

## LATTICE-INADMISSIBLE INCIDENCE STRUCTURES \*

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### Abstract

Join-independent and meet-independent sets in complete lattices were defined in [6]. According to [6], to each complete lattice  $(L, \leq)$  and a cardinal number  $p$  one can assign (in a unique way) an incidence structure  $\mathcal{J}_L^p$  of independent sets of  $(L, \leq)$ . In this paper some lattice-inadmissible incidence structures are founded, i.e. such incidence structures that are not isomorphic to any incidence structure  $\mathcal{J}_L^p$ .

**Keywords:** complete lattices, join-independent and meet-independent sets, incidence structures.

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Let  $(L, \leq)$  be a complete lattice and let  $\bigvee, \bigwedge$  be the supremum and the infimum of any subset of  $L$ , respectively. The least and the greatest elements in  $(L, \leq)$  are denoted by  $0, 1$  respectively. If  $x, y \in L$ , then  $x \parallel y$  means that  $x, y$  are incomparable in  $(L, \leq)$ . If  $X \subseteq L$ , then we put  $X_x := X \setminus \{x\}$  for  $x \in X$  and

$$J(X) = \left\{ \bigvee X_x \mid x \in X \right\}, \quad M(X) = \left\{ \bigwedge X_x \mid x \in X \right\}.$$

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**Definition 1.** A subset  $X \subseteq L$  is said to be *join-independent* (*meet-independent*) if and only if  $x \not\leq \bigvee X_x$  ( $\bigwedge X_x \not\leq x$ , resp.) for all  $x \in X$ .

**Remark 1.** The concept of independence have been studied in various types of lattices motivated by applications in algebra and geometry (refer to [1]–[3], [5], [12]). Our approach is explained in [6] in detail and it is used also in [11].

**Remark 2.** A set  $X = \{x\}$  is *join-independent* (*meet-independent*) if and only if  $x \neq 0$  ( $x \neq 1$ ). If  $\text{card}(X) = |X| \geq 2$ , then  $X$  is join-independent (meet-independent) if and only if  $x \parallel \bigvee X_x$  ( $x \parallel \bigwedge X_x$ , resp.) for all  $x \in X$ .

To avoid trivial cases we will suppose that  $|X| > 2$  in what follows. The notions of join- and meet-independent sets are dual in complete lattices. Each assertion about join-independent sets admits its corresponding dual one which will not be stated explicitly.

The set of all join-independent (meet-independent) sets of cardinality  $p > 2$  will be denoted by  $G^p$  ( $M^p$ , respectively).

The following proposition is obvious:

**Proposition 1.** *Let  $x, y$  be distinct elements of a set  $X \in G^p$ . Then  $x \parallel y$  and  $\bigvee X_x \parallel \bigvee X_y$ .* ■

To every subset  $X \subseteq L$  we assign a system  $U_X$  of subsets of  $L$  by setting  $Y \in U_X$  iff there exists a bijective mapping  $\alpha : X \rightarrow Y$  such that  $\bigvee X_x \leq \alpha(x)$  and  $\alpha(x) \parallel x$  for all  $x \in X$ . This mapping is called a *U-mapping*.

Dually, to a subset  $X \subseteq L$  we assign a system  $V_X$  of subsets of  $L$  by setting  $Z \in V_X$  iff there exists a bijective mapping  $\beta : X \rightarrow Z$  such that  $\beta(x) \leq \bigwedge X_x$  and  $\beta(x) \parallel x$  for all  $x \in X$ . This mapping is called a *V-mapping*. It is easy to show: If  $\alpha$  is a *U-mapping*, then  $\alpha^{-1}$  is a *V-mapping*.

The proof of the following proposition is straightforward.

