Discussiones Mathematicae
General Algebra and Applications 24 (2004) 177–184

A GROUPOID CHARACTERIZATION OF BOOLEAN ALGEBRAS

Ivan Chajda

Department of Algebra and Geometry
Palacký University of Olomouc
Tomkova 40; 779 00 Olomouc, Czech Republic
e-mail:chajda@inf.upol.cz

Abstract

We present a groupoid which can be converted into a Boolean algebra with respect to term operations. Also conversely, every Boolean algebra can be reached in this way.

Keywords: groupoid, Boolean algebra, semi-bolean algebra, involution, dualautomorphism (or antiautomorphism).

2000 Mathematics Subject Classification: 20N02, 03G05 03G25, 06A11, 06F99, 08C10.

There are several characterizations of Boolean algebras by means of algebras having various similarity types. Some of such characterizations are based on groupoids (with constants). We can mention that J. Dudek in [3] described some classes of (commutative) groupoids by number sequences of distinct term operations (called there polynomials). Our starting point will be the characterization of semi-boolean algebras done by J.C. Abbott in [1]. His characterization is using a binary operation which is an algebraic counterpart of the logic propositional connective "implication". Although this characterization is simple, the axiom of exchange:

$$x \circ (y \circ z) = y \circ (x \circ z)$$

can be difficult to verify and also its interpretation is not immediately clear.

We are going to substitute it by a weaker form

$$x \circ (y \circ 0) = y \circ (x \circ 0)$$

(where 0 is a constant) which is formulated in two variables only and hence easily to check in groupoids; moreover, its interpretation is almost evident because it is in fact the contraposition law

$$x \Rightarrow y \quad \text{iff} \quad \neg y \Rightarrow \neg x$$

whenever "$\circ$" is interpret as the implication. Of course, it works only if the list of axioms is completed by one more axiom concerning the antitony of the right multiplication.

On the other hand, our axioms (1)–(3) (see below) can be used also for ortholattices when the mentioned weakened exchange axiom is deleted.

A mapping f of a set $A \neq \emptyset$ onto itself is called an *involution* if $f(f(x)) = x$. Let $(A, \leq)$ be an ordered set. A mapping f of A into itself is called *antitone* if $x \leq y$ implies $f(y) \leq f(x)$.

The following result is a folklore in Lattice Theory (see, e.g., [2]).

Proposition. *Let $\mathcal{L} = (L, \vee, \wedge)$ be a lattice and $\leq$ its induced order and let the mapping $x \mapsto x^\circ$ be an antitone involution on $(L, \leq)$. Then this mapping is a dualautomorphism of $\mathcal{L}$, i.e., the so-called De Morgan laws*

$$(x \vee y)^\circ = x^\circ \wedge y^\circ \quad \textit{and} \quad (x \wedge y)^\circ = x^\circ \vee y^\circ$$

are satisfied.

Remark that the dualautomorphism is also often called the antiautomorphism in other papers.

From now on, we will study an algebra $\mathcal{A} = (A, \circ, 0)$ of type $(2, 0)$ satisfying various axioms. Of course, $\mathcal{A}$ has an algebraic constant $0 \circ 0$ which will be denoted by 1 in the whole paper.

Lemma 1. *Let $\mathcal{A} = (A, \circ, 0)$ satisfy the axioms*

(1) $x \circ 1 = 1$, $1 \circ x = x$, $0 \circ x = 1$;

(2) $(x \circ y) \circ y = (y \circ x) \circ x$;

(3) $(((x \circ y) \circ y) \circ z) \circ (x \circ z) = 1$.

Then $\mathcal{A}$ satisfies also $x \circ x = 1$. Define a binary relation $\leq$ on A as follows:

$$(*) \qquad x \leq y \quad if \quad and \quad only \quad if \quad x \circ y = 1.$$

Then $\leq$ is an order on A and $0 \leq x \leq 1$ for each $x \in A$.

Proof. We can easily compute by (1) and (2):

$$x \circ x = (1 \circ x) \circ x = (x \circ 1) \circ 1 = 1.$$

Hence, $\leq$ is reflexive. Suppose $x \leq y$ and $y \leq x$. Then $x \circ y = 1$ and $y \circ x = 1$, thus

$$x = 1 \circ x = (y \circ x) \circ x = (x \circ y) \circ y = 1 \circ y = y$$

proving antisymmetry of $\leq$.

