$

Example 3.7. Let $\mathcal{B}$ be a left zero band with base set $B = \{a, b, c, d\}$. Let $\mathcal{A}$ be the generalized inflation of $\mathcal{B}$ with four new elements x, y, z, w, where $\hat{x} = b$, $\hat{y} = a$, $\hat{z} = d$ and $\hat{w} = c$, and $\Gamma(p, k) = pk$ for all $p \in A \setminus B$ and all $k \in B$, as shown in the multiplication table below. Now consider the set $C = \{y, z, b, c, d\}$. It can be checked that $\mathcal{C}$ is a subalgebra of $\mathcal{A}$. But $\mathcal{C}$ itself is not a generalized inflation of any left zero band. If it were, it would have to be a generalized inflation of $D = Im(\mathcal{C}) = \{b, c, d\}$. But the set H_y is empty here, so no possible choice for $\hat{y}$ exists from the base D. This example shows us that a subalgebra of an algebra in $GInf(LZ)$ is not in $GInf(LZ)$, and hence that $GInf(LZ)$ is not a variety.

$\mathcal{A}$	a	b	c	d	x	y	z	w
a	a	a	a	a	a	a	a	a
b	b	b	b	b	b	b	b	b
c	c	c	c	c	c	c	c	c
d	d	d	d	d	d	d	d	d
x	a	b	b	b	b	a	b	b
y	b	c	b	c	c	b	c	b
z	c	d	b	b	d	c	b	b
w	c	a	c	c	a	c	c	c

Next, we characterize which generalized inflations of left zero semigroups are in fact themselves semigroups. We do this first for the case that only one new element is added to the base algebra, then use this case to prove the general result.

Lemma 3.8. *Let $\mathcal{A} = \mathcal{B} \cup \{x\}$ be a null extension of a left zero semigroup $\mathcal{B}$. Then the following are equivalent:*

(1) *$\mathcal{A} = (\mathcal{B}, \Gamma, \hat{\ })$ is an associative generalized inflation of $\mathcal{B}$;*

(2) *for any $p \in B$, $x(xp) = xx = x(xx)$;*

(3) *$x(xx) = xx$ and for all $p \in B$, either $xp = xx$ or $x(xp) = xx$.*

Proof. (1) $\Rightarrow$ (2) : If $\mathcal{A}$ is an associative generalized inflation of the left zero semigroup $\mathcal{B}$, then, by associativity and the LZ axioms, we have $x(xp) = (xx)p = xx$, for any $p \in B$. Also $x(xx) = \Gamma(x, xx) = \Gamma(x, \Gamma(x, \hat{x})) = x(x\hat{x}) = (xx)\hat{x} = xx$.

$(2) \Leftrightarrow (3)$: This is obvious.

$(2) \Rightarrow (1)$: Suppose that $A = B \cup \{x\}$ satisfies the identities in (2). Then since $xx = x(xx)$, we have $xx \in H_x$, and, by Theorem 3.4, we can make $\mathcal{A}$ into a generalized inflation by taking $\hat{x} = xx$ and setting $\Gamma(x,p) = xp$ for all $p \in B$. Thus, there remains only to show that $\mathcal{A}$ is associative. By Lemma 3.3, we have $x(xx) = xx = (xx)x$, and $x(xp) = xx = (xx)p$ for all $p \in B$. Similarly, $x(px) = \Gamma(x,px)px = \Gamma(x,px) = \Gamma(x,p\Gamma(x,p)) = \Gamma(x,p) = xp = (xp)x$. All the remaining cases needed for associativity can be verified similarly. ∎

We recall that for a new element x in a generalized inflation, and for any base element p, the element $\Gamma(x,p)$ can be viewed as the role model of x when used with p. In the left zero case, we have $xp = \Gamma(x,p)p = \Gamma(x,p)$. We will denote by $R_x = \{xp : p \in \mathcal{B}\}$ the set of all role models of the element x.

Theorem 3.9. *Let $\mathcal{A}$ be a null extension of a left zero semigroup $\mathcal{B}$. Then the following are equivalent:*

(1) *$\mathcal{A}$ is an associative generalized inflation of $\mathcal{B}$.*

(2) *For all $x, y \in A \setminus B$, $xR_y = \{xy\}$ and $xx = x(xx)$.*

Proof. $(1) \Rightarrow (2)$: When $\mathcal{A}$ is an associative generalized inflation of $\mathcal{B}$, we have $x(yp) = (xy)p = xy$ for any $x, y \in A \setminus B$. Therefore, $xR_y = \{xy\}$. The condition $xx = x(xx)$ follows from Lemma 3.8 and the fact that if $\mathcal{A}$ is an associative generalized inflation, so is $\mathcal{B} \cup \{x\}$ for any $x \in A \setminus B$.

