

## ON TERNARY SEMIFIELDS

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### Abstract

In this paper, we introduce the notion of ternary semi-integral domain and ternary semifield and study some of their properties. In particular we also investigate the maximal ideals of the ternary semiring  $\mathbf{Z}_0^-$ .

**Keywords:** ternary semiring, prime ideal, maximal ideal, ternary semi-integral domain, ternary division semiring, ternary semifield.

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### 1. INTRODUCTION

The notion of ternary algebraic system was introduced by D.H. Lehmer in [7]. He investigated certain ternary algebraic systems called triplexes which turn out to be commutative *ternary groups* (in the sense of W. Dörnte or D.H. Lehmer see [7]). The notion of ternary semigroups was introduced by S. Banach (cf. [9]). He showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup. These two algebraic structures were further studied by different authors in the middle part of the twentieth century. In [10], M.L. Santiago developed the theory of ternary semigroups and semiheaps. He devoted his attention mainly to the study of regular ternary semigroups, completely regular ternary semigroups, bi-ideals

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and intersection ideals in ternary semigroups, the standard embedding of a ternary semigroup and a semiheap with some of their applications. In [8], W.G. Lister characterized those additive subgroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. He also studied the imbedding of ternary rings, representation of ternary rings in terms of modules, semisimple ternary rings with minimum condition, and radical theory of such rings.

In [1], we have introduced the notion of ternary semiring which is a generalization of the ternary ring introduced by Lister [8]. Ternary semiring arises naturally as follows – consider the ring of integers  $\mathbf{Z}$  which plays a vital role in the theory of ring. The subset  $\mathbf{Z}^+$  of all positive integers of  $\mathbf{Z}$  is an additive semigroup which is closed under the ring product i.e.  $\mathbf{Z}^+$  is a semiring. Now, if we consider the subset  $\mathbf{Z}^-$  of all negative integers of  $\mathbf{Z}$ , then we see that  $\mathbf{Z}^-$  is an additive semigroup which is closed under the triple ring product (however,  $\mathbf{Z}^-$  is not closed under the binary ring product), i.e.  $\mathbf{Z}^-$  forms a ternary semiring. Thus, we see that in the ring of integers  $\mathbf{Z}$ ,  $\mathbf{Z}^+$  forms a semiring whereas  $\mathbf{Z}^-$  forms a ternary semiring. More generally, in an ordered ring, we can see that its positive cone forms a semiring, whereas its negative cone forms a ternary semiring. Thus a ternary semiring may be considered as a counterpart of semiring in an ordered ring.

The main purpose of this paper is to study ternary semifield. In Section 2, we give some preliminary definitions and examples. In Section 3, we introduce the notion of ternary semi-integral domain and ternary semifield and study some of their properties. In Section 4, we characterize the maximal ideals of the ternary semiring  $\mathbf{Z}_0^-$ .

## 2. SOME BASIC DEFINITIONS AND EXAMPLES

**Definition 2.1** (see [1]). A non-empty set  $S$  together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a *ternary semiring* if  $S$  is an additive commutative semigroup satisfying the following conditions:

- (i)  $(abc)de = a(bcd)e = ab(cde)$ ,
- (ii)  $(a + b)cd = acd + bcd$ ,
- (iii)  $a(b + c)d = abd + acd$ ,
- (iv)  $ab(c + d) = abc + abd$

for all  $a, b, c, d, e \in S$ .

**Example.** Let  $S$  be the set of all continuous functions  $f : X \rightarrow \mathbf{R}^-$ , where  $X$  is a topological space and  $\mathbf{R}^-$  is the set of all negative real numbers. Now we define a binary addition and a ternary multiplication on  $S$  in the following way:

$$\text{i) } (f + g)(x) = f(x) + g(x),$$

$$\text{ii) } (fgh)(x) = f(x)g(x)h(x),$$

for all  $f, g, h \in S$  and  $x \in X$ .

Then together with the binary addition and the ternary multiplication,  $S$  forms a ternary semiring.

**Remark.** We note that the positive real valued continuous functions form a semiring, whereas the negative real valued continuous functions form a ternary semiring.

**Definition 2.2.** A ternary semiring  $S$  is said to be

(i) *commutative* if  $abc = bac = bca$  for all  $a, b, c \in S$ ;

(ii) *laterally commutative* if  $abc = cba$  for all  $a, b, c \in S$ .