Suppose $x \leq y$ and $y \leq z$. Then $x \circ y = 1$ and $y \circ z = 1$. We can substitute these in (3) to obtain $x \circ z = 1$ thus $x \leq z$. Hence, $\leq$ is also transitive, i.e. it is an order on A. Applying (1), we conclude $0 \leq x$, $x \leq 1$ for each $x \in A$. ■

The order $\leq$ involved by $(\star)$ of Lemma 1 will be called the *induced order of* $\mathcal{A} = (A, \circ, 0)$.

Lemma 2. *Let $\mathcal{A} = (A, \circ, 0)$ be an algebra satisfying* (1),(2),(3) *and $\leq$ be its induced order. Then*

$$x \leq y \quad implies \quad y \circ z \leq x \circ z \qquad (antitonity).$$

Proof. Suppose $x \leq y$. Then $x \circ y = 1$ and, substituting this into (3), we get $(y \circ z) \circ (x \circ z) = 1$, and thus, $y \circ z \leq x \circ z$. ■

Remark. In accordance with Lemma 2, the identity (3) is in fact the antitonity.

Theorem 1. *Let $\mathcal{A} = (A, \circ, 0)$ satisfy* (1),(2),(3) *and $\leq$ be its induced order. Then $(A; \leq)$ is a bounded lattice where for each $p \in A$ the mapping $x \mapsto x \circ p$ is an antitone involution of the interval $[p, 1]$ onto itself. The lattice join is the term operation $x \vee y = (x \circ y) \circ y$. If, moreover, $\mathcal{A}$ satisfies*

$$((x \circ 0) \circ x) \circ x = 1, \tag{4}$$

then $(A, \leq)$ is an ortholattice where $x \circ 0$ is an orthocomplement of $x \in A$.

Proof. By Lemma 1 and Lemma 2, $y \leq 1$ and hence $x = 1 \circ x \leq y \circ x$, i.e. $\mathcal{A}$ satisfies

$$x \circ (y \circ x) = 1. \tag{$**$}$$

Let $a, b \in A$. Then by (2) we have

$$a \circ ((a \circ b) \circ b) = a \circ ((b \circ a) \circ a) = 1, \quad \text{i.e.} \quad a \leq (a \circ b) \circ b,$$

$$b \circ ((a \circ b) \circ b) = 1, \qquad \text{i.e.} \quad b \leq (a \circ b) \circ b,$$

thus $(a \circ b) \circ b$ is a common upper bound of a, b. Suppose $a \leq c$, $b \leq c$. Then $b \circ c = 1$ and, by Lemma 2, $c \circ b \leq a \circ b$, and hence $(a \circ b) \circ b \leq (c \circ b) \circ b = (b \circ c) \circ c = 1 \circ c = c$. Thus, $(a \circ b) \circ b$ is the least common upper bound of a, b, i.e. $a \vee b = (a \circ b) \circ b$.

Let $p \in A$ and $x \in [p, 1]$. Then $p \leq x$ and $(x \circ p) \circ p = x \vee p = x$, i.e. the mapping $x \mapsto x \circ p$ is an involution on $[p, 1]$. Suppose $x, y \in [p, 1]$, $x \leq y$. By Lemma 2, we have $y \circ p \leq x \circ p$ thus the involution is antitone. Consider $p = 0$, the least element of $(A, \leq)$. Then the mapping $x \mapsto x \circ 0$ is an antitone involution on $(A, \leq)$, and by the Proposition,

$$x \wedge y = ((x \circ 0) \vee (y \circ 0)) \circ 0.$$

Thus, $(A, \leq)$ is a bounded lattice.

If $\mathcal{A}$ satisfies also (4), then $(x \circ 0) \vee x = ((x \circ 0) \circ x) \circ x = 1$ and, due to De Morgan law, $(x \circ 0) \wedge x = 0$, i.e. $x \circ 0$ is a complement of x. By Lemma 2, $x \leq y$ implies $y \circ 0 \leq x \circ 0$, and thus, $x \circ 0$ is an orthocomplement of x and $(A, \leq)$ is an ortholattice. ∎

Corollary 1. *Let $\mathcal{A} = (A, \circ, 0)$ satisfy (1),(2),(3) and $\leq$ be its induced order. Then*

$$(x \vee y) \circ y = x \circ y,$$

i.e. $\mathcal{A}$ satisfies the identity $((x \circ y) \circ y) \circ y = x \circ y$.

