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CONVERGENCE WITH A REGULATOR IN DIRECTED GROUPS

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Abstract

There is defined and studied a convergence with a fixed regulator u in directed groups. A u-Cauchy completion of an integrally closed directed group is constructed.

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B.Z. Vulikh [6] has defined the notion of a convergence of sequences with a regulator (r-convergence) in a vector lattice V. A convergence regulator depends on a sequence in V. In the book [5], W.A.J. Luxemburg and A.C. Zaanen introduced the notion of a convergence in V with a fixed regulator for all sequences in V.

The definition from [5] was formally changed in [1] and then applied in a lattice ordered group G to define the notion of a convergence with a fixed regulator u (u-convergence) in G. For an Archimedean lattice ordered group G there was defined and investigated the concept of a u-Cauchy completion of G.

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In the present note the notion of a u-convergence is introduced in directed groups in such a way that it coincides with that in lattice ordered groups. Further, there is constructed a u-Cauchy completion G^* of an integrally closed directed group G.

1. Preliminaries

We recall some relevant basic notions and results (cf. [2], [3], [4]) concerning ordered groups.

Let G be a partially ordered group and let $\mathbb{N}$ ($\mathbb{Z}$, resp.) be the set of all positive integers (integers, resp.). An element $0 < e \in G$ is called a *strong unit* of G if for each $x \in G$ there exists $n \in \mathbb{N}$ with $ne > x$. A partially ordered group G will be said to be *integrally closed* if for all $x, y \in G$, $nx \leq y$ for each $n \in \mathbb{N}$ implies $x \leq 0$. We will call G *Archimedean* if for all $x, y \in G, nx \leq y$ for each $n \in \mathbb{Z}$ implies $x = 0$. If G is integrally closed, then G is Archimedean. The converse does not hold in general (for example the additive group G of all complex numbers with $G^+ = \{x + iy : x = y = 0$ or $x > 0$ and $y > 0\}$ is Archimedean and it fails to be integrally closed), but it does for lattice ordered groups. G is called *directed* if its partial order is upward (or, equivalently downward) directed. Equivalently, G is directed if and only if every element $g \in G$ may be expressed in the form $g = x - y$ for suitable elements $x, y \in G,\ x, y \geq 0$.

Assume that G is a lattice ordered group and $x, y \in G$. Putting $|x| = x \vee (-x)$ we define the *absolute value* of x. Evidently, $|x| \leq y$ if and only if $-y \leq x \leq y$.

G is *torsion free*, i.e., $x \neq 0$ entails $nx \neq 0$ for each $n \in \mathbb{N}$.

If G is Abelian and $n \in \mathbb{N}$, then

$$nx < ny \text{ implies } x < y. \tag{1}$$

A partially ordered group G is said to be *complete* if every nonempty subset of G bounded from above has a least upper bound in G. Every directed complete partially ordered group is a lattice ordered group. For a subset X of $G, U(X)$ ($L(X)$, resp.) will denote the set of all upper (lower, resp.) bounds of X in G.

Theorem 1.1 (cf. [4], Theorem 9.A). *Let G be a directed group. Then G can be embedded into a complete lattice ordered group H if and only if G is integrally closed.*

Theorem 1.2 ([3], Theorem 18). *A complete lattice ordered group is Abelian.*

By using Theorems 1.1 and 1.2, we immediately obtain

Corollary 1.3 ([3], Corollary 20). *Every integrally closed directed group is Abelian.*

Assume that G is an integrally closed directed group. Then, by Theorems 1.1 and 1.2, it follows that the implication (1) is valid in H and also in G. Since H is torsion-free, G is as well.

2. u-convergence in directed groups

In this section we recall the notion of a convergence with a fixed regulator in vector lattices and formally change it to be applicable in directed groups. The notion of a convergence with a fixed regulator u (u-convergence) is defined and studied in directed groups.