$(2) \Rightarrow (1)$: From the identities in (2), it follows from Lemma 3.8 that for any $x \in A \setminus B$, $\mathcal{B} \cup \{x\}$ is an associative generalized inflation of $\mathcal{B}$, using $\hat{x} = xx$. Thus, we can make a generalized inflation of $\mathcal{B}$ on the set A, using $\hat{x} = xx$ and $\Gamma(x,p) = xp$, and the first part of (2) means that this generalized inflation coincides with $\mathcal{A}$. As in Lemma 3.8, we can now verify all cases needed to show that $\mathcal{A}$ is associative. First, $xR_y = \{xy\}$ implies that $x(yp) = xy = (xy)p$ for all $x, y \in A \setminus B$ and $p \in B$. Similarly, $p(xy) = p = (px)y$ and $x(py) = xp = (xp)y$. Moreover, $x(yz) = \Gamma(x,yz)(yz) = \Gamma(x,yz) = \Gamma(x,\Gamma(y,\hat{z})) = x(y\hat{z}) = xy = (xy)z$. Hence, $\mathcal{A}$ is an associative generalized inflation of $\mathcal{B}$. ∎

The symmetric result is also true for right zero semigroups $\mathcal{B}$. Using the above result we can construct examples of associative generalized inflations of left or right zero semigroups.

Example 3.10. Let $\mathcal{B}$ be a finite left zero semigroup with elements b_1, b_2, b_3, b_4, and b_5. Let $\mathcal{A}$ be the infinite null extension of $\mathcal{B}$ shown in the table below. Note that $R_{a_1} = R_{a_6} = \{b_1, b_3\}$, $R_{a_2} = R_{a_3} = \{b_2, b_4\}$, $R_{a_4} = \{b_4, b_5\}$, and $R_{a_5} = \{b_4\}$. We can check that $x(xx) = xx$ and $xR_y = \{xy\}$ for any $x, y \in A \smallsetminus B$. For example, $a_1 R_{a_1} = a_1 R_{a_6} = \{b_1\} = \{a_1 a_5\} = \{a_1 a_6\}$. Thus, $\mathcal{A}$ is an associative generalized inflation of $\mathcal{B}$.

$\mathcal{A}$	b_1	b_2	b_3	b_4	b_5	a_1	a_2	a_3	a_4	a_5	a_6	$\cdots$
b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1
b_2	b_2	b_2	b_2	b_2	b_2	b_2	b_2	b_2	b_2	b_2	b_2	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_1	b_1	b_3	b_1	b_3	b_3	b_1	b_3	b_3	b_3	b_3	b_1	$\cdots$
a_2	b_4	b_2	b_4	b_2	b_2	b_4	b_2	b_2	b_2	b_2	b_4	$\cdots$
a_3	b_2	b_4	b_2	b_4	b_4	b_2	b_4	b_4	b_4	b_4	b_2	$\cdots$
a_4	b_5	b_4	b_5	b_4	b_4	b_5	b_4	b_4	b_4	b_4	b_5	$\cdots$
a_5	b_4	b_4	b_4	b_4	b_4	b_4	b_4	b_4	b_4	b_4	b_4	$\cdots$
a_6	b_3	b_1	b_3	b_1	b_1	b_3	b_1	b_1	b_1	b_1	b_3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Similar examples can be constructed in which some of the new elements have more than two role models, and in which the base algebra $\mathcal{B}$ is infinite.

Corollary 3.11. *Let $\mathcal{A} = \mathcal{B} \cup \{x\}$ be an associative generalized inflation of $\mathcal{B}$ with $xx = b_i \in B$, such that there is exactly one $b_l \in B$ such that $xb_l = b_m \neq b_i$. If $\mathcal{A} \cup \{y\}$ is an associative generalized inflation of $\mathcal{B}$ with $\hat{y} = yy = b_l$, then $A \cup \{y\}$ is an inflation of $\mathcal{A}$.*

Proof. Suppose that $A \cup \{y\}$ is not an inflation of $\mathcal{A}$, so that the new element y has more than one role model. Then there exist base elements $b_k \neq b_j \neq b_i$ such that $yb_k = b_l$ and $yb_j = b_k$ where $b_k \neq b_l$. But now, we have $x(yb_j) = xb_k \neq b_m$, because b_l is the only element satisfying $xb_l = b_m$; but $(xy)b_j = \Gamma(x, \hat{y}) = \Gamma(x, b_l) = xb_l = b_m$, which contradicts associativity. Hence, $A \cup \{y\}$ is an inflation of $\mathcal{A}$. ∎

4. GENERALIZED INFLATIONS OF RECTANGULAR BANDS

In this section, we characterize which null extensions of a rectangular band are generalized inflations, and which generalized inflations of a rectangular band are semigroups.