We note that if  $S$  is commutative, then also  $abc = acb = cba = cab$  for all  $a, b, c \in S$ .

**Definition 2.3.** Let  $S$  be a ternary semiring. If there exists an element  $0 \in S$  such that  $0 + x = x$  and  $0xy = xy0 = 0$  for all  $x, y \in S$ , then '0' is called the *zero element* (or simply the *zero*) of the ternary semiring  $S$ . In this case we say that  $S$  is a ternary semiring with zero.

We note that a ternary semiring may not contain an identity but there are certain ternary semirings which generate identities in the sense defined below:

**Definition 2.4.** A ternary semiring  $S$  admits an *identity* provided that there exist elements  $e_i, f_i$  in  $S$  ( $i = 1, 2, \dots, n$ ) such that  $\sum_{i=1}^n e_i f_i t = \sum_{i=1}^n e_i t f_i = \sum_{i=1}^n t e_i f_i = t$  for all  $t \in S$ . In this case, the ternary semiring  $S$  is said to be a *ternary semiring with identity*  $(e_i, f_i)$ .

**Example.** Let  $\mathbf{Z}_0^-$  be the set of all negative integers with zero. Then with the usual binary addition and ternary multiplication,  $\mathbf{Z}_0^-$  forms a commutative ternary semiring with zero and  $(-1, -1)$  is the identity of  $\mathbf{Z}_0^-$ .

Throughout this paper,  $S$  will always denote a ternary semiring with zero and unless otherwise stated a ternary semiring means a ternary semiring with zero.

Let  $A, B, C$  be three subsets of  $S$ . Then by  $ABC$ , we mean the set of all finite sums of the form  $\sum a_i b_i c_i$ , with  $a_i \in A, b_i \in B, c_i \in C$ .

**Definition 2.5.** An additive subsemigroup  $I$  of  $S$  is called a *left (right, lateral) ideal* of  $S$  if  $s_1 s_2 i \in I$  (respectively  $i s_1 s_2, s_1 i s_2 \in I$ ) for all  $s_1, s_2 \in S$  and  $i \in I$ . If  $I$  is simultaneously a left, a right, and a lateral ideal of  $S$ , then  $I$  is called an *ideal* of  $S$ .

**Definition 2.6** (see also [6], p. 79, for binary semirings). An ideal  $I$  of  $S$  is called a *k-ideal* if  $x + y \in I$ , and  $x \in S, y \in I$  imply that  $x \in I$ .

Let  $A$  be an ideal of  $S$ . Then the *k-closure* of  $A$ , denoted by  $\bar{A}$ , is defined by  $\bar{A} = \{a \in S : a + b = c \text{ for some } b, c \in A\}$ . We can easily show that  $A \subseteq \bar{A}$ ,  $A \subseteq B \implies \bar{A} \subseteq \bar{B}$  and  $\overline{\bar{A}} = \bar{A}$ . We note that an ideal  $A$  of  $S$  is a *k-ideal* if and only if  $A = \bar{A}$ .

**Remark.** Let  $A$  be an ideal of a ternary semiring  $S$ . Then the *k-closure* of  $A$ , i.e.  $\bar{A}$ , is a *k-ideal* of  $S$ .

**Note.** We note that the intersection of any set of *k-ideals* of a ternary semiring  $S$  is a *k-ideal* of  $S$ .

**Definition 2.7.** A ternary semiring  $S$  is called a *simple (k-simple) ternary semiring* if it contains no non-zero proper ideal (*k-ideal*, resp.) of  $S$ .

Note that every simple ternary semiring is a *k-simple* ternary semiring.

**Definition 2.8** (see also [5], p. 78, for binary semirings). Let  $I$  be a proper ideal of the ternary semiring  $S$ . Then the congruence on  $S$ , denoted by  $\rho_I$  and defined by  $s \rho_I s'$  if and only if  $s + a_1 = s' + a_2$  for some  $a_1, a_2 \in I$ , is called the *Bourne ternary congruence* on  $S$  defined by the ideal  $I$ .

We denote the Bourne ternary congruence ( $\rho_I$ ) class of an element  $r$  of  $S$  by  $r/\rho_I$  (or simply by  $r/I$ ) and denote the set of all such congruence classes of  $S$  by  $S/\rho_I$  (or simply by  $S/I$ ).