**Proposition 2.** *Let  $X \subseteq L$ . Then the following statements are equivalent:*

(1)  $X \in G^p$ ,

(2)  $J(X) \in U_X$ ,

(3)  $U_X \neq \emptyset$ . ■

**Proposition 3.** *Let  $X \subseteq L$  where  $|X| = p$ . If  $Y \in U_X$ , then  $Y \in M^p$  and  $X \in V_Y$ .*

**Proof.** Let  $Y \in U_X$ . Then a  $U$ -mapping  $\alpha : X \rightarrow Y$  exists. Let us put  $Y_{\alpha(x)} = Y \setminus \{\alpha(x)\}$  for all  $x \in X$ . If  $\alpha(y) \in Y_{\alpha(x)}$ , then  $y \in X_x$  and  $x \in X_y$  which yields  $x \leq \bigvee X_y \leq \alpha(y)$ . Hence,  $x \leq \bigwedge Y_{\alpha(x)}$ . If  $\bigwedge Y_{\alpha(x)} \leq \alpha(x)$ , then  $x \leq \alpha(x)$  which is a contradiction. Thus,  $Y \in M^p$ . Since  $\alpha^{-1} : Y \rightarrow X$  is a  $V$ -mapping we get  $X \in V_Y$ . ■

**Proposition 4.** *Let  $X \subseteq L$ . Then the following statements are equivalent:*

- (1)  $X \in G^p$ ,
- (2)  $J(X) \in M^p$ .

**Proof.** (1)  $\Rightarrow$  (2) : It follows from Proposition 2 and 3.

(2)  $\Rightarrow$  (1) : Let  $J(X) \in M^p$ . If we put  $P_x = J(X) \setminus \bigvee X_x$  for  $x \in X$ , then  $\bigwedge P_x \not\leq \bigvee X_x$  and  $\bigwedge P_x \leq \bigvee X_y$  for each  $y \in X_x$ . Let us assume that  $x \leq \bigvee X_x$ . Then  $\bigvee X_x = \bigvee X$  and  $\bigvee X_y \leq \bigvee X_x$  for all  $y \in X_x$ . Thus,  $\bigwedge P_x \leq \bigvee X_x$  which is a contradiction. Hence,  $x \not\leq \bigvee X_x$  and  $X \in G^p$ . ■

**Proposition 5.** *Let  $X \in G^p$  and  $Y \subseteq L$ . Then*

- (1)  $Y \in U_X$

*if and only if*

- (2) *there exists a bijective mapping  $\gamma : J(X) \rightarrow Y$  such that  $m \leq \gamma(m)$  for each  $m \in J(X)$  and  $\gamma(m) \parallel n$  for all  $n \in J(X)$  distinct from  $m$ .*

**Proof.** Since  $X$  is a join-independent set the mapping  $\beta : x \mapsto \bigvee X_x$ ,  $x \in X$ , is a bijection of  $X$  onto  $J(X)$ .

(1)  $\Rightarrow$  (2) : It follows from  $Y \in U_X$  that there exists a  $U$ -mapping  $\alpha : X \rightarrow Y$ . Let us put  $\gamma = \alpha\beta^{-1}$ . If  $m \in J(X)$ , then  $m = \bigvee X_x$  for a certain  $x \in X$  and  $\gamma(\bigvee X_x) = \alpha(x)$ . Thus,  $\bigvee X_x \leq \gamma(\bigvee X_x)$ . Consider  $n \in J(X)$  where  $n \neq m$ . Then  $n = \bigvee Y_y$  where  $y \neq x$ . If  $\alpha(x) \leq \bigvee X_y$ , then  $\bigvee X_x \leq \alpha(x) \leq \bigvee X_y$  which contradicts Proposition 1. If  $\bigvee X_y \leq \alpha(x)$ , then  $x \leq \bigvee X_y \leq \alpha(x)$ , a contradiction again. Hence,  $\alpha(x) \parallel \bigvee X_y$  and  $\gamma(m) \parallel n$ .