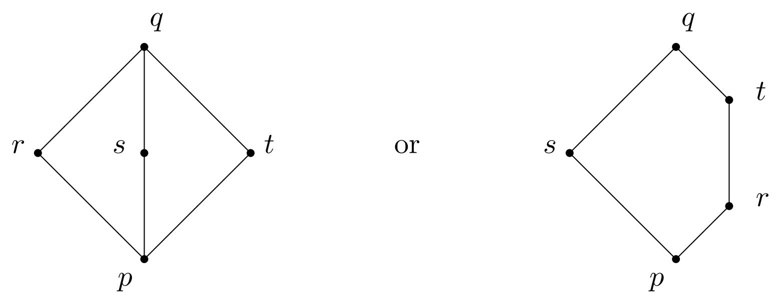
Proof. By $(**)$, $x \circ y \geq y$ thus $x \circ y \in [y,1]$ and, by Theorem 1, $((x \circ y) \circ y) \circ y = x \circ y$ since the mapping $a \mapsto a \circ y$ (for $a \in [y,1]$) is an involution. However, $(x \circ y) \circ y = x \vee y$ and thus, we conclude $(x \vee y) \circ y = x \circ y$. ■

Theorem 2. *Let* $\mathcal{A} = (A, \circ, 0)$ *satisfy* (1),(2),(3) *and*

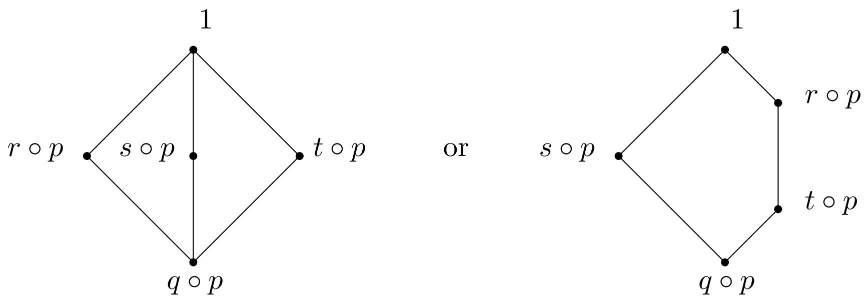
$$x \circ (y \circ 0) = y \circ (x \circ 0). \tag{5}$$

Moreover, let $\leq$ *be the induced order. Then* $(A, \leq)$ *is a distributive lattice.*

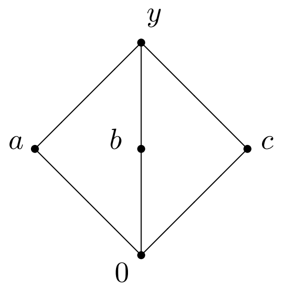
Proof. By Theorem 1, $(A, \leq)$ is a bounded lattice where for each $p \in A$, the mapping $x \mapsto x \circ p$ is an antitone involution on the sublattice $[p,1]$. Suppose that $\mathcal{A}$ satisfies (1),(2),(3),(5) and $(A, \leq)$ is not distributive. Then it contains a sublattice isomorphic to



Consider the interval $[p,1]$. The antitone involution $x \mapsto x \circ p$ is a dualautomorphism (or, in another terminology, an antiautomorphism) of $[p,1]$ thus $(A, \leq)$ contains also the sublattice

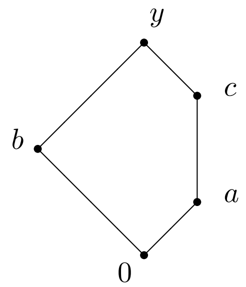


We can apply the involution $x \mapsto x \circ 0$ to show that $(A, \leq)$ contains also the sublattice



M_3

or



N_5

where $y = (q \circ p) \circ 0$, $a = (r \circ p) \circ 0$, $b = (s \circ p) \circ 0$, $c = (t \circ p) \circ 0$.