Definition 2.1 (cf. [5]). Let V be a vector lattice and $0 \leq u \in V$. It is said that a sequence (x_n) in V *u-converges* to an element $x \in V$ if the following condition is satisfied:

(c) for every real number $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|x_n - x| \leq \varepsilon u \ \text{ for each } \ n \in \mathbb{N}, n \geq n_0.$$

In the condition (c), ε can be equivalently replaced by $\frac{1}{p}$ $(p \in \mathbb{N})$. Then

$$|x_n - x| \leq \frac{1}{p}u, \ \text{ i.e.,} \quad -\frac{1}{p}u \leq x_n - x \leq \frac{1}{p}u.$$

Therefore, the condition (c) is equivalent to the condition

(c') for each $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ with

$$-u \leq p(x_n - x) \leq u \ \text{ for each } \ n \in \mathbb{N}, n \geq n_0.$$

The condition (c') can be applied also in directed groups.

Definition 2.2. Let G be a directed group, and $0 < u \in G$. We say that a sequence (x_n) in G *u-converges* to an element $x \in G$ (or x *is a u-limit of* (x_n)), written $x_n \xrightarrow{u} x$, if the condition (c‘) is satisfied.

The element u is called a *convergence regulator*. Let R be the additive group of all reals with the natural linear order. If $G = R$, then for each $0 < u \in G$ the notion of u-convergence coincides with the usual convergence.

Theorem 2.3. *Let G be an integrally closed directed group and $0 < u \in G$. Then u-limits in G are uniquely determined.*

Proof. Assume that (x_n) is a sequence in G such that $x_n \xrightarrow{u} x$ and $x_n \xrightarrow{u} y$. Let $p \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ with

$$-u \leq 2p(x_n - x) \leq u \ \text{ and } \ -u \leq 2p(x_n - y) \leq u \ \text{ for each } \ n \in \mathbb{N}, n \geq n_0.$$

We have

$$-u \leq 2p(x_{n_0} - x) \leq u \quad \text{and} \quad -u \leq 2p(x_{n_0} - y) \leq u.$$

Therefore,

$$2p(y - x) = 2p(y - x_{n_0} + x_{n_0} - x) \leq 2u$$

and

$$2p(x - y) = 2p(x - x_{n_0} + x_{n_0} - y) \leq 2u.$$

From this, we infer that $p(y - x) \leq u$ and $p(x - y) \leq u$. The hypothesis yields $y - x \leq 0$ and $x - y \leq 0$. Thus $x = y$. ■

Lemma 2.4. *Let G be a non-Archimedean directed group. Then there exists $0 < u \in G$ such that u-limits are not uniquely determined.*

Proof. There are $a, b \in G, a \neq 0, b > 0$ such that $ka \leq b$ for each $k \in \mathbb{Z}$. Since G is directed, there exist $0 \leq x, y \in G, x \neq y$ with $a = x - y$. Consider the sequence $(x_n) = (x, y, x, y, \ldots)$. Let $p \in \mathbb{N}$. We obtain

$$p(x_n - x) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \\ p(y - x) = (-p)a, & \text{if } n \text{ is odd.} \end{cases}$$

Then $-b \leq (-p)a \leq b$. If we put $u = b$, we get $x_n \xrightarrow{u} x$. Analogously, we derive that $x_n \xrightarrow{u} y$. ∎

Assume that G is a non-integrally closed directed group. If G is non-Archimedean, then, by Lemma 2.4, u-limits are not uniquely determined for some $0 < u \in G$. If G is Archimedean, the question of the uniqueness of u-limits remains open.

The idea of the proofs of Theorem 2.3 and Lemma 2.4 is the same as for corresponding results in [1], where it was used in the case of lattice-ordered groups.

In what follows, G is assumed to be an integrally closed directed group and $0 < u \in G$ a fixed convergence regulator in G. We shall write $x_n \to x$ (or $x_n \to x$ in G) instead of $x_n \xrightarrow{u} x$. By a convergent sequence, a u-convergent sequence is meant and a limit will mean a u-limit.

Lemma 2.5. *Let (x_n) and (y_n) be sequences in $G, x_n \to x, y_n \to y$. Then $x_n + y_n \to x + y$.*

Proof. Let $p \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ with

$$-u \leq 2p(x_n - x) \leq u, \ -u \leq 2p(y_n - y) \leq u \ \text{ for each } \ n \in \mathbb{N}, \ n \geq n_0.$$