Lemma 4.1. *Let $\mathcal{A} = (\mathcal{B}, \Gamma, \hat{\ })$ be any generalized inflation of a rectangular band $\mathcal{B}$. Then for all $x, y \in A \setminus B$ and all $b \in B$, the following hold:*

(i) $\Gamma(x, b) = (xb)(bx)$,

(ii) $\Gamma(x, \hat{y}) = (xy)(yx) = (x\hat{y})(\hat{y}x)$.

Proof. (i): Let $x, y \in A \setminus B$ and let $b \in B$. Then in the generalized inflation we have $(xb)(bx) = \Gamma(x, \hat{b})\Gamma(b, \hat{x})\Gamma(b, \hat{x})\Gamma(x, \hat{b}) = \Gamma(x, b)bb\Gamma(x, b)$, and in a rectangular band this is equal to $\Gamma(x, b)$.

(ii): The equality $\Gamma(x, \hat{y}) = (x\hat{y})(\hat{y}x)$ follows from part (i), and the other equality can be verified in a similar fashion. ■

It follows from part (i) of this Lemma that if a null extension $\mathcal{A}$ of a rectangular band $\mathcal{B}$ is to be a generalized inflation of $\mathcal{B}$, then there is a unique way to define the function $\Gamma : A \times B \to B$, namely by $\Gamma(x, b) = (xb)(bx)$.

Theorem 4.2. *Let $\mathcal{A}$ be a null extension of $\mathcal{B} \in RB$. For each $x \in A \setminus B$, let*

$$H_x = \{b \in B \mid (wx)(xw) = (wb)(bw), \textit{ for all } w \in A \setminus B\}.$$

Then $\mathcal{A}$ is a generalized inflation of $\mathcal{B}$ iff for all $x \in A \setminus B$, the set H_x is non-empty.

Proof. When $\mathcal{A}$ is a generalized inflation of $\mathcal{B}$, it follows from Lemma 4.1 (ii) above that for any $x \in A \setminus B$, we have $(wx)(xw) = (w\hat{x})(\hat{x}w)$ for all $w \in A \setminus B$, and hence that $\hat{x} \in H_x$. Conversely, suppose that the set H_x is non-empty, for each $x \in A \setminus B$. Then we choose for $\hat{x}$ any element of H_x. Using this choice of "hats", along with a function Γ defined by $\Gamma(x, b) = (xb)(bx)$, gives us a generalized inflation of $\mathcal{B}$. Moreover, the multiplication in this generalized inflation satisfies $xy = \Gamma(x, \hat{y})\Gamma(y, \hat{x}) = (x\hat{y})(\hat{y}x)(y\hat{x})(\hat{x}y) = (xy)(yx)(yx)(xy)$ in B, by Lemma 4.1, and this gives the product xy in our original algebra $\mathcal{A}$. Thus, the generalized inflation of $\mathcal{B}$ created here is the algebra $\mathcal{A}$. ■

Theorem 4.3. *Let $\mathcal{A} = (\mathcal{B}, \Gamma, \hat{\ })$ be a generalized inflation of $\mathcal{B} \in RB$. Then $\mathcal{A}$ is a semigroup iff $\mathcal{A}$ satisfies*

$$\Gamma(x,a)\Gamma(x,b) = \Gamma(x,ab) \quad \textit{and} \quad \Gamma(x,\Gamma(y,a)) = \Gamma(x,\hat{y}),$$

for all $a, b \in B$ and all $x, y \in A \setminus B$.

Proof. Let $\mathcal{A} = (\mathcal{B}, \Gamma, \hat{\ })$ be a generalized inflation of $\mathcal{B}$ which is a semigroup. Let $a, b \in B$ and $x, y \in A \setminus B$. Then $(xa)b = x(ab)$ implies that $\Gamma(x,a)ab = \Gamma(x,ab)ab$, and, dually, $a(bx) = (ab)x$ implies that $ab\Gamma(x,b) = ab\Gamma(x,ab)$. Multiplying these two equations gives $\Gamma(x,a)abab\Gamma(x,b) = \Gamma(x,ab)abab\Gamma(x,ab)$. In RB this gives $\Gamma(x,a)\Gamma(x,b) = \Gamma(x,ab)$. For the second property, we start with the equation $(xy)a = x(ya)$ in A. We have $x(ya) = \Gamma(x,\Gamma(y,a)a)\Gamma(y,a)a = \Gamma(x,\Gamma(y,a))\Gamma(x,a)\Gamma(y,a)a$, using the first property, while $(xy)a = \Gamma(x,\hat{y})\Gamma(y,\hat{x})a$. Equating these and using the axioms of RB shows that we must have $\Gamma(x,\hat{y})a = \Gamma(x,\Gamma(y,a))a$. A dual argument starting from the equation $(ay)x = a(yx)$ gives the equation $a\Gamma(x,\Gamma(y,a)) = a\Gamma(x,\hat{y})$. Now, multiplying these two equations together and using the RB identities gives $\Gamma(x,\Gamma(y,a)) = \Gamma(x,\hat{y})$.