(2)  $\Rightarrow$  (1) : The mapping  $\alpha = \gamma\beta$  is a bijection of  $X$  onto  $Y$  with  $\alpha(x) = \gamma(\bigvee X_x)$  for  $x \in X$ . It suffices to show that  $\alpha$  is a  $U$ -mapping. ■

**Proposition 6.** *If  $X \subseteq L$  and  $Y \in V_X$ , then  $U_X \cap U_Y = \emptyset$ .*

**Proof.** If  $|X| = p$ , then  $Y \in V_X$  yields  $Y \in G^p$  and  $J(Y) \in M^p$ . By Proposition 3,  $X \in U_Y$  and there exists a mapping  $\gamma : J(Y) \rightarrow X$  given in Proposition 5. Assume that  $A \in U_X$ . According to Proposition 5, for each  $a \in A$  there is a unique element  $\bigvee X_x \in J(X)$  such that  $\bigvee X_x \leq a$ . Then  $z \leq a$  for all  $z \in X_x$ . It follows from  $p > 2$  that  $X_x$  contains at least two distinct elements  $z_1, z_2$ . If we put  $\gamma^{-1}(z_1) = m_1$ ,  $\gamma^{-1}(z_2) = m_2$ , then we obtain  $m_1 \leq z_1 \leq a$ ,  $m_2 \leq z_2 \leq a$ . Thus, by Proposition 5,  $A \notin U_Y$ . ■

**Proposition 7.** *Let  $X, Y \in G^p$ . Then  $J(X) = J(Y)$  if and only if  $U_X = U_Y$ .*

**Proof.**

1. Let  $J(X) = J(Y)$  and consider  $C \in U_X$ . Then, by Proposition 5, there exists a mapping  $\gamma : J(X) \rightarrow C$ . Since  $J(X) = J(Y)$ , we obtain  $C \in U_Y$  and thus,  $U_X \subseteq U_Y$ . It is also obvious that  $U_Y \subseteq U_X$ .
2. Let  $U_X = U_Y$ . Since  $J(X) \in U_X$  and  $J(Y) \in U_Y$ , we get  $J(X) \in U_Y$  and  $J(Y) \in U_X$ . It follows from  $J(X) \in U_Y$  that there exists a bijection  $\gamma : J(X) \rightarrow J(Y)$  established in Proposition 5 and for each  $\bigvee X_x \in J(X)$  there exists a unique element  $\bigvee Y_y$  such that  $\bigvee X_x \leq \bigvee Y_y$ . If we put  $\xi_1(x) = y$ , we get a bijective mapping of  $X$  onto  $Y$ . Similarly, with the help of  $J(Y) \in U_X$  we define a bijective mapping  $\xi_2 : Y \rightarrow X$  such that  $\xi_2(m) = n$  if and only if  $\bigvee Y_m \leq \bigvee X_n$ . For  $x \in X$  we get  $\bigvee X_x \leq \bigvee Y_{\xi_1(x)} \leq \bigvee X_{\xi_2 \xi_1(x)}$  and, by Proposition 1,  $x = \xi_2 \xi_1(x)$ . Consider  $\bigvee X_x \in J(X)$ . Then  $\bigvee X_x \leq \bigvee Y_{\xi_1(x)}$  and, with respect to  $\xi_1(x) \in Y$ , we obtain  $\bigvee Y_{\xi_1(x)} \leq \bigvee X_{\xi_2 \xi_1(x)} = \bigvee X_x$ . Thus,  $\bigvee X_x = \bigvee Y_{\xi_1(x)}$  and  $\bigvee X_x \in J(Y)$ . Therefore,  $J(X) \subseteq J(Y)$  and  $J(Y) \subseteq J(X)$  can be obtained similarly. ■

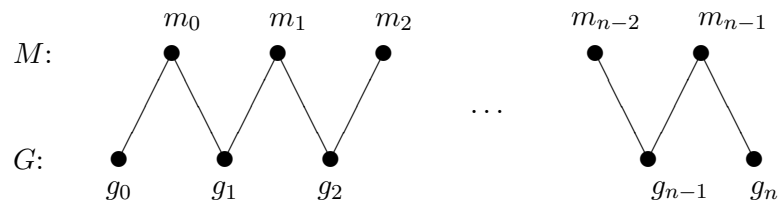
As in [6], to  $(L, \leq)$  and  $p$  an incidence structure can be assigned. We recall the definition and some basic facts (more thoroughly, see [4]) about incidence structures needed in what follows.

