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ORTHORINGS

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Abstract

Certain ring-like structures, so-called orthorings, are introduced which are in a natural one-to-one correspondence with lattices with 0 every principal ideal of which is an ortholattice. This correspondence generalizes the well-known bijection between Boolean rings and Boolean algebras. It turns out that orthorings have nice congruence and ideal properties.

Keywords: ortholattice, generalized ortholattice, sectionally complemented lattice, orthoring, arithmetical variety, weakly regular variety, congruence kernel, ideal term, basis of ideal terms, subtractive term.

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1. Introduction

The well-known natural bijective correspondence between Boolean algebras and Boolean rings is widely used in applications, see [1] for details. This correspondence was generalized in different ways thus giving rise to natural connections between certain lattice structures on the one hand and certain ring-like structures on the other hand. On the lattice-theoretical side the following structures were considered: orthomodular lattices ([8] and [18]), ortholattices ([2]), bounded lattices with an involutory antiautomorphism ([9], [10], [11], [12], [13] and [14]), pseudocomplemented semilattices ([5]) and MV-algebras ([6]). The corresponding ring-like structures were called orthomodular Boolean quasirings or orhomodular pseudorings, orthopseudorings, orthopseudosemirings, Boolean quasirings and pseudorings, respectively. (No name was assigned to the ring-like structures induced by pseudocomplemented semilattices.) In [3] the ring-like structures introduced in [2] and [9], respectively, are related to each other. However, each one of the derived ring-like structures considered so far in this context was endowed with a constant 1 which plays a role similar to the unit element in rings. On the other hand, starting with so-called generalized Boolean algebras, one can derive Boolean rings which need not have a unit element (see [1]). A similar approach was used in [7], where so-called generalized orthomodular lattices (introduced by M.F. Janowitz in [17], see also [16]) were considered. Our aim is to investigate ring-like structures (so-called orthorings) which correspond to lattices with 0 such that every principal lattice ideal is an ortholattice. It should be pointed out that though these lattices do not form a variety, the term equivalent orthorings form a variety and hence allow the application of universal algebraic methods and results. Moreover, we are going to show that – in spite of their generality – orthorings have nice properties.

We recall that an *ortholattice* is an algebra $(L; \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and $(x')' = x$, $(x \vee y)' = x' \wedge y'$, $(x \wedge y)' = x' \vee y'$, $x \vee x' = 1$ and $x \wedge x' = 0$ for all $x, y \in L$.

2. Orthorings

First we introduce the concept of a generalized ortholattice and distinguish generalized ortholattices from other classes of lattices.

Definition 2.1. (cf., e.g., [15]). A lattice $(L; \vee, \wedge, 0)$ with 0 is called a *sectionally complemented lattice* if for each $a \in L$, $([0, a]; \vee, \wedge)$ is a complemented lattice, i.e. for every $a, b \in L$ with $b \leq a$ there exists an element c of L with $c \leq a$, $b \vee c = a$ and $b \wedge c = 0$.

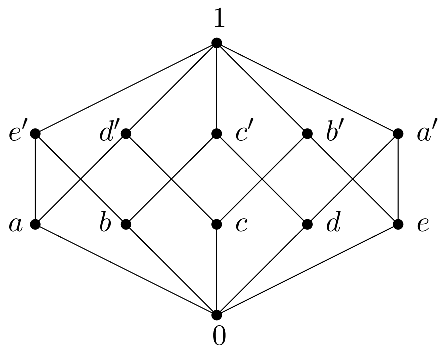
Definition 2.2. $(L; \vee, \wedge, ({}^a\,; a \in L), 0)$ where $(L; \vee, \wedge, 0)$ is a lattice with 0 is called a *generalized ortholattice* if for each $a \in L$, $([0, a]; \vee, \wedge, {}^a, 0, a)$ is an ortholattice.

Of course, if $(L; \vee, \wedge, ({}^a\,; a \in L), 0)$ is a generalized ortholattice, then $(L; \vee, \wedge, 0)$ is a sectionally complemented lattice.

Example 2.1. The five-element modular non-distributive lattice is sectionally complemented but it cannot be considered as a generalized ortholattice since it has an odd number of elements.

Example 2.2. If $(L; \vee, \wedge, ', 0, 1)$ is an orthomodular lattice, i.e. an ortholattice satisfying $y = x \vee (y \wedge x')$ for all $x, y \in L$ with $x \leq y$, then $(L; \vee, \wedge, (x \mapsto x' \wedge a; a \in L), 0)$ is a generalized ortholattice.