Conversely, suppose that $\mathcal{A}$ is a generalized inflation of $\mathcal{B}$ which satisfies the two given equations. Then for any $x, y, z \in A$, we have

$$\begin{aligned}
(xy)z &= \Gamma(x,\hat{y})\Gamma(y,\hat{x})\ \Gamma(z,\Gamma(x,\hat{y})\ \Gamma(y,\hat{x}))\\
&= \Gamma(x,\hat{y})\ \Gamma(y,\hat{x})\ \Gamma(z,\Gamma(x,\hat{y}))\ \Gamma(z,\Gamma(y,\hat{x})) && \text{by the first property,}\\
&= \Gamma(x,\hat{y})\ \Gamma(y,\hat{x})\ \Gamma(z,\hat{x})\ \Gamma(z,\hat{y}) && \text{by the second property,}\\
&= \Gamma(x,\hat{y})\ \Gamma(z,\hat{y}) && \text{by the } RB \text{ identities,}\\
&= \Gamma(x,\hat{y})\ \Gamma(x,\hat{z})\ \Gamma(y,\hat{z})\ \Gamma(z,\hat{y}) && \text{by the } RB \text{ identities,}\\
&= \Gamma(x,\Gamma(y,\hat{z}))\ \Gamma(x,\Gamma(z,\hat{y}))\ \Gamma(y,\hat{z})\ \Gamma(z,\hat{y}) && \text{by the second property,}\\
&= \Gamma(x,\Gamma(y,\hat{z})\ \Gamma(z,\hat{y}))\ \Gamma(y,\hat{z})\ \Gamma(z,\hat{y}) && \text{by the first property,}\\
&= x(yz).
\end{aligned}$$

This shows that $\mathcal{A}$ satisfies the associative identity, and is a semigroup. ∎

5. Generalized inflations of normal bands

Now we consider generalized inflations of base algebras which are free normal bands. Let $\mathcal{B}$ be a free algebra in the variety NB on some generating set $X = \{x_1, x_2, x_3, \ldots\}$. Let s and t be any two elements of $\mathcal{B}$, which we can consider as "words" on the alphabet X. We will say that s is a left factor of t if $t = su$ for some (possibly empty) word u, and dually for right factors. Note that in order to discuss longest left and right factors of an element in $\mathcal{B}$, we need to assume that $\mathcal{B}$ is a free normal band on a set of generating letters.

Theorem 5.1. *Let $\mathcal{B}$ be a free normal band, and let $\mathcal{A}$ be a null extension of $\mathcal{B}$. Then A is a generalized inflation of $\mathcal{B}$ iff the following two conditions are satisfied by $\mathcal{A}$:*

(1) *For all $p \in B$ and all $x \in A \smallsetminus B$, $p(xp)(px) = px$ and $(xp)(px)p = xp$;*

(2) *For every $x \in A \smallsetminus B$, there exists an element $x_b \in B$ such that:*

  (a) *For all $y \in A \smallsetminus B$, the elements yx_b and yx have a common left factor, and the elements x_by and xy have a common right factor.*

  (b) *If $x_b = z_b$ for two elements $x, z \in A \smallsetminus B$, then $(xx)(zz) = xz$, and for all $y \in A \smallsetminus B$, the elements yx_b, yx and yz have a common left factor and the elements x_by, xy and zy have a common right factor.*

  (c) *Let $L(x, y)$ be the longest left common factor of the elements yx_b, yx, and any yz for which $z_b = x_b$. Dually, let $R(x, y)$ be the longest right common factor of x_by, xy, and any zy for which $z_b = x_b$. Let Γ' be a function from $A \times B$ to B for which $\Gamma'(y, x_b) = L(x, y)R(x, y)$, and $\Gamma'(y, p) = (yp)(py)$ for any element $p \in B$ which is not in the set $\{x_b \mid x \in A \smallsetminus B\}$. Then for all new elements x and y, $\mathcal{A}$ satisfies $xy = \Gamma'(x, y_b)\Gamma'(y, x_b)$ and $x_b\Gamma'(y, x_b) = x_by$ and $\Gamma'(y, x_b)x_b = yx_b$.*

Proof. First, suppose that $\mathcal{A}$ is a generalized inflation $(\mathcal{B}, \Gamma, \hat{\ })$ of $\mathcal{B}$. We verify that the specified conditions are all met in $\mathcal{A}$.

(1): In the generalized inflation multiplication, we have $p(xp)(px) = p\Gamma(x, p)pp\Gamma(x, p) = p\Gamma(x, p) = px$ and $(xp)(px)p = \Gamma(x, p)pp\Gamma(x, p)p = \Gamma(x, p)p = xp$, using the associativity and idempotence of the base $\mathcal{B}$.

(2): We show that for any $x \in A \smallsetminus B$, the element $\hat{x}$ has the properties needed for x_b.