**Definition 2.** An *incidence structure (context)* is a triple of sets  $\mathcal{J} = (G, M, I)$ , where  $I \subset G \times M$ . An incidence structure  $\mathcal{J}_1 = (G_1, M_1, I_1)$  is a *substructure* of  $\mathcal{J}$  if  $G_1 \subseteq G$ ,  $M_1 \subseteq M$  and  $I_1 = I \cap (G_1 \times M_1)$ .

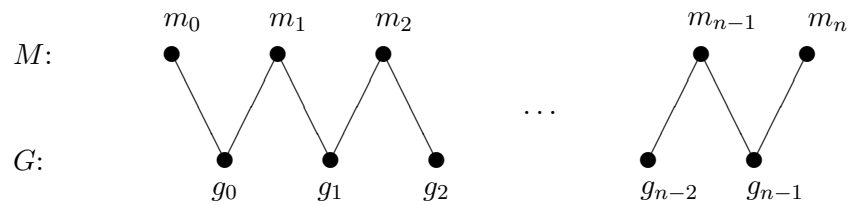
**Remark 3.** Incidence structures are often given by their graphs: The elements of sets  $G, M$  are represented by points and those corresponding to elements  $g \in G, m \in M$  are joined by a line-segment iff  $gIm$ .

**Definition 3.** An incidence structure  $\mathcal{J} = (G, M, I)$  having the following incidence graph is called a *simple connection*

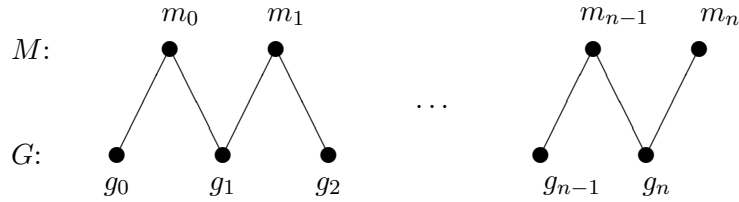
(a) of type 1:



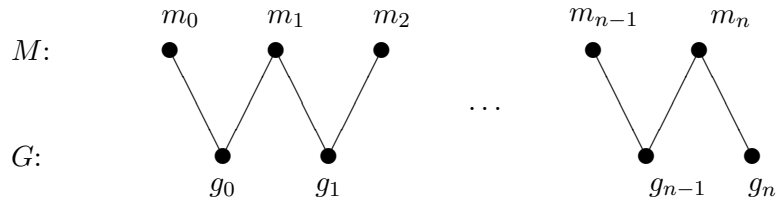
(b) of type 1':



(c) of type 2:



(d) of type 2':



The positive integer  $n$  is said to be a *length* of this connection.

Let  $\mathcal{J} = (G, M, I)$  be an incidence structure. Then for every subset  $A \subseteq G$ , respectively  $B \subseteq M$ , we put  $A^\uparrow = \{m \in M \mid (\forall g \in A)[gIm]\}$ ,  $B^\downarrow = \{g \in G \mid (\forall m \in B)[gIm]\}$ . In [7], *independent sets* in  $G$  and  $M$  are defined and to each cardinal number  $p$  the incidence structure  $\mathcal{J}^p$  of independent sets of cardinality  $p$  is assigned.