Then

$$-2u \leq 2p(x_n + y_n - (x + y)) \leq 2u,$$

$$-u \leq p(x_n + y_n - (x + y)) \leq u,$$

i.e.,

$$x_n + y_n \to x + y.$$

∎

Lemma 2.6. *If $x_n \to x$ and $k \in \mathbb{Z}$, then $kx_n \to kx$.*

Proof. Let $p \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ with

$$-u \leq p|k|(x_n - x) \leq u \text{ for each } n \in \mathbb{N}, n \geq n_0.$$

This yields

$$-u \leq pk(x_n - x) \leq u, \text{ for each } n \in \mathbb{N}, n \geq n_0.$$

Thus, $kx_n \to kx$. ∎

Definition 2.7. A sequence (x_n) in G is said to be *u-fundamental* (abbreviate to *fundamental*) in G if for each $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that

$$-u \leq p(x_n - x_m) \leq u, \text{ for each } m, n \in \mathbb{N}, m \geq n \geq n_0.$$

Lemma 2.8. *Every convergent sequence in G is fundamental in G.*

Proof. Assume that (x_n) is a convergent sequence in $G, x_n \to x$. Let $p \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ with

$$-u \leq 2p(x_n - x) \leq u, \text{ for each } n \in \mathbb{N}, n \geq n_0.$$

For each $m, n \in \mathbb{N}, m \geq n \geq n_0,$ we get $-u \leq 2p(x - x_m) \leq u$. Therefore, $-2u \leq 2p(x_n - x_m) \leq 2u$. Hence, $-u \leq p(x_n - x_m) \leq u$ and the proof is finished. ■

In general, a fundamental sequence is not convergent. Indeed, it suffices to put $G = Q$, where Q is the additive group of all rationals with the natural linear order. If all fundamental sequences in G are convergent, then we shall refer to G as *u-Cauchy complete* (shortly *C-complete*).

3. u-Cauchy completion of G

Remind that G stands for an integrally closed directed group. In this section a construction of a u-Cauchy completion of G will be presented.

Lemma 3.1. *Every fundamental sequence in G is bounded.*

Proof. Suppose that (x_n) is a fundamental sequence in G. Let $p \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ such that

$$-u \leq p(x_n - x_m) \leq u \text{ for each } m, n \in \mathbb{N}, m \geq n \geq n_0.$$

Therefore,

$$x_{n_0} - u \leq x_m \leq x_{n_0} + u \text{ for each } m \in \mathbb{N}, m \geq n_0.$$

Let $h \in U(\{x_1, x_2, \dots, x_{n_0-1}, x_{n_0} + u\})$ and $l \in L(\{x_1, x_2, \dots, x_{n_0-1}, x_{n_0} - u\})$. Then we obtain $l \leq x_m \leq h$ for each $m \in \mathbb{N}$. ■

By a *zero sequence* is understood a sequence (x_n) with $x_n \to 0$. The set of all fundamental (zero, resp.) sequences in G is denoted by F (E, resp.).

Define the operation $+$ in F by putting $(x_n)+(y_n) = (x_n+y_n)$. Further, set $(x_n) \leq (y_n)$ if and only if $x_n \leq y_n$ for each $n \in \mathbb{N}$. It is clear that $(F, \leq)$ is a partially ordered set. Moreover, we have

Lemma 3.2. *$(F, +, \leq)$ is an integrally closed directed group.*

Proof. Let $(x_n), (y_n) \in F$ and $p \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ with

$$-u \leq 2p(x_n - x_m) \leq u \quad \text{and} \quad -u \leq 2p(y_n - y_m) \leq u$$

for each $$m, n \in \mathbb{N}, m \geq n \geq n_0.$$

Then

$$-2u \leq 2p(x_n + y_n) - (x_m + y_m) \leq 2u.$$

Thus,

$$-u \leq p((x_n + y_n) - (x_m + y_m)) \leq u.$$

Hence, $(x_n + y_n) \in F$.

From $(x_n) \in F$, it follows $(-x_n) \in F$ as well; $(-x_n)$ is the inverse to (x_n). Hence, F is a group.