It should be noted that for any  $s \in S$  and for any proper ideal  $I$  of  $S$ ,  $s/I$  is not necessarily equal to  $s + I = \{s + a : a \in I\}$  but surely contains it.

**Definition 2.9** (see [3]). A proper ideal  $P$  of a ternary semiring  $S$  is called a *prime ideal* of  $S$  if  $ABC \subseteq P$  implies  $A \subseteq P$  or  $B \subseteq P$  or  $C \subseteq P$  for any three ideals  $A, B, C$  of  $S$ .

**Definition 2.10.** An element  $a$  in a ternary semiring  $S$  is called an *idempotent element* if  $a^3 = a$ . A ternary semiring  $S$  is called an *idempotent ternary semiring* if every element of  $S$  is idempotent.

An ideal  $I$  of  $S$  is called *idempotent* if  $I^3 = I$ .

**Definition 2.11.** An element  $e$  in a ternary semiring  $S$  is called a *bi-unital element* if  $eex = xee = x$  for all  $x \in S$ .

**Example.** In the ternary semiring  $\mathbf{Z}_0^-$ , the element  $(-1)$  is a bi-unital element.

**Definition 2.12** (see [1]). An element  $a$  in a ternary semiring  $S$  is called *regular* if there exists an element  $x$  in  $S$  such that  $axa = a$ . A ternary semiring is called *regular* if all of its elements are regular.

**Definition 2.13.** A ternary semiring  $S$  is said to be *semi-subtractive* if for any elements  $a, b \in S$ ; there is always some  $x \in S$  or some  $y \in S$  such that  $a + y = b$  or  $b + x = a$ .

**Note.** Each ternary ring is a semi-subtractive ternary semiring.

**Definition 2.14.** A ternary semiring  $S$  is called *zero-sum free* if  $a + b = 0$  always implies that  $a = b = 0$ .

**Remark.** Every additively-idempotent ternary semiring  $S$  is zero-sum free.

The notions of congruence, maximal ideal, and also additively cancellative (AC, for short) or multiplicatively cancellative (MC) ternary semiring are introduced similarly as in classical cases.

### 3. TERNARY SEMI-INTEGRAL DOMAIN AND TERNARY SEMIFIELD

**Definition 3.1.** A ternary semiring (ring)  $S$  is said to be *zero divisor free* (ZDF, for short) if for  $a, b, c \in S$ ,  $abc = 0$  implies that  $a = 0$  or  $b = 0$  or  $c = 0$ .

**Definition 3.2.** A commutative ternary semiring (ring) is called a *ternary semi-integral* (*integral*, resp.) *domain* if it is zero divisor free.

**Lemma 3.3.** *An MC ternary semiring  $S$  is ZDF.*

**Proof.** Let  $S$  be an MC ternary semiring and  $abc = 0$  for  $a, b, c \in S$ . Suppose  $b \neq 0$  and  $c \neq 0$ . Then by right cancellativity,  $abc = 0 = 0bc$  implies that  $a = 0$ . Similarly, we can show that  $b = 0$  if  $a \neq 0$  and  $c \neq 0$  or  $c = 0$  if  $a \neq 0$  and  $b \neq 0$ . Consequently,  $S$  is ZDF. ■

For the converse part we have the following result:

**Lemma 3.4.** *A ZDF ternary semiring  $S$  is MC whenever it is AC and semi-subtractive.*

**Proof.** Let  $S$  be a ZDF, AC and semi-subtractive ternary semiring. Let  $a, b \in S \setminus \{0\}$  be such that  $abx = aby$  for  $x, y \in S$ . Since  $S$  is semi-subtractive, for  $x, y \in S$  there is always some  $c \in S$  or some  $d \in S$  such that  $y + c = x$  or  $x + d = y$ . Let  $y + c = x$ . Then  $aby + abc = abx$  implies  $abc = 0$  (by AC) which again implies that  $c = 0$  (since  $S$  is ZDF). Similarly, we can show that  $d = 0$  when  $x + d = y$ . Consequently, we have  $x = y$  and hence  $S$  is multiplicatively left cancelletive (MLC, for short). Similarly, it can be proved that  $S$  is multiplicatively right cancelletive (MRC) and multiplicatively laterally cancelletive (MLLC). Thus  $S$  is MC. ■

**Definition 3.5.** An element  $a$  of a ternary semiring  $S$  is said to be *invertible* in  $S$  if there exists an element  $b$  in  $S$  (called the *ternary semiring-inverse* of  $a$ ) such that  $abt = bat = tab = tba = t$  for all  $t \in S$ .