If  $(L, \leq)$  is a complete lattice, then  $\mathcal{J}_L = (L, L, I)$  is an incidence structure in which  $aIb$  iff  $a \leq b$  for  $a, b \in L$ . Join- and meet-independent sets in  $(L, \leq)$  are independent in  $\mathcal{J}_L$  in the sense of [7]. To  $(L, \leq)$  and a cardinal  $p$  the incidence structure  $\mathcal{J}_L^p = (G^p, M^p, I^p)$  is assigned, where  $A I^p B$  iff  $B \in U_A$  for any  $A \in G^p$ ,  $B \in M^p$  (see [6]). It is obvious that  $A^\uparrow = U_A$ ,  $B^\downarrow = V_B$  for  $A \in G^p$ ,  $B \in M^p$ .

**Definition 4.** An incidence structure  $\mathcal{J}$  is said to be *lattice-inadmissible* if there do not exist a complete lattice  $L$  and a cardinal number  $p > 2$  such that the associated incidence structure  $\mathcal{J}_L^p$  is isomorphic to  $\mathcal{J}$ . Otherwise,  $\mathcal{J}$  is called *lattice-admissible*.

**Remark 4.** Each incidence structure  $\mathcal{J} = (G, M, I)$  with  $\{g\}^\uparrow = \emptyset$  ( $\{m\}^\downarrow = \emptyset$ , respectively) for some  $g \in G$  ( $m \in M$ ) is lattice-inadmissible, since  $U_A \neq \emptyset$  ( $V_B \neq \emptyset$ ) for every  $A \in G^p$  ( $B \in M^p$ , resp.).

Some other examples of lattice-inadmissible incidence structures are given below.

**Proposition 8.** *Let  $X \in G^p \cap M^p$ . Then*

$$(1) \quad X \not I^p X,$$

and

$$(2) \quad \text{if } X I^p C \text{ and } B I^p X, \text{ then } B \not I^p C.$$

**Proof.** From  $B I^p X$ , we get  $B \in V_X$  and, by Proposition 6,  $U_X \cap U_B = \emptyset$ . If  $X I^p C$  and  $B I^p X$ , then  $C \in U_X \cap U_B$  which is a contradiction. Obviously,  $X I^p J(X)$  and  $M(X) I^p X$ . Since  $M(X) \in V_X$ , we obtain  $U_X \cap U_{M(X)} = \emptyset$  again. If  $X I^p X$ , then  $X \in U_X \cap U_{M(X)}$  which is a contradiction. ■

**Corollary 1.**

1. *If an incidence structure  $\mathcal{J} = (G, M, I)$  contains an element  $x \in G \cap M$  such that  $x I x$ , then  $\mathcal{J}$  is lattice-inadmissible. In particular, for any (non-empty) complete lattice  $(L, \leq)$ , the incidence structure  $\mathcal{J}_L$  is lattice-inadmissible, since  $a I a$  for all  $a \in L$ .*
2. *If  $\mathcal{J} = (G, M, I)$  contains elements  $x \in G \cap M$ ,  $b \in G$ ,  $c \in M$  such that  $x I c$ ,  $b I x$  and  $b I c$ , then  $\mathcal{J}$  is lattice-inadmissible.* ■

**Theorem 1.** *Let  $(L, \leq)$  be a complete lattice and  $p > 2$ . Then, in  $L$ , there do not exist pairwise distinct subsets  $A, B, C \in G^p$ ,  $X, Y, Z \in M^p$  such that  $U_A = \{X\}$ ,  $U_B = \{X, Y\}$ ,  $U_C = \{Y, Z\}$ ,  $V_X = \{A, B\}$ ,  $V_Y = \{B, C\}$ .*

**Proof.** Let us suppose that such subsets exist. Then obviously  $X = J(A)$ . If furthermore  $X = J(B)$ , then  $U_A = U_B$ , by Proposition 7, which is a contradiction. Hence,  $Y = J(B)$  and similarly  $Z = J(C)$ . Since  $X = J(A) = \{\bigvee A_x \mid x \in A\} \in M^p$ , we get  $M(X) = \{\bigwedge P_x \mid x \in A\}$ , where