Example 2.3. The following Hasse diagram shows a non-orthomodular generalized ortholattice:



Next we introduce ring-like structures corresponding to generalized ortholattices.

Definition 2.3. An *orthoring* is an algebra $(R; +, \cdot, 0)$ of type $(2, 2, 0)$ satisfying

(O1) $x+y=y+x$,

(O2) $x+0=x$,

(O3) $xy=yx$,

(O4) $(xy)z=x(yz)$,

(O5) $xx=x$,

(O6) $x0=0$,

(O7) $(xy+x)+x=xy$,

(O8) $((x+y)+xy)+xy=x+y$,

(O9) $(xy+x)x=xy+x$,

(O10) $(x+y)xy=0$,

(O11) $((x+y)+xy)x=x$,

(O12) $((xy+xz)+xyz)x=(xy+xz)+xyz$

and

(O13) $(xyz+x)(xy+x)=xy+x$.

Remark 2.1. Orthorings $\mathcal{R}=(R;+,\cdot,0)$ are of characteristic 2, i.e. $x+x=0$ for all $x\in R$.

Proof.

$$x+x \overset{(O1),(O2)}{=} (0+x)+x \overset{(O4),(O10)}{=} ((x+x)xx+x)+x \overset{(O3)-(O5)}{=}$$

$$\overset{(O3)-(O5)}{=} (x(x+x)+x)+x \overset{(O7)}{=} x(x+x) \overset{(O3)-(O5)}{=} (x+x)xx \overset{(O10)}{=} 0.$$

■

Now we can state our main result describing a natural bijective correspondence between generalized ortholattices and orthorings.

Theorem 2.1. *For fixed set L the formulas*

$$x+y:=(x\wedge y)^{x\vee y},$$

$$xy:=x\wedge y$$

and

$$x \vee y := (x+y) + xy,$$

$$x \wedge y := xy,$$

$$x^y := x + y$$

induce mutually inverse bijections between the set of all generalized ortholattices on L and the set of all orthorings on L.

Proof. Let $\mathcal{L} = (L; \vee, \wedge, (^a\,;\, a \in L), 0)$ be a generalized ortholattice and put $x + y := (x \wedge y)^{x \vee y}$ and $xy := x \wedge y$ for all $x, y \in L$. Let $x, y, z \in L$. Then

$$(x+y) + xy = ((x \wedge y)^{x \vee y} \wedge x \wedge y)^{(x \wedge y)^{x \vee y} \vee (x \wedge y)} = 0^{x \vee y} = x \vee y,$$

$$x + 0 = 0^x = x,$$

$$(xy + x) + x = ((x \wedge y)^x)^x = x \wedge y = xy,$$

$$((x+y) + xy) + xy = (x \wedge y)^{x \vee y} = x + y,$$

$$(xy + x)x = (x \wedge y)^x \wedge x = (x \wedge y)^x = xy + x,$$

$$(x+y)xy = (x \wedge y)^{x \vee y} \wedge x \wedge y = 0,$$

$$((x+y) + xy)x = (x \vee y) \wedge x = x,$$

$$((xy + xz) + xyz)x = ((xy + xz) + (xy)(xz))x = ((x \wedge y) \vee (x \wedge z)) \wedge x =$$

$$= (x \wedge y) \vee (x \wedge z) = (xy + xz) + (xy)(xz) = (xy + xz) + xyz \text{ and}$$

$$(xyz + x)(xy + x) = (x \wedge y \wedge z)^x \wedge (x \wedge y)^x = (x \wedge y)^x = xy + x.$$

Hence, $(L; +, \cdot, 0)$ is an orthoring. Moreover,

$$(x+y) + xy = x \vee y,$$

$$xy = x \wedge y$$

and

$$x \leq y \text{ implies } x + y = x^y.$$

Therefore, the algebra induced by $(L;+,\cdot,0)$ according to the formulas given in the theorem coincides with $\mathcal{L}$.