(a) For any $y \in A \smallsetminus B$, we have $y\hat{x} = \Gamma(y, \hat{x})\hat{x}$ and $yx = \Gamma(y, \hat{x})\Gamma(x, \hat{y})$. Thus these two elements have a common left factor of $\Gamma(y, \hat{x})$. The other part of condition (2a) is verified similarly.

(b) Let $z \in A \smallsetminus B$ such that $\hat{z} = \hat{x}$. Then

$$(xx)(zz) = \Gamma(x, \hat{x})\Gamma(x, \hat{x})\Gamma(z, \hat{z})\Gamma(z, \hat{z}) = \Gamma(x, \hat{x})\Gamma(z, \hat{z})$$
$$= \Gamma(x, \hat{z})\Gamma(z, \hat{x}) = xz.$$

Moreover, for any y, $yx = \Gamma(y, \hat{x})\Gamma(x, \hat{y})$ and $yz = \Gamma(y, \hat{z})\Gamma(z, \hat{y})$, and these two elements have a common left factor of $\Gamma(y, \hat{x})$. The other part of condition (2b) is verified similarly.

(c) From part (2a), we have $yx = \Gamma(y, \hat{x})\Gamma(x, \hat{y})$, which means that the longest left common factor $L(x, y)$ has the form $\Gamma(y, \hat{x})\gamma$, for some possibly empty word γ which is a left factor of $\Gamma(x, \hat{y})$. Similarly, $xy = \Gamma(x, \hat{y})\Gamma(y, \hat{x})$ means that the longest right common factor $R(x, y)$ has the form $\delta\Gamma(y, \hat{x})$, for some possibly empty word δ which is a right factor of $\Gamma(x, \hat{y})$. Dual arguments allow us to write $L(y, x) = \Gamma(x, \hat{y})\alpha$ and $R(y, x) = \beta\Gamma(x, \hat{y})$, where α and β are left and right factors respectively of $\Gamma(y, \hat{x})$. Now, using the normality properties of the base, we have

$$\Gamma'(x, \hat{y})\ \Gamma'(y, \hat{x}) = \Gamma(x, \hat{y})\alpha\beta\Gamma(x, \hat{y})\Gamma(y, \hat{x})\gamma\delta\Gamma(y, \hat{x})$$
$$= \Gamma(x, \hat{y})\alpha\Gamma(y, \hat{x})\beta\gamma\Gamma(x, \hat{y})\delta\Gamma(y, \hat{x}),$$
$$= \Gamma(x, \hat{y})\Gamma(y, \hat{x})\Gamma(x, \hat{y})\Gamma(y, \hat{x})$$
$$= \Gamma(x, \hat{y})\Gamma(y, \hat{x}) = xy.$$

The final two properties follow from a similar argument. From part (2a), we have $y\hat{x} = \Gamma(y, \hat{x})\hat{x}$, which means that the longest left common factor $L(x, y)$ has the form $\Gamma(y, \hat{x})\theta$ for some possibly empty word θ which is a left factor of $\hat{x}$. Similarly, $\hat{x}y = \hat{x}\Gamma(y, \hat{x})$ means that the longest right common factor $R(x, y)$ has the form $\rho\Gamma(y, \hat{x})$, for some possibly empty word ρ which is a right factor of $\hat{x}$. Now, using the normality properties of the base, including the fact that $wlr = w$ for any words w, l and r for which l is a left factor and r is a right factor of w, we have

$$\hat{x}\Gamma'(y,\hat{x}) = \hat{x}\Gamma(y,\hat{x})\theta\rho\Gamma(y,\hat{x}) = \hat{x}\Gamma(y,\hat{x})\Gamma(y,\hat{x}) = \hat{x}\Gamma(y,\hat{x}) = \hat{x}y.$$

This shows that any generalized inflation of a free normal band has all the properties given here, using $x_b = \hat{x}$. Conversely, we show that any null extension $\mathcal{A}$ of a free normal band which has these properties will be a generalized inflation. For any new element $x \in A \setminus B$, we chose $\hat{x}$ to be the given element x_b, and we define the function Γ' as in condition (2c). We claim that the generalized inflation $(\mathcal{B}, \Gamma', \hat{\ })$ agrees with the original null extension algebra $\mathcal{A}$. Condition (2c) means precisely that for any two new elements x and y, the multiplication for xy in the generalized inflation agrees with the product xy in $\mathcal{A}$. We also need to verify that this holds for products of the form px or xp, for a new element x and a base element p. If p is not equal to $\hat{y}$ for any $y \in A \setminus B$, then in the generalized inflation we have $xp = \Gamma'(x,p)p = (xp)(px)p = xp$, and dually $px = p\Gamma'(x,p) = p(xp)(px) = px$, by condition (1). Otherwise, if $p = \hat{y}$ for some y, then we have $xp = \Gamma'(x,p)p = \Gamma'(x,\hat{y})\hat{y} = x\hat{y} = xp$, and similarly for px, by condition (2c). ■

6. GENERALIZED INFLATIONS OF SEMILATTICES

In this section we study generalized inflations of base algebras which are semilattices. In both the left zero and rectangular band cases, we saw that the Γ function of a generalized inflation is completely determined by the multiplication table; but for semilattices, as with normal bands, it is possible for an algebra $\mathcal{A}$ to be a generalized inflation of the base under several different Γ functions. We shall show that for a given choice of hats in a generalized inflation of a semilattice, there is one Γ function which is canonical.