According to Lemma 3.1, there are $h_i, l_i (i = 1, 2)$ with

$$l_1 \leq x_n \leq h_1 \quad \text{and} \quad l_2 \leq y_n \leq h_2 \quad \text{for each} \quad n \in \mathbb{N}.$$

Let $h \in U(\{h_1, h_2\})$ and $l \in L(\{l_1, l_2\})$.Constant sequences $(l, l, \ldots)$ and $(h, h, \ldots)$ belong to F. We have

$$(l, l, \ldots) \leq (x_n) \leq (h, h, \ldots) \quad \text{and} \quad (l, l, \ldots) \leq (y_n) \leq (h, h, \ldots).$$

This shows that F is a directed set. Since the group operation is performed componentwise, F is an integrally closed partially ordered group and the proof is complete. ∎

Lemma 3.3. *E is a convex subgroup of F.*

Proof. In view of Lemma 2.8, we have $E \subseteq F$. According to Lemmas 2.5 and 2.6, E is a subgroup of F. Assume that $(x_n) \in E, (y_n) \in F$ and $0 \le (y_n) \le (x_n)$. Then for each $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ with $-u \le py_n \le px_n \le u$ each $n \in \mathbb{N}, n \ge n_0$. Therefore $(y_n) \in E$ and E is convex in F. ■

Since F is Abelian, we can form the factor group $G^* = F/E$. We use $(x_n)^*$ to denote the class of G^* containing a sequence $(x_n) \in F$. Let $(x_n)^*, (y_n)^* \in G^*$. By Lemma 3.2, we obtain $(x_n + y_n) \in F$. We have $(x_n)^* + (y_n)^* = (x_n + y_n)^*$. The group G^* can be made into a partially ordered group by putting $(x_n)^* \le (y_n)^*$ if and only if there exist $(x'_n) \in (x_n)^*$ and $(y'_n) \in (y_n)^*$ such that $(x'_n) \le (y'_n)$ (equivalently, if for each $(x'_n) \in (x_n)^*$ there is $(y'_n) \in (y_n)^*$ with $(x'_n) \le (y'_n)$).

It is easy to see that $(x_n)^* \le (y_n)^*$ if and only if there exists a sequence $(t_n) \in E$ with $(x_n) \le (y_n) + (t_n)$. By using Lemma 3.2, we obtain that G^* is an Abelian directed group.

Now, we are interested in the question whether G^* is integrally closed. The following theorem offers a partial answer.

Theorem 3.4. *Let G be an integrally closed directed group and let u be a strong unit of G. Then G^* is integrally closed.*

Proof. Let $(x_n)^*, (y_n)^* \in G^*$ such that $k(x_n)^* \le (y_n)^*$ for each $k \in \mathbb{N}$. We are going to show that $(x_n)^* \le E$.

For each $k \in \mathbb{N}$, there exists a sequence $(t_n^k) \in E$ with

$$k(x_n) \le (y_n) + (t_n^k).$$

With respect to Lemma 3.1, there exists $g \in G$ with $y_n \le g$ for each $n \in \mathbb{N}$. Since u is a strong unit of G, there exists $m \in \mathbb{N}$ with $mu > g$.

We have

$$(m+1)kx_n \le y_n + t_n^{(m+1)k} \le g + t_n^{(m+1)k} < mu + t_n^{(m+1)k}$$

for each $n \in \mathbb{N}$.

From $(t_n^{(m+1)k}) \in E$, we infer that there exists $n_0 \in \mathbb{N}$ with $t_n^{(m+1)k} \le u$ for each $n \in \mathbb{N}, n \ge n_0$. Let $n \in \mathbb{N}, n \ge n_0$. We get

$$(m+1)kx_n < mu + u = (m+1)u.$$

Hence, $kx_n < u$. From the assumption that G is integrally closed, it follows $x_n \leq 0$ for each $n \in \mathbb{N}, n \geq n_0$, proving $(x_n)^* \leq E$. ∎

The proof of Theorem 3.4 is similar to that in [1] applied for an Archimedean lattice ordered group.

Consider the class T of integrally closed directed group G such that G^* is again such one. This class is rich. Indeed, every Archimedean linearly ordered group, $G \in T$. It is a consequence of the fact that every Archimedean linearly ordered group is a subgroup of R. Another groups from the class T can be obtained by using Theorem 3.4. Let G be an integrally closed directed group and $0 < g \in G$. Then $G(g) = \bigcup[-ng, ng](n \in \mathbb{N})$ is a convex subgroup of G generated by g. It is easily seen that $G(g)$ is an integrally closed directed group and g is a strong unit of $G(g)$. Applying Theorem 3.4, we conclude $G(g) \in T$.

The question whether Theorem 3.4 holds without assuming that u is a strong unit of G, remains open.