**Definition 3.6.** A ternary semiring (ring)  $S$  with  $|S| \geq 2$  is called a *ternary division semiring* (*ring*, resp.) if every non-zero element of  $S$  is invertible.

**Remark.** From the Definition 3.5, it follows that a bi-unital element of  $S$  is invertible and any invertible element is regular. Hence every ternary division semiring is a regular ternary semiring.

**Definition 3.7.** A commutative ternary division semiring (ring) is called a *ternary semifield* (*field*, resp.), i.e. a commutative ternary semiring (ring)  $S$  with  $|S| \geq 2$  is a ternary semifield (field) if for every non-zero element  $a$  of  $S$ , there exists an element  $b$  in  $S$  such that  $abx = x$  for all  $x \in S$ .

Note that a ternary semifield (field)  $S$  has always an identity.

**Example.** Denote by  $\mathbf{R}_0^-$ ,  $\mathbf{Q}_0^-$  and  $\mathbf{Z}_0^-$  the sets of all non-positive real numbers, non-positive rational numbers and non-positive integers, respectively. Then  $\mathbf{R}_0^-$  and  $\mathbf{Q}_0^-$  form ternary semifields with usual binary addition and ternary multiplication and  $\mathbf{Z}_0^-$  forms only a ternary semi-integral domain but not a ternary semifield.

**Lemma 3.8.** *A ternary semifield  $S$  is MC.*

**Proof.** Let  $S$  be a ternary semifield and let  $a, b \in S \setminus \{0\}$  be such that

$$(i) \quad abx = aby$$

where  $x, y \in S$ . Since  $S$  is a ternary semifield, there exist elements  $r, s \in S$  such that

$$(ii) \quad aru = u$$

and

$$(iii) \quad bsv = v$$

for all  $u, v \in S$ .

Now, by (i)–(iii) and by associativity and commutativity of the ternary multiplication, we have  $x = wx = ar(bsx) = (abx)rs = (aby)rs = y$ . Consequently,  $S$  is MCL. Since  $S$  is commutative, it follows that  $S$  is MC. ■

By Definitions 3.1, 3.2 and 3.7, and by Lemmas 3.3 and 3.8, we have:

**Lemma 3.9.** *A ternary semifield  $S$  is ZDF and  $S$  is a ternary semi-integral domain.* ■

**Proposition 3.10.** *A ternary semifield  $S$  does not possess any non-zero proper ideal, i.e.  $S$  is simple.*

**Proof.** Let  $P$  be any non-zero ideal of  $S$  and  $a(\neq 0) \in P$ . Then there exists an element  $b \in S$  such that  $abx = x$  for all  $x \in S$ . Then for any  $t \in S$ ,  $t = abt \in P$ . Consequently,  $P = S$ , and the proof is complete. ■

**Proposition 3.11.** *A commutative ternary semiring  $S$  is a ternary semifield if and only if it is ZDF and has no non-zero proper ideals.*

**Proof.** Let the commutative ternary semiring  $S$  be ZDF and have no non-zero proper ideals. Let  $a(\neq 0) \in S$ . Then  $aSx$  is an ideal of  $S$  for any non-zero  $x \in S$ . Since  $S$  is ZDF,  $aSx \neq \{0\}$ . Thus  $aSx$  is a non-zero ideal of  $S$ . Consequently, by hypothesis,  $aSx = S$  and hence for  $x(\neq 0) \in S$  there exists  $b \in S$  such that  $abx = x$ . Let  $y$  be any element of  $S$ . Then there exists  $c \in S$  such that  $acx = y$ . Thus  $aby = ab(acx) = (abx)ac = xac = acx = y$  for all  $y \in S$ . This shows that  $S$  is a ternary semifield.

The converse implication of the theorem follows from Lemma 3.9 and Proposition 3.10. ■

So, it is easy to observe that a commutative simple ZDF ternary semiring is a ternary semifield. Moreover, a commutative ZDF ternary semiring  $S$  is a ternary semifield if and only if  $\{0\}$  is a maximal ideal of  $S$ .

By using Zorn's Lemma we can state:

If  $S$  is any commutative ternary semiring with identity, then every ideal  $I \neq S$  is contained in a maximal ideal  $S$ .