$P_x = X \setminus \{\bigvee A_x\}$ . Moreover,  $a \leq \bigwedge P_a$  for all  $a \in A$  and  $a \parallel \bigwedge P_x$  for all  $x \in A_a$ . It follows from  $V_X = \{A, B\}$  that either  $A = M(X)$  or  $B = M(X)$ . Let  $B = M(X)$ . Then there is a unique  $a \in A$  such that  $B = \{\bigwedge P_a\} \cup A_a$ , where  $a < \bigwedge P_a$  and  $x = \bigwedge P_x$  for all  $x \in A_a$ . Obviously,  $B \setminus \{\bigwedge P_a\} = A_a$  and  $\bigvee B \wedge P_a = \bigvee A_a$ . For  $y \in A_a$ , we get  $x \leq \bigvee A_y$  for all  $x \in A_y \setminus \{a\}$  and also  $a \leq \bigwedge P_a \leq \bigvee A_y$ . This yields  $\bigvee A_y = \bigvee B_y$  and  $X = J(B)$ , which is a contradiction. Thus,  $A = M(X)$ . In a similar way, from  $V_Y = \{B, C\}$ , we show that  $B = M(Y)$ .

Since  $V_X = \{A, B\}$  and  $A = M(X)$ , there exists precisely one element  $a \in A$  such that  $B = \{b\} \cup A_a$ , where  $b < a$  and  $b \parallel x$  for all  $x \in A_a$ . Then  $B_b = A_a$  and  $\bigvee B_b = \bigvee A_a$ . It follows from  $U_B = \{X, Y\}$  and  $Y = J(B)$  that there exists a unique  $y \in A_a$  such that  $\bigvee B_y < \bigvee A_y$  and  $\bigvee B_x = \bigvee A_x$  for each  $x \in A_a \setminus \{y\}$ . Hence,  $Y = \{\bigvee B_y\} \cup \{\bigvee A_a\} \cup \{\bigvee A_x \mid x \in A_a \setminus \{y\}\}$ . Since  $Y \in M^p$ , we get  $\bigvee B_y \parallel \bigvee A_x$  for all  $x \in A_y$ .

It follows from  $V_Y = \{B, C\}$  and  $B = M(Y)$  that  $C = \{c\} \cup B_z$  for some  $z \in B$ , where  $c < z$  and  $c \parallel x$  for all  $x \in B_z$ .

Since  $Z = J(C)$ , it is obvious that  $Z = \{\bigvee C_q \mid q \in C\}$ . It follows from  $U_C = \{Y, Z\}$  that  $|Y \cap Z| = p - 1$ . Let us prove that  $X \in U_C$  by assigning a mapping  $\gamma$  of the set  $J(C) = Z$  onto the set  $X$  (from Proposition 5). We examine all particular cases.