Definition 3.5. Let $G \in T$ and u be a convergence regulator in G. Then G^* is called a *u-Cauchy completion* (shortly a *C-completion)* of G.

Throughout, it will be supposed that G^* is a C-completion of G.

The element $U = (u, u, \ldots)^*$ is considered as a convergence regulator in G^*. For $(x_n) \in F$, denote $X_n = (x_n, x_n, \ldots)^*$.

Define the mapping $\Psi : G \to G^*$ by the rule $\Psi(x) = (x, x, \ldots)^*$ for each $x \in G$. Then Ψ is an isomorphism of a directed group G into G^*.

Theorem 3.6. *Every element of G^* is a U-limit of some sequence in $\Psi(G)$.*

Proof. Let $(x_n)^* \in G^*$. Then (X_n) is a sequence in $\Psi(G)$. We intend to show that $X_n \xrightarrow{U} (x_n)^*$.

Let $n_1 \in \mathbb{N}$ be fixed. An easy verification establishes that

$$X_{n_1} - (x_n)^* = (x_{n_1},\ x_{n_1},\ \ldots)^* - (x_1,\ x_2,\ \ldots,\ x_n,\ x_{n+1},\ x_{n+2},\ \ldots)^* =$$

$$(x_{n_1} - x_1,\ x_{n_1} - x_2,\ \ldots,\ x_{n_1} - x_{n_1},\ x_{n_1} - x_{n_1+1},\ x_{n1} - x_{n_1+2},\ \ldots)^* =$$

$$(0,\ x_{n_1} - x_{n_1+1},\ x_{n_1} - x_{n_1+2}, \ldots)^* = (x_{n_1} - x_m)^* \text{ (where } m \geq n_1).$$

Let $p \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ with

$$-u \leq p(x_n - x_m) \leq u \text{ for each } m, n \in \mathbb{N}, m \geq n \geq n_\circ.$$

Suppose that $n \in \mathbb{N}, n \geq n_\circ$ is fixed and $m \in \mathbb{N}, m \geq n$. Then for the sequence $(x_n - x_m)$ $(m \in \mathbb{N})$, we obtain $(x_n - x_m) \in F$ and

$$-U \leq p(x_n - x_m)^* = p(X_n - (x_n)^*) \leq U,$$

completing the proof. ∎

F^* will denote the set of all U-fundamental sequences in G^*.

Theorem 3.7. *Let (x_n) be a sequence in G. Then $(x_n) \in F$ if and only if $(X_n) \in F^*$.*

Proof. Assume that $(x_n) \in F$ and $p \in \mathbb{N}$. There exists $n_\circ \in \mathbb{N}$ with

$$-u \leq p(x_n - x_m) \leq u \text{ for each } m, n \in \mathbb{N}, m \geq n \geq n_\circ.$$

Then

$$(-u, -u, \ldots) \leq p(x_n - x_m, x_n - x_m, \ldots) \leq (u, u, \ldots),$$

$$-U \leq p(x_n - x_m, x_n - x_m, \ldots)^* \leq U,$$

$$-U \leq p((x_n, x_n \ldots)^* - (x_m, x_m, \ldots)^*) \leq U,$$

$$-U \leq p(X_n - X_m) \leq U$$

and thus, $(X_n) \in F^*$.

Conversely, suppose that $(X_n) \in F^*$ and $p \in \mathbb{N}$. Then there exists $n_0 \in \mathbb{N}$ with

$$-U \leq 2p(X_n - X_m) \leq U \text{ for each } m, n \in \mathbb{N}, m \geq n \geq n_0.$$

We have

$$X_n - X_m = (x_n, x_n, \ldots)^* - (x_m, x_m, \ldots)^* = (x_n - x_m, x_n - x_m, \ldots)^*.$$

Then

$$-(u, u, \ldots)^* \leq 2p(x_n - x_m, x_n - x_m, \ldots)^* \leq (u, u, \ldots)^*.$$