**Note.** The set  $\mathbf{Z}_0^- \setminus \{-1\}$  is a maximal ideal of the ternary semiring  $\mathbf{Z}_0^-$  with identity, which contains all ideals of  $\mathbf{Z}_0^-$ . Note that this ideal is not a principal ideal.

By standard verifications, we have

**Lemma 3.12.** *For any proper  $k$ -ideal  $P$  of a commutative ternary semiring  $S$  with identity, the factor ternary semiring  $S/P$  is also a commutative ternary semiring with identity.* ■

**Theorem 3.13.** *Let  $M$  be a proper  $k$ -ideal of a commutative ternary semiring  $S$ . Then  $M$  is a maximal ideal of  $S$  if and only if the factor ternary semiring  $S/M$  has only two ideals  $\{0/M\}$  and  $S/M$ .*

**Proof.** By Lemma 3.12, it follows that the factor ternary semiring  $S/M$  is a commutative ternary semiring with zero element  $0/M$ . Let the proper  $k$ -ideal  $M$  be a maximal ideal of  $S$ . Then  $S/M \neq \{0/M\}$ . Let  $I$  be an ideal of  $S/M$  properly containing  $\{0/M\}$ . Now let  $I_0 = \{x \in S : x/M \in I\}$ . Then  $I_0$  is an ideal of  $S$ . Let  $x \in M$ . Then  $x/M = 0/M \in I$ . So  $x \in I_0$ . Hence  $M \subseteq I_0$ . Since  $I$  properly contains  $\{0/M\}$ , there exists  $a/M \in I$  such that  $a/M \neq 0/M$ . Then  $a \notin M$  but  $a \in I_0$ . Hence  $I_0$  properly contains  $M$ . Since  $M$  is a maximal ideal of  $S$ ,  $I_0 = S$ . Thus  $I = S/M$ . This implies that the zero ideal  $\{0/M\}$  of  $S/M$  is maximal. Hence  $S/M$  has only two ideals  $\{0/M\}$  and  $S/M$ .



Conversely, suppose that  $S/M$  has only two ideals  $\{0/M\}$  and  $S/M$ . Then  $\{0/M\}$  is a maximal ideal of  $S/M$ . Next let  $J$  be an ideal of  $S$  properly containing  $M$  and  $J_1 = \{x/M \in S/M : x \in J\}$ . Then  $J_1$  is an ideal of  $S/M$ . Since  $J \supset M$ , there exists  $x \in J \setminus M$ . Then  $x/M \neq 0/M \in J_1/M$ . Consequently,  $J_1 \neq \{0/M\}$ . Therefore,  $J_1 = S/M$ . Thus  $J = S$  and hence  $M$  is a maximal ideal of the ternary semiring  $S$ . Hence the theorem. ■

**Theorem 3.14** ([3]). *Let  $S$  be a commutative ternary semiring. Then a proper  $k$ -ideal  $P$  of  $S$  is prime if and only if the factor ternary semiring  $S/P$  is a ternary semi-integral domain.* ■

**Theorem 3.15.** *Let  $S$  be a commutative ternary semiring with identity. Then a proper  $k$ -ideal  $M$  of  $S$  is maximal if and only if the factor ternary semiring  $S/M$  is a ternary semifield.*

**Proof.** Let  $M$  be a maximal  $k$ -ideal of a ternary semiring  $S$  with identity  $(e_i, f_i)(i = 1, 2, \dots, m)$ . Since  $S$  is a commutative ternary semiring with identity  $(e_i, f_i)$ , it follows, by Lemma 3.12, that  $S/M$  is also a commutative ternary semiring with identity. Let  $a/M \neq 0/M$ . Then  $a \notin M$ . Now  $M + SSa$  is an ideal of  $S$  which properly contains  $M$ . Since  $M$  is a maximal ideal, we have  $M + SSa = S$ . This implies that there exist  $m_i \in M$  and  $s_{ij}, t_{ij} \in S$  such that

$$e_1 = m_1 + \sum_{j=1}^n s_{1j}t_{1j}a;$$

$$e_2 = m_2 + \sum_{j=1}^n s_{2j}t_{2j}a;$$

.....

$$e_i = m_i + \sum_{j=1}^n s_{ij}t_{ij}a.$$

From above we have,

$$e_1 f_1 x = m_1 f_1 x + \left( \sum_{j=1}^n s_{1j} t_{1j} a \right) f_1 x = m_1 f_1 x + \left( \sum_{j=1}^n s_{1j} t_{1j} f_1 \right) a x$$

$$e_2 f_2 x = m_2 f_2 x + \left( \sum_{j=1}^n s_{2j} t_{2j} a \right) f_2 x = m_2 f_2 x + \left( \sum_{j=1}^n s_{2j} t_{2j} f_2 \right) a x$$

.....

$$e_i f_i x = m_i f_i x + \left( \sum_{j=1}^n s_{ij} t_{ij} a \right) f_i x = m_i f_i x + \left( \sum_{j=1}^n s_{ij} t_{ij} f_i \right) a x$$

for all  $x \in S$ .