1. Suppose that  $z = b$ . Then  $c < b < a \leq \bigvee A_x$  for all  $x \in A_a$  and  $C = \{c\} \cup A_a$ . Obviously  $c \parallel \bigvee A_a$  and  $\bigvee C_c = \bigvee A_a$ . Moreover,  $\bigvee C_y \leq \bigvee B_y < \bigvee A_y$  and  $\bigvee C_x \leq \bigvee A_x$  for all  $x \in A_a \setminus \{y\}$ , where, since  $|Y \cap Z| = p - 1$ , precisely one inequality  $\leq$  is replaced by the strict one. Thus,  $Z = \{\bigvee A_a\} \cup \{\bigvee C_x \mid x \in A_a\}$ . Consider a mapping  $\gamma : Z \rightarrow X$  defined by setting  $\gamma(\bigvee A_a) = \bigvee A_a$ ,  $\gamma(\bigvee C_x) = \bigvee A_x$  for all  $x \in A_a$ . It is easy to see that  $m \leq \gamma(m)$  for all  $m \in Z$ . We prove that  $\gamma(m) \parallel n$  for all  $n \in Z \setminus \{m\}$ .
  - a) Let  $\bigvee C_y < \bigvee B_y$ . Then  $Z = \{\bigvee C_y\} \cup \{\bigvee A_x \mid x \in A_y\}$ . It suffices to show that  $\bigvee C_y \parallel \bigvee A_q$  for  $q \in A_y$ . Let  $\bigvee C_y \leq \bigvee A_a$ . Then, from  $c \leq \bigvee C_y$ , we get  $c \leq \bigvee A_a$ , which is a contradiction. Let  $\bigvee C_y \leq \bigvee A_x$  for  $x \in A_a \setminus \{y\}$ . Then  $x \leq \bigvee C_y$ , which is a contradiction again.
  - b) Let  $\bigvee C_q < \bigvee A_q$  for a certain  $q \in A_a \setminus \{y\}$ . Then  $Z = \{\bigvee B_y\} \cup \{\bigvee C_q\} \cup \{\bigvee A_x \mid x \in A \setminus \{q, y\}\}$ . It suffices to show that  $\bigvee C_q \parallel \bigvee A_x$  for  $x \in A_q$ . Suppose that  $\bigvee C_q \leq \bigvee A_a$ . Then, from  $c \leq \bigvee C_q$ , we get  $c \leq \bigvee A_a$ , which is a contradiction. If  $\bigvee C_q \leq \bigvee A_x$  for  $x \in A_a \setminus \{q\}$ , then we obtain a contradiction again, because of  $x \leq \bigvee C_q$ .



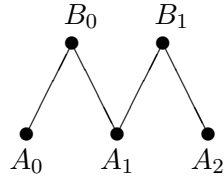
2. Let  $z = y$ . Then  $c \parallel \bigvee A_y$  and  $\bigvee C_c = \bigvee B_y < \bigvee A_y$ ,  $\bigvee C_b \leq \bigvee A_a$ ,  $\bigvee C_x \leq \bigvee A_x$  for all  $x \in A_a \setminus \{y\}$ . It is easy to see that  $Z = \{\bigvee B_y\} \cup \{\bigvee C_q \mid q \in B_y\}$ . The mapping  $\gamma$  is defined by setting  $\gamma(\bigvee B_y) = \bigvee A_y$ ,  $\gamma(\bigvee C_b) = \bigvee A_a$ ,  $\gamma(\bigvee C_x) = \bigvee A_x$  for  $x \in A_a \setminus \{y\}$ . Further, we proceed similarly to the case 1.
- a) Let  $\bigvee C_b < \bigvee A_a$ . Then  $Z = \{\bigvee B_y\} \cup \{\bigvee C_b\} \cup \{\bigvee A_x \mid x \in A_a \setminus \{y\}\}$ . If  $\bigvee C_b \leq \bigvee A_y$ , then  $c \leq \bigvee C_b$  yields  $c \leq \bigvee A_y$ , which is a contradiction. If  $\bigvee C_b \leq \bigvee A_x$  for  $x \in A \setminus \{y\}$ , then  $x \in \bigvee A_x$ .
- b) Let  $\bigvee C_q < \bigvee A_q$  for a certain  $q \in A_a \setminus \{y\}$ . Then  $Z = \{\bigvee B_y\} \cup \{\bigvee C_q\} \cup \{\bigvee A_x \mid x \in B \setminus \{q, y\}\}$ . Similarly to the preceding case, we show that  $\bigvee C_x \parallel \bigvee A_x$  for  $x \in A_q$ .
3. Let  $z \in A_a \setminus \{y\}$ . Then  $c \parallel \bigvee A_z$  and  $\bigvee C_c = \bigvee B_z = \bigvee A_z$ ,  $\bigvee C_b \leq \bigvee A_a$ ,  $\bigvee C_y \leq \bigvee B_y < \bigvee A_y$  and  $\bigvee C_x \leq \bigvee A_x$  for remaining  $x \in A$ . Let us put  $\gamma(\bigvee C_c) = \bigvee A_z$ ,  $\gamma(\bigvee C_b) = \bigvee A_a$ ,  $\gamma(\bigvee C_y) = \bigvee A_y$  and  $\gamma(\bigvee C_x) = \bigvee A_x$  for remaining  $x \in A$ .
- a) Let  $\bigvee C_b < \bigvee A_a$ . If  $\bigvee C_b \leq \bigvee A_z$ , then  $c \leq \bigvee A_z$ , which is a contradiction. For  $x \in A_a \setminus \{z\}$ , it follows from  $\bigvee C_b \leq \bigvee A_x$  that  $x \leq \bigvee A_x$ .
- b) Let  $\bigvee C_y < \bigvee B_y$ . Then  $\bigvee C_y \leq \bigvee A_a$  implies  $b \leq \bigvee A_a$ ,  $\bigvee C_y \leq \bigvee A_z$  implies  $c \leq \bigvee A_z$ , and for remaining  $x \in A$ , we get  $x \leq \bigvee A_x$ , which is a contradiction in all cases.
- c) Let  $\bigvee C_q < \bigvee A_q$  for  $q \in A_a \setminus \{y, z\}$ . Similarly to the preceding cases, we show that  $\bigvee C_x \parallel \bigvee A_x$  for  $x \in A_q$ .