Let $m, n \geq n_0$ be fixed. For each $p \in \mathbb{N}$ there exists a sequence $(t_s^p) \in E$ $(s \in \mathbb{N})$ such that

$$2p(x_n - x_m) \leq u + t_s^p \quad \text{for each} \quad s \in \mathbb{N}.$$

There exists $s_0 \in \mathbb{N}$ such that $t_s^p \leq u$ for each $s \in \mathbb{N}, s \geq s_0$. We set $t'^p_s = t_s^p$ if $s \in \mathbb{N}, s \geq s_0$ and $t'^p_s = t'^p_{s_0}$, if $s \in \mathbb{N}, s < s_0$. Then $(t'^p_s) \in E$ and $t'^p_s \leq u$ for each $s \in \mathbb{N}$. We get $2p(x_n - x_m) \leq 2u$. Hence, $p(x_n - x_m) \leq u$. In a similar way, we get $-u \leq p(x_n - x_m)$. Whence, $(x_n) \in F$. ■

Theorem 3.8. *G^* is C-complete.*

Proof. Let $X^1 = (x_m^1)^*, X^2 = (x_m^2)^*, \ldots$ be a sequence from G^*.We have to show that this sequence is U-convergent. According to Theorem 3.6, every element $X^n = (x_m^n)^*$ of the sequence is a U-limit of some sequence in $\Psi(G)$, namely $X_m^n = (x_m^n, x_m^n, \ldots)^* \xrightarrow{U} X^n$. For each $n \in \mathbb{N}$ can be found $m_n \in \mathbb{N}$ such that $-U \leq k(X_m^n - X^n) \leq U$ for each $m, k \in \mathbb{N}, m \geq m_n, k \leq n$. Whence $-U \leq k(X_{m_n}^n - X^n) \leq U$ for each $k \in \mathbb{N}, k \leq n$. If we denote $\mathbb{Z}_n = X_{m_n}^n$, then

$$-U \leq k(\mathbb{Z}_n - X^n) \leq U \text{ for each } k \in \mathbb{N}, k \leq n.$$

We are going to prove that $(\mathbb{Z}_n) \in F^*$.

We have

$$\mathbb{Z}_n - \mathbb{Z}_m = (\mathbb{Z}_n - X^n) + (X^n - X^m) + (X^m - \mathbb{Z}_m).$$

Let $p \in \mathbb{N}$. There is $n_0 \in \mathbb{N}, n_0 \geq 3p$, such that for each $m, n \in \mathbb{N}, m \geq n \geq n_0$, we get

$$-U \leq 3p(\mathbb{Z}_n - X^n) \leq U, -U \leq 3p(X^n - X^m) \leq U, -U \leq 3p(X^m - \mathbb{Z}_m) \leq U$$

which entails

$$-3U \leq 3p(\mathbb{Z}_n - \mathbb{Z}_m) \leq 3U.$$

Therefore,

$$-U \leq p(\mathbb{Z}_n - \mathbb{Z}_m) \leq U,$$

that means $(\mathbb{Z}_n) \in F^*$.

Under the notation $z_n = x^n_{m_n}$, we have $\mathbb{Z}_n = (z_n, z_n, \dots)^*$. By Theorem 3.7 from $(\mathbb{Z}_n) \in F^*$, it follows $(z_n) \in F$. We intend to show that $X^n \to (z_n)^*$.

By Theorem 3.6, $\mathbb{Z}_n \to (z_n)^*$. We have

$$X^n - (z_n)^* = (X^n - \mathbb{Z}_n) + (\mathbb{Z}_n - (z_n)^*).$$

Let $p \in \mathbb{N}$. There is $n_0 \in \mathbb{N}, n_0 \geq 2p$ such that for each $n \in \mathbb{N}, n \geq n_0$, we obtain

$$-U \leq 2p(X^n - \mathbb{Z}_n) \leq U, -U \leq 2p(\mathbb{Z}_n - (z_n)^*).$$

This yields

$$-2U \leq 2p(X^n - (z_n)^*) \leq 2U \text{ for each } n \in \mathbb{N}, n \geq n_0.$$

Whence,

$$-U \leq p(X^n - (z_n)^*) \leq U \text{ for each } n \in \mathbb{N}, n \geq n_0$$

and the proof is complete. ■

If x and $\Psi(x)$ an identified for each $x \in G$, then u is a convergence regulator in G^* and it follows at once from Theorems 3.8 and 3.6.

Theorem 3.9. *If* $G \in T$, *then*

(i) *C-completion G^* of G is C-complete;*

(ii) *G is a subgroup of G^* with the induced partial order;*

(iii) *every element of G^* is a limit of some sequence in G.* ■

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