Adding all these equations, we have

$$\sum_{i=1}^m e_i f_i x = \sum_{i=1}^m m_i f_i x + \sum_{i=1}^m \left( \sum_{j=1}^n s_{ij} t_{ij} f_i \right) a x$$

for all  $x \in S$ .

This implies that  $x = m' + bax$  for all  $x \in S$ ,  $m' = \sum_{i=1}^m m_i f_i x \in M$  and  $b = \sum_{i=1}^m \sum_{j=1}^n s_{ij} t_{ij} f_i \in S$ . Thus  $x + 0 = bax + m'$ . Consequently,  $x/M = (bax)/M$  that is  $x/M = (b/M)(a/M)(x/M)$  for all  $x/M \in S/M$ . Hence  $S/M$  is a ternary semifield.

Conversely, let the factor ternary semiring  $S/M$  be a ternary semifield. Since  $S/M$  is a ternary semifield,  $S \neq M$ . Let  $I$  be a  $k$ -ideal of  $S$  such that  $M \subset I \subseteq S$ . Then there exists an element  $a \in I$  such that  $a \notin M$ . Thus  $a/M \neq 0/M$  and hence there exists  $b/M \in S/M$  such that  $(a/M)(b/M)(x/M) = x/M$  for all  $x/M \in S/M$  that is  $(abx)/M = x/M$  for all  $x/M \in S/M$ . This implies that  $abx + s_1 = x + s_2$  for some  $s_1, s_2 \in M \subset I$  and for all  $x \in S$ . Since  $I$  is an ideal of  $S$ ,  $a \in I$  implies that  $abx \in I$  and hence  $abx + s_1 = x + s_2 \in I$  for all  $x \in S$ . Since  $I$  is a  $k$ -ideal,  $x + s_2 \in I$ ,  $s_2 \in I$  implies that  $x \in I$  for all  $x \in S$ . Consequently,  $I = S$ . Hence  $M$  is a maximal ideal of  $S$ . This completes the proof of the theorem. ■

Since every ternary semifield is always a ternary semi-integral domain, from Theorems 3.14 and 3.15 we have the following corollary:

**Corollary 3.16.** *Every maximal ideal of a ternary semiring  $S$  is a prime ideal of  $S$ .* ■

**Theorem 3.17.** *A finite commutative MC ternary semiring  $S$  with  $|S| \geq 2$  is a ternary semifield.*

**Proof.** Let  $S$  be a finite commutative MC ternary semiring. Let  $S = \{a_1, a_2, \dots, a_n\}$ . Suppose  $a \in S$  and  $a \neq 0$ . Then for a fixed  $a_1 \neq 0 \in S$ ,  $aa_1a_i \in S$  for all  $i$  and hence  $\{aa_1a_1, aa_1a_2, \dots, aa_1a_n\} \subseteq S$ . Since  $S$  is MC, if  $aa_1a_i = aa_1a_j$  then  $a_i = a_j$ . Therefore, the elements  $aa_1a_1, aa_1a_2, \dots, aa_1a_n$  must be distinct and hence  $S = \{aa_1a_1, aa_1a_2, \dots, aa_1a_n\}$ . This implies that one of the product must be equal to  $a$ , say  $aa_1a_i = a$ . Let  $x$  be any element of  $S$ . Then there exists  $a_j \in S$  such that  $x = aa_1a_j$ . Since  $S$  is commutative,  $a_1a_ix = a_1a_i(aa_1a_j) = (a_1a_ia) a_1a_j = (aa_1a_i) a_1a_j = aa_1a_j = x$  for all  $x \in S$ . Since  $a_1$  is arbitrary, we find that  $S$  is a ternary semifield. ■

**Theorem 3.18.** *A ternary semifield  $S$  is a ternary field if and only if it is not zero-sum free.*

**Proof.** Let a ternary semifield  $S$  be a ternary field. Then  $S$  is an additive group that is for every  $a(\neq 0) \in S$ , there exists  $a'(\neq 0) \in S$  (the additive inverse of  $a$ ) such that  $a + a' = 0$ . Consequently,  $S$  is not zero-sum free.