Suppose $(A, \leq)$ contains M_3. Then, by Corollary 1, $(a \circ 0) \circ (c \circ 0) = ((a \circ 0) \vee (c \circ 0)) \circ (c \circ 0) = ((r \circ p) \vee (t \circ p)) \circ (c \circ 0) = 1 \circ (c \circ 0) = c \circ 0$, $c \circ ((a \circ 0) \circ 0) = c \circ a = (c \vee a) \circ a = y \circ a$. By (5), we have $(a \circ 0) \circ (c \circ 0) = c \circ ((a \circ 0) \circ 0)$ thus $c \circ 0 = y \circ a$. Interchanging b and c, we obtain analogously $b \circ 0 = y \circ a$, thus $b \circ 0 = c \circ 0$, i.e. $b = (b \circ 0) \circ 0 = (c \circ 0) \circ 0 = c$, a contradiction.

Suppose $(A, \leq)$ contains N_5. By Corollary 1, we compute $(b \circ 0) \circ (a \circ 0) = ((b \circ 0) \vee (a \circ 0)) \circ (a \circ 0) = ((s \circ p) \vee (r \circ p)) \circ (a \circ 0) = 1 \circ (a \circ 0) = a \circ 0$ and $a \circ ((b \circ 0) \circ 0) = a \circ b = (a \vee b) \circ b = y \circ b$.

Analogously, we obtain $(b \circ 0) \circ (c \circ 0) = c \circ 0$ and $c \circ ((b \circ 0) \circ 0) = y \circ b$.

Applying (5), we conclude

$$a \circ 0 = y \circ b = c \circ 0$$

whence $a = (a \circ 0) \circ 0 = (c \circ 0) \circ 0 = c$, a contradiction again. Hence, the lattice $(A, \leq)$ is distributive. ■

Corollary 2. *Let $\mathcal{A} = (A, \circ, 0)$ satisfy (1)–(5) and $\leq$ be its induced order. Then $(A, \leq)$ is a Boolean algebra, where*

$$1 = 0 \circ 0,$$

$$x \vee y = (x \circ y) \circ y,$$

$$x' = x \circ 0,$$

$$x \wedge y = (x' \vee y')'.$$

Proof. By Theorem 1, $(A, \leq)$ is an ortholattice, by Theorem 2, it is distributive. Together, $(A, \leq)$ is a Boolean algebra. The expressions for $\vee$, $\wedge$ and the complementation follow by Theorem 1 and the De Morgan law. ■

Now, we are going to prove the converse.

Theorem 3. *Let $\mathcal{B} = (B, \vee, \wedge, ', 0, 1)$ be a Boolean algebra. Define a term operation $\circ$ as follows*

$$x \circ y = x' \vee y.$$

Then $\mathcal{A}(\mathcal{B}) = (B, \circ, 0)$ is an algebra satisfying te identities (1)–(5).

Proof.

(1): $x \circ 1 = x' \vee 1 = 1,$

$1 \circ x = 1' \vee x = 0 \vee x = x,$

$0 \circ x = 0' \vee x = 1 \vee x = 1.$

(2): $(x \circ y) \circ y = (x' \vee y)' \vee y = (x \wedge y') \vee y = x \vee y.$

Analogously, $(y \circ x) \circ x = y \vee x = x \vee y$.

(3): By (2), we have $(x \circ y) \circ y = x \vee y$, thus (3) can be rewritten as

$$((x \vee y) \circ z) \circ (x \circ z) = 1.$$

To prove this, we compute $(x \vee y) \circ z = (x \vee y)' \vee z$, $x \circ z = x' \vee z$, thus $((x \vee y) \circ z) \circ (x \circ z) = ((x \vee y)' \vee z)' \vee (x' \vee z) = ((x \vee y) \wedge z') \vee (x' \vee z) = (x \vee y \vee x' \vee z) \wedge (z' \vee x' \vee z) = 1.$

(4): $((x \circ 0) \circ x) \circ x = (x \circ 0) \vee x = x' \vee x = 1.$

(5): $x \circ (y \circ 0) = x \circ y' = x' \vee y' = y' \vee x' = y \circ x' = y \circ (x \circ 0).$ ■

References

[1] J.C. Abbott, *Semi-boolean algebra*, Mat. Vestnik **4** (1967), 177–198.

[2] G. Birkhoff, *Lattice Theory*, third edition, Publ. Amer. Math. Soc., Providence, RI, 1967.

[3] J. Dudek, *Polynomials in idempotent commutative groupoids*, Dissert. Math. **286** (1989), 1–59.

Received 23 December 2003
Revised 29 December 2004