We use the following construction. Let $\mathcal{B}$ be a semilattice, and suppose we have a set of new elements $x \in A \setminus B$, each attached to some element $\hat{x} \in B$, and a Γ function from $A \times B$ to B, given by a table.

Step 1. Construct a partial table for Γ, showing only those entries of the form $\Gamma(x, \hat{y})$.

$\Gamma(x,\hat{y})$	x	y	z	$\cdots$
$\hat{x}$				
$\hat{y}$		$\Gamma(y,\hat{y})$		
$\hat{z}$				
$\vdots$				

Step 2. Construct the multiplication table of $\mathcal{A}$ as follows.

A	B	x	y	z	$\cdots$
B		$px = p\Gamma(x,p)$			
x					
y	$yp = \Gamma(y,p)p$	$xy = \Gamma(x,\hat{y})\Gamma(y,\hat{x})$			
z					
$\vdots$					

Note that the generalized inflation rules $px = p\Gamma(x,p)$ and $yp = \Gamma(y,p)p$ mean that two sections of the multiplication table, those showing products of a new element with an old one, are obtained from the original Γ function table by multiplying table entries by the corresponding old element. If we view the entries of the table obtained in Step 1 as a matrix, we obtain the last section of the multiplication table for $\mathcal{A}$, the products of new elements with new elements, by multiplying each Γ entry by its "transpose" entry, since $xy = \Gamma(x,\hat{y})\Gamma(y,\hat{x})$.

Since $\mathcal{B}$ is a semilattice, there is a natural partial order $\preceq$ defined on the universe set B: we set $x \preceq y$ if x is a sub-word of y (or $xz = y$ for some z). Let $gcd(x,y,\ldots)$ denote the longest common subwords of $x, y, \ldots$. Of course, $gcd(x,y,\ldots) \preceq x$ and $gcd(x,y,\ldots) \preceq y$. Our construction of a canonical Γ function, using these greatest common divisors, will be similar to that used for normal bands; however for semilattices we need consider only the common content, rather than left and right common factors.

Lemma 6.1. *Let $\mathcal{A} = (\mathcal{B}, \Gamma, \hat{\ })$ be a generalized inflation of $\mathcal{B} \in SL$. Construct Γ' as follows:*

$$\Gamma'(x,p) = \begin{cases} p\Gamma(x,p), & \text{if } p \neq \hat{y} \text{ for any } y \in A \setminus B; \\ gcd(px, yx, zx, \ldots), & \text{if } p = \hat{y} = \hat{z} = \ldots \text{ for some } y, z \in A \setminus B. \end{cases}$$

Then $\mathcal{A} = (\mathcal{B}, \Gamma, \hat{\ }) = (\mathcal{B}, \Gamma', \hat{\ })$.

Proof. From the construction of Γ', we have $\Gamma(x,p) \preceq \Gamma'(x,p) \preceq p\Gamma(x,p)$. Therefore, $p\Gamma(x,p) \preceq p\Gamma'(x,p) \preceq p\Gamma(x,p)$ implies that $px = xp = p\Gamma(x,p) = p\Gamma'(x,p)$. Similarly, $\Gamma(x,\hat{y}) \preceq \Gamma'(x,\hat{y}) \preceq yx$ and $\Gamma(y,\hat{x}) \preceq \Gamma'(y,\hat{x}) \preceq xy$ imply that

$$xy = \Gamma(x,\hat{y})\Gamma(y,\hat{x}) \preceq \Gamma'(x,\hat{y})\Gamma'(y,\hat{x}) \preceq (yx)(xy) = xy.$$

Hence, $\mathcal{A}$ is also a generalized inflation of $\mathcal{B}$ using the same hat function and the new Γ function Γ', that is, $\mathcal{A} = (\mathcal{B}, \Gamma', \hat{\ })$. ■

Corollary 6.2. *Let $\mathcal{A}$ be a null extension of $\mathcal{B} \in SL$ with a given 'hat' function $\hat{\ }$. Define*

$$\Gamma'(x,p) = \begin{cases} px = xp, & \text{if } p \neq \hat{y} \text{ for any } y \in A \setminus B; \\ gcd(px, yx, zx, \ldots), & \text{if } p = \hat{y} = \hat{z} = \cdots \text{ for some } y, z \in A \setminus B. \end{cases}$$

Then $\mathcal{A}$ is a generalized inflation of $\mathcal{B}$ if and only if $\mathcal{A}$ is also a generalized inflation of $\mathcal{B}$ by using Γ'.