Conversely, let $\mathcal{R}=(L;+,\cdot,0)$ be an orthoring and put $x\vee y:=(x+y)+xy$, $x\wedge y:=xy$ and $x^y:=x+y$ for all $x,y\in L$. Let $(L;\leq)$ denote the poset corresponding to the meet-semilattice $(L;\cdot)$ and $x,y,z\in L$. Then

$$x(x\vee y)=x((x+y)+xy)\overset{(\mathrm{O3})}{=}((x+y)+xy)x\overset{(\mathrm{O11})}{=}x,\ \text{i.e. } x\leq x\vee y,$$

$$y(x\vee y)=y((x+y)+xy)\overset{(\mathrm{O1}),(\mathrm{O3})}{=}((y+x)+yx)y\overset{(\mathrm{O11})}{=}y,\ \text{i.e. } y\leq x\vee y,$$

$$\begin{aligned}x,y\leq z \text{ implies } (x\vee y)z&=((x+y)+xy)z\overset{(\mathrm{O3}),(\mathrm{O4})}{=}((zx+zy)+zxy)z\\&\overset{(\mathrm{O4}),(\mathrm{O12})}{=}(zx+zy)+zxy\overset{(\mathrm{O3}),(\mathrm{O4})}{=}(x+y)+xy=x\vee y,\ \text{i.e. } x\vee y\leq z,\end{aligned}$$

$$x\leq y \text{ implies } x^y y=(x+y)y\overset{(\mathrm{O3})}{=}(yx+y)y\overset{(\mathrm{O9})}{=}yx+y\overset{(\mathrm{O3})}{=}x+y=x^y,$$

$$\text{i.e. } x^y\leq y,$$

$$x\leq y \text{ implies } (x^y)^y=(x+y)+y\overset{(\mathrm{O3})}{=}(yx+y)+y\overset{(\mathrm{O7})}{=}yx\overset{(\mathrm{O3})}{=}x,$$

$$\begin{aligned}x\leq y\leq z \text{ implies } y^z x^z&=(y+z)(x+z)\overset{(\mathrm{O3}),(\mathrm{O4})}{=}(zyx+z)(zy+z)\\&\overset{(\mathrm{O4}),(\mathrm{O13})}{=}zy+z\overset{(\mathrm{O3})}{=}y+z=y^z,\ \text{i.e. } y^z\leq x^z \text{ and}\end{aligned}$$

$$x\leq y \text{ implies } x\wedge x^y=x(x+y)\overset{(\mathrm{O3}),(\mathrm{O4})}{=}(x+y)xy\overset{(\mathrm{O10})}{=}0.$$

Hence $(L;\vee,\wedge,({}^a;\,a\in L),0)$ is a generalized ortholattice. Moreover,

$$(x\wedge y)^{x\vee y}=xy+((x+y)+xy)\overset{(\mathrm{O1})}{=}((x+y)+xy)+xy\overset{(\mathrm{O8})}{=}x+y \text{ and}$$

$$x\wedge y=xy.$$

This shows that the algebra induced by $(L;\vee,\wedge,({}^a;\,a\in L),0)$ according to the formulas of the theorem coincides with $\mathcal{R}$. ∎

Remark 2.2. If $(L; \vee, \wedge, ', 0, 1)$ is a Boolean algebra, then $x + y = (x \wedge y)^{x \vee y}$ is the well-known symmetric difference since $(x \wedge y') \vee (x' \wedge y) = (x \wedge y)' \wedge (x \vee y) = (x \wedge y)^{x \vee y}$ for all $x, y \in L$.

Comparing the definition of an orthoring to the definition of a Boolean pseudoring introduced in [7] and comparing the definition of a generalized ortholattice with that of a generalized orthomodular lattice (cf. [17]), we obtain

Theorem 2.2. *An orthoring $(R; +, \cdot, 0)$ is a Boolean pseudoring if and only if $(x + y)x = x + xy$ and $(xyz + x)y = xyz + xy$ for all $x, y, z \in R$. A generalized ortholattice $(L; \vee, \wedge, ({}^a; a \in L), 0)$ is a generalized orthomodular lattice if and only if $x^z \wedge y = x^y$ for all $x, y, z \in L$ with $x \leq y \leq z$.*

Proof. The second assertion can be proved as follows: Let $\mathcal{L} = (L; \vee, \wedge, ({}^a; a \in L), 0)$ be a generalized ortholattice. If $\mathcal{L}$ is a generalized orthomodular lattice, then $x^z \wedge y = x^y$ for all $x, y, z \in L$ with $x \leq y \leq z$ according to the definition of a generalized orthomodular lattice. Conversely, if $x^z \wedge y = x^y$ for all $x, y, z \in L$ with $x \leq y \leq z$, then $x \vee (y \wedge x^z) = x \vee x^y = y$ for all $x, y, z \in L$ with $x \leq y \leq z$ and, hence, $([0, z]; \vee, \wedge, {}^z, 0, z)$ is orthomodular for all $z \in L$. Therefore, $\mathcal{L}$ is a generalized orthomodular lattice. ■

3. Congruence and ideal properties

For an overview on congruence conditions, their characterizations and the theory of ideals in universal algebras, see [4].