Thus, we have obtained  $X \in U_C$ , which contradicts our assumption  $U_C = \{Y, Z\}$ . ■

**Remark 5.** The dual statement also holds, where  $V_X = \{A\}$ ,  $V_Y = \{A, B\}$ ,  $V_Z = \{B, C\}$  and  $U_A = \{X, Y\}$ ,  $U_B = \{Y, Z\}$ .

**Corollary 2.** *Every simple connection (of type 1, 1', 2, 2') of the length greater than 1 is a lattice-inadmissible incidence structure.*

**Proof.** Consider a complete lattice  $(L, \leq)$ . Let  $\mathcal{J}_L^p = (G^p, M^p, I^p)$  be a simple connection of type 1 and of the length 2. Thus, its graph can be sketched as follows:



Obviously,  $B_0 = J(A_0)$ . If  $B_0 = J(A_1)$ , then  $U_{A_0} = U_{A_1}$ , which is a contradiction. Hence,  $B_1 = J(A_1)$ . However, it means that  $B_1 = J(A_2)$ , which is a contradiction again. Dually, we can proceed for any simple connection of type 1' and of the length 2.

Consider a simple connection  $\mathcal{J}_L^p$  of type 1 and of the length greater than 2 or a simple connection of type 2 and of the length at least 2. Then  $\mathcal{J}_L^p$  contains sets  $A_0, A_1, A_2 \in G^p$  and  $B_0, B_1, B_2 \in M^p$  such that  $U_{A_0} = \{B_0\}$ ,  $U_{A_1} = \{B_0, B_1\}$ ,  $U_{A_2} = \{B_1, B_2\}$ ,  $V_{B_0} = \{A_0, A_1\}$ ,  $V_{B_1} = \{A_1, A_2\}$ . According to Theorem, such sets cannot exist. Similar assertion for simple connections of types 1', 2' holds dually. ■

**Remark 6.** Simple connections of the length 1 are lattice-admissible incidence structures (refer to [6] for an example of a simple connection of type 2).

**Remark 7.** There exists a complete lattices  $(L, \leq)$  and a cardinal  $p$  such that the incidence structure  $\mathcal{J}_L^p$  contains a simple connection of the length greater than 1 as its substructure.

There exist (general) incidence structures  $\mathcal{J}$  such that their corresponding incidence structures  $\mathcal{J}^p$  of independent sets are simple connections. In [8]–[10], there are such incidence structures  $\mathcal{J}$  investigated that  $\mathcal{J}^p$  are simple connections of type 1.

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