We shall call the function Γ' thus defined the *canonical* Γ function for a generalized inflation $\mathcal{A}$ of a semilattice $\mathcal{B}$. Next, we characterize when a null extension is a generalized inflation of a semilattice, without having a specified 'hat' or Γ function.

Theorem 6.3. *Let $\mathcal{A}$ be a null extension of $\mathcal{B} \in SL$. Then $\mathcal{A}$ is a generalized inflation of $\mathcal{B}$ if and only if the following conditions hold:*

(1) *$\mathcal{A}$ is commutative and $p(px) = px$ for any $p \in B$ and $x \in A \setminus B$;*

(2) *$H_x = \{p \in B : px = p(xx)\} \neq \emptyset$ for any $x \in A \setminus B$;*

(3) *$\forall y \in A \setminus B$, there exists $b_y \in H_y$ such that $b_y(yy) = b_y y$ and $gcd(b_y x, yx) \neq \emptyset$ for any $x \in A \setminus B$. Moreover, $b_y \cdot gcd(b_y x, yx) = b_y x$;*

(4) *If $b_y = b_z$, then $gcd(yx, zx) \neq \emptyset$ for any $x \in A \setminus B$. Moreover $(yy)(zz) = yz$;*

(5) *For any $x, y \in A \setminus B$, $gcd(b_y x, yx, \ldots) gcd(b_x y, xy, \ldots) = xy$.*

Proof. Let $\mathcal{A} = (\mathcal{B}, \Gamma, \hat{\ })$ be a generalized inflation of $\mathcal{B}$. Then $\mathcal{A}$ must be commutative and $p(px) = p(p\Gamma(x,p)) = p\Gamma(x,p) = px$. Obviously, $\hat{x} \in H_x \neq \emptyset$ for any $x \in A \setminus B$, and we choose $b_x = \hat{x}$ for each x. Therefore, $gcd(b_y x, yx)$ is the greatest common factor of $\hat{y}\Gamma(x,\hat{y})$ and $\Gamma(y,\hat{x})\Gamma(x,\hat{y})$, with a common subword $\Gamma(x,\hat{y})$. Moreover, it follows from $\Gamma(x,\hat{y}) \preceq gcd(b_y x, yx) \preceq b_y x$ and Condition (1) that $\hat{y}\Gamma(x,\hat{y}) = \hat{y}(\hat{y}x) = b_y(b_y x) = b_y x$. From this we conclude that $b_y \cdot gcd(b_y x, yx) = b_y x$. If $\hat{y} = \hat{z}$, then $\Gamma(x,\hat{y}) \preceq gcd(yx, zx) = gcd\left(\Gamma(y,\hat{x})\, \Gamma(x,\hat{y}),\ \Gamma(z,\hat{x})\, \Gamma(x,\hat{z})\right) \neq \emptyset$. Furthermore, $(yy)(zz) = (\Gamma(y,\hat{y})\Gamma(y,\hat{y}))(\Gamma(z,\hat{z})\Gamma(z,\hat{z})) = \Gamma(y,\hat{y})\Gamma(z,\hat{z}) = yz$. Finally, $\Gamma(x,\hat{y}) \preceq gcd(b_y x, yx, \ldots) \preceq yx$ and $\Gamma(y,\hat{x}) \preceq gcd(b_x y, xy, \ldots) \preceq xy$, so it follows that

$$xy = \Gamma(x,\hat{y})\Gamma(y,\hat{x}) \preceq gcd(b_y x, yx, \ldots) gcd(b_x y, xy, \ldots) \preceq (yx)(xy) = xy$$

for any $x, y \in A \setminus B$.

Conversely, let $\mathcal{A}$ be a null extension satisfying these five conditions. We choose $\hat{x} = b_x$ for any $x \in A \setminus B$. We define a Γ function as follows:

$$\Gamma(x,p) = \begin{cases} px, & \text{if } p \neq \hat{y} \text{ for any } y \in A \setminus B; \\ gcd(px, yx, zx, \ldots), & \text{if } p = \hat{y} = \hat{z} = \cdots \text{ for some } y, z \in A \setminus B. \end{cases}$$

Now we show that the generalized inflation $(\mathcal{B}, \Gamma, \hat{\ })$ coincides with the algebra $\mathcal{A}$. For this we compare products in the generalized inflation with products in $\mathcal{A}$, and we note that it suffices to consider only products of the form px or xy, where p is an element of the base B and x and y are new elements. If p is not equal to any b_y, then $p\Gamma(x,p) = p(px) = px$. Otherwise, if $p = b_y$ for some y, then $p \cdot gcd(px, yx, \ldots) = px$ implies that $p\Gamma(x,p) = px$, by Condition (3). Therefore, $px = p\Gamma(x, p)$ in both cases. Condition (5) then says that $xy = \Gamma(x, \hat{y})\Gamma(y, \hat{x})$. Therefore, $\mathcal{A}$ is indeed a generalized inflation of $\mathcal{B}$. ∎

The next example shows that Condition (5) of this theorem is necessary, by giving an algebra $\mathcal{A}$ which satisfies the first four conditions but not the fifth.