Conversely, let a ternary semifield  $S$  be not zero-sum free. To show  $S$  is a ternary field we have to show that  $S$  is an additive group. Since  $S$  is not zero-sum free, there are elements  $a(\neq 0), b(\neq 0) \in S$  such that  $a + b = 0$ . Let  $x(\neq 0) \in S$ . Then  $axS = \{axy | y \in S\}$  is a non-zero proper ideal of  $S$ . Since  $S$  is a ternary semifield, by Proposition 3.11, it follows that  $S = axS$ . Then for any  $c \in S$ , there exists  $y \in S$  such that  $c = axy$ . Now  $a + b = 0$  implies that  $axy + bxy = 0$  which again implies that  $c + bxy = 0$ . Hence  $c$  has an additive inverse. This shows that each element of  $S$  has an additive inverse in  $S$ . Thus  $S$  is an additive group and hence  $S$  is a ternary ring. Consequently,  $S$  is a ternary field. ■

**Theorem 3.19.** *Every finite ternary semifield  $S$  with  $|S^*| \geq 2$ , where  $S^* = S \setminus \{0\}$ , is a ternary field.*

**Proof.** Suppose  $a \neq 0 \in S$ . Since  $S$  is a ternary semifield, there exists a unique  $b(\neq 0) \in S$  such that  $abx = x$  for all  $x \in S$ . Since  $|S^*| \geq 2$ , there exists  $c \in S \setminus \{0, b\}$ . Then  $(a, c)$  is not the identity of  $S$ . Let  $d \neq 0 \in S$ . Since  $S$  is ZDF,  $(ac)^i d \neq 0$  for  $i \geq 1$ . Since  $S$  is finite,  $\{acd, (ac)^2d, (ac)^3d, \dots\}$  is finite. Then there exist  $i$  and  $j$  such that  $(ac)^i d = (ac)^j d$ . Suppose  $i > j$ . Then by MC, we have  $(ac)^k d = d$ , where  $k = i - j$ . Suppose  $S$  is zero-sum free. Let  $p = acd + (ac)^2d + (ac)^3d + \dots + (ac)^k d$ . Then since  $S$  is zero-sum free,  $p \neq 0$ . Therefore,  $pac = p$ . Then by associativity and commutativity of the ternary multiplication, we have  $p(acx)a = (pac)xa = pxa$ . By MC, we get  $acx = x$  for all  $x \in S$ , which is a contradiction, since  $(a, c)$  is not the identity of  $S$ . Consequently,  $S$  is not zero-sum free. Therefore, by Theorem 3.18,  $S$  is a ternary field. ■

**Proposition 3.20.** *Every finite commutative ternary semiring  $S$ , with  $|S^*| \geq 2$ , which is MC is a ternary field.*

**Proof.** By Theorem 3.17,  $S$  is a ternary semifield. Since  $S$  is finite, the proof of the theorem follows from Theorem 3.19. ■

#### 4. REMARKS ON MAXIMAL IDEALS OF THE TERNARY SEMIRING $\mathbf{Z}_0^-$

In this section we recall some results proved in [3] and [4], and add a few new ones concerning maximal ideals. Through out this section  $\mathbf{Z}_0^-$  denotes the set of all negative integers with zero and  $\mathbf{Z}_0^+$  denotes the set of all positive integers with zero.

For a subset  $I$  of  $\mathbf{Z}_0^-$  we define a subset  $I^*$  of  $\mathbf{Z}_0^+$  as follows :  $I^* = \{n \in \mathbf{Z}_0^+ \mid -n \in I\}$ . Then we have:

**Lemma 4.1** ( see [3]).

- (i)  $I$  is an ideal (a  $k$ -ideal) of  $\mathbf{Z}_0^-$  if and only if  $I^*$  is an ideal (a  $k$ -ideal, resp.) of  $\mathbf{Z}_0^+$ .
- (ii)  $I = (-n)\mathbf{Z}_0^- \mathbf{Z}_0^-$  if and only if  $I^* = n\mathbf{Z}_0^+$ .
- (iii) Let  $\mathcal{I}(\mathbf{Z}_0^-)$  denote the set of all ideals of  $\mathbf{Z}_0^-$  and  $\mathcal{I}^*(\mathbf{Z}_0^+)$  denote the set of all ideals of  $\mathbf{Z}_0^+$ . Then there exists an order-preserving bijective correspondence  $I \leftrightarrow I^*$  from  $\mathcal{I}(\mathbf{Z}_0^-)$  to  $\mathcal{I}^*(\mathbf{Z}_0^+)$ .
- (iv) (see [3] and [4])  $I$  is a prime (semiprime) ideal of  $\mathbf{Z}_0^-$  if and only if  $I^*$  is a prime (semiprime, resp.) ideal of  $\mathbf{Z}_0^+$ .

From these results, we have immediately.

**Corollary 4.2.**  *$I$  is a maximal ideal ( $k$ -ideal) of  $\mathbf{Z}_0^-$  if and only if  $I^*$  is a maximal ideal ( $k$ -ideal, resp.) of  $\mathbf{Z}_0^+$ . ■*

**Remark** (see [3]). Not all ideals of the ternary semiring  $\mathbf{Z}_0^-$  are of the form  $(-n)\mathbf{Z}_0^-\mathbf{Z}_0^-$ , since every ideal of the semiring  $\mathbf{Z}_0^+$  is not of the form  $n\mathbf{Z}_0^+$  ([6], Example 8.3 (a), p. 77).

But since all  $k$ -ideals of the semiring  $\mathbf{Z}_0^+$  are of the form  $n\mathbf{Z}_0^+$  for  $n \in \mathbf{Z}_0^+$  ([5], Example 6.6, p. 66; [6], Corollary 8.10, p. 82), we have the following result:

**Proposition 4.3** (see [3]). *All  $k$ -ideals of the ternary semiring  $\mathbf{Z}_0^-$  are of the form  $(-n)\mathbf{Z}_0^-\mathbf{Z}_0^-$  and hence  $\mathbf{Z}_0^-$  is a principal  $k$ -ideal ternary semiring. ■*

**Remark.** In [3], we have proved that the prime  $k$ -ideals of the ternary semiring  $\mathbf{Z}_0^-$  are of the form  $(-p)\mathbf{Z}_0^-\mathbf{Z}_0^-$ , where  $p$  is a positive prime. In [4], we have proved that the semiprime  $k$ -ideals of the ternary semiring  $\mathbf{Z}_0^-$  are of the form  $(-n)\mathbf{Z}_0^-\mathbf{Z}_0^-$ , where  $n$  is a square free positive integer.

In [11], Sen and Adhikari proved that all maximal  $k$ -ideals of the semiring  $\mathbf{Z}_0^+$  are of the form  $p\mathbf{Z}_0^+$ , where  $p$  is a positive prime. But none of the maximal  $k$ -ideals  $p\mathbf{Z}_0^+$  of  $\mathbf{Z}_0^+$  is a maximal ideal of  $\mathbf{Z}_0^+$ , since each ideal  $A^* = p\mathbf{Z}_0^+$  is properly contained in the proper ideal  $B^* = \{b \in \mathbf{Z}_0^+ : b \geq p\}$  of  $\mathbf{Z}_0^+$ .

Now we have the following result regarding the maximal  $k$ -ideals of the ternary semiring  $\mathbf{Z}_0^-$  :

**Proposition 4.4.** *All maximal  $k$ -ideals of the ternary semiring  $\mathbf{Z}_0^-$  are of the form  $(-p)\mathbf{Z}_0^-\mathbf{Z}_0^-$ , where  $p$  is a positive prime. But none of the maximal  $k$ -ideals  $(-p)\mathbf{Z}_0^-\mathbf{Z}_0^-$  of  $\mathbf{Z}_0^-$  is a maximal ideal of  $\mathbf{Z}_0^-$ , since each ideal  $A = (-p)\mathbf{Z}_0^-\mathbf{Z}_0^-$  is properly contained in the proper ideal  $B = \{b \in \mathbf{Z}_0^- : b \leq (-p)\}$ .*

**Remark.** In the ternary semiring  $\mathbf{Z}_0^-$ , both the prime  $k$ -ideals and the maximal  $k$ -ideals are of the form  $(-p)\mathbf{Z}_0^-\mathbf{Z}_0^-$ , where  $p$  is a positive prime.

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