Example 6.4. Let $\mathcal{B} = \{a, b, c, ab, ac, bc, abc\}$ be the free semilattice on three generators a, b and c. Let $\mathcal{A}$ be the null extension formed by attaching three new elements x, y, z to $\mathcal{B}$, with multiplication as shown in the table below.

A	a	b	c	ab	ac	bc	abc	x	y	z
a	a	ab	ac	ab	ac	abc	abc	ab	a	abc
b	ab	b	bc	ab	abc	bc	abc	bc	bc	bc
c	ac	bc	c	abc	ac	bc	abc	ac	ac	ac
ab	ab	ab	abc	ab	abc	abc	abc	ab	ab	abc
ac	ac	abc	ac	abc	ac	abc	abc	ac	ac	ac
bc	abc	bc	bc	abc	abc	bc	abc	bc	bc	bc
abc	abc	abc	abc	abc	abc	abc	abc	abc	abc	abc
x	ab	bc	ac	ab	ac	bc	abc	ab	abc	abc
y	a	bc	ac	ab	ac	bc	abc	abc	ab	abc
z	abc	bc	ac	abc	ac	bc	abc	abc	abc	ab

Obviously $\mathcal{A}$ is commutative and $p(px) = px$ for any $p \in B$ and $x, y, z \in A \setminus B$. It can be checked that $H_x = \{a, ab, abc\}$, $H_y = \{ab, abc\}$ and $H_z = \{abc\}$. For $\mathcal{A}$ to be a generalized inflation of $\mathcal{B}$, by condition (4), we need $\hat{x} \neq \hat{y} \neq \hat{z}$. Hence the only possible choice is to have $\hat{x} = b_x = a$, $\hat{y} = b_y = ab$, and $\hat{z} = b_z = abc$. It is straightforward to check that the required *gcd* conditions are also met, so $\mathcal{A}$ satisfies the first four conditions of the theorem. But note that we have the following table:

Γ	x	y	z
$\hat{x}$	ab	a	abc
$\hat{y}$	ab	ab	abc
$\hat{z}$	abc	abc	abc

Thus, we have $\Gamma(x, \hat{y})\Gamma(y, \hat{x}) = (ab)(a) = ab$ which is not $xy = abc$ in the original multiplication table. Therefore, $\mathcal{A}$ can not be a generalized inflation of $\mathcal{B}$.

REFERENCES

[1] S. Bogdanović and S. Milić, *Inflation of semigroups*, Publ. Inst. Math. (N.S.) **41** (**55**) (1987), 63–73.

[2] I. Chajda, *Normally presented varieties*, Algebra Universalis **34** (1995), 327–335.

[3] A. Christie, Q. Wang and S.L. Wismath, *Class pperators involving inflations*, preprint 2003.

[4] G.T. Clarke, *Semigroup varieties of inflations of unions of groups*, Semigroup Forum **23** (1981), 311–319.

[5] G.T. Clarke and R.A.R. Monzo, *A generalization of the concept of an inflation of a semigroup*, Semigroup Forum **60** (2000), 172–186.

[6] A.H. Clifford and G.B. Preston, *The Algebraic Theory of Semigroups*, Vol. 1, Amer. Math. Soc., Providence, R.I, 1961.

[7] K. Denecke and S.L. Wismath, *A characterization of k-normal varieties*, Algebra Universalis **51** (2004), 395–409.

[8] E. Graczyńska, *On normal and regular identities and hyperidentities*, p. 107–135 in: *"Universal and Applied Algebra, (Turawa, 1988)"*, World Scientific Publishing Co., Teaneck, NJ, 1989.

[9] I.I. Mel'nik, *Nilpotent shifts of varieties*, (in Russian), Mat. Zametki, **14** (1973), 703–712; English translation in: Math. Notes **14** (1973), 962–966.

[10] S. Milić, *Inflation of algebras*, p. 89–98 in: *"Procedings of the Conference Algebra and Logic, Sarajevo 1987"*, Univ. Novi Sad, Novi Sad 1989.

[11] T. Tamura, *Semigroups satisfying identity $xy = f(x, y)$*, Pacific J. Math. **31** (1969), 513–521.

[12] M. Yamada, *A note on middle unitary semigroups*, Kodai Math. Sem. Rep. **7** (1955), 49–52.

Received 17 May 2004
Revised 8 July 2004