A variety is called *arithmetical* if it is both congruence permutable and congruence distributive. A variety with a constant term 0 is called *weakly regular* if any congruence of an algebra belonging to this variety is determined by its 0-class.

It is easy to see that the congruence lattice of an orthoring is a sublattice of the congruence lattice of the corresponding sectionally complemented lattice. Since it is well known that sectionally complemented lattices are arithmetical and weakly regular (see, e.g., [15]), this carries over to orthorings.

Here we will provide a different (and direct) proof of this result.

Theorem 3.1. *Orthorings are arithmetical and weakly regular.*

Proof. Consider the terms

$$
\begin{aligned}
t_1(x,y) &:= xy + x,\\
t_2(x,y) &:= xy + y,\\
t(x,y,z,u) &:= (y+u)+z \text{ and}\\
m(x,y,z) &:= (xy+yz)+zx.
\end{aligned}
$$

We show that t_1, t_2 and t satisfy the identities

$$
\begin{aligned}
&t_1(x,x) = t_2(x,x) = 0,\\
&t(x,y,t_1(x,y),t_2(x,y)) = x \text{ and}\\
&t(x,y,0,0) = y
\end{aligned}
$$

from which it follows that orthorings are permutable and weakly regular according to Theorem 6.4.11 of [4]. Moreover, we prove that m is a majority term, i.e. it satisfies

$$m(x,x,y) = m(x,y,x) = m(y,x,x) = x$$

from which we obtain that ortholattices are congruence distributive according to Corollary 3.2.4 of [4].

The following calculations yield the desired identities:

$$(x+xy)+xy = ((x\wedge y)^x \wedge (x\wedge y))^{(x\wedge y)^x \vee (x\wedge y)} = 0^x = x,$$

$$t_1(x,x) = xx + x \overset{(O5)}{=} x + x = 0,$$

$$t_2(x,x) = xx + x \overset{(O5)}{=} x + x = 0,$$

$$
\begin{aligned}
&t(x,y,t_1(x,y),t_2(x,y)) = (y+(xy+y)) + (xy+x) \overset{(O1),(O3)}{=}\\
&\quad \overset{(O1),(O3)}{=} ((yx+y)+y) + (xy+x) \overset{(O7)}{=} yx + (xy+x) \overset{(O1),(O3)}{=}\\
&\quad \overset{(O1),(O3)}{=} (x+xy)+xy = x,
\end{aligned}
$$

$$t(x,y,0,0)=(y+0)+0\overset{(O2)}{=}y,$$

$$m(x,x,y)=(xx+xy)+yx\overset{(O3),(O5)}{=}(x+xy)+xy=x,$$

$$m(x,y,x)=(xy+yx)+xx\overset{(O3),(O5)}{=}(xy+xy)+x\overset{(O1),(O2)}{=}x \text{ and}$$

$$m(y,x,x)=(yx+xx)+xy\overset{(O1),(O3),(O5)}{=}(x+xy)+xy=x.$$

■

Let $\mathcal{V}$ be a variety with a constant term 0, $\mathcal{A}$ an algebra belonging to $\mathcal{V}$ and B a subset of the carrier set of $\mathcal{A}$. A term $t(x_1,\ldots,x_n)$ is called an *ideal term* with respect to the variables $x_i, i\in I$, $(I\subseteq\{1,\ldots,n\})$ if $t(x_1,\ldots,x_n)=0$ whenever $x_i=0$ for all $i\in I$. Let $t(x_1,\ldots,x_n)$ be an ideal term with respect to the variables $x_i, i\in I$. B is called *closed with respect to* t if $t(a_1,\ldots,a_n)\in B$ provided $a_1,\ldots,a_n\in A$ and $a_i\in B$ for all $i\in I$. B is called an *ideal* of $\mathcal{A}$ if B is closed with respect to all ideal terms. A set T of ideal terms is called a *basis of ideal terms* if a subset of the carrier set of an algebra $\mathcal{C}$ belonging to $\mathcal{V}$ is an ideal of $\mathcal{C}$ whenever it is closed with respect to all ideal terms belonging to T. If Θ is a congruence on $\mathcal{A}$, then the congruence class $[0]\Theta$ of 0 with respect to Θ is called the *conguence kernel* of Θ. It is easy to see that every congruence kernel is an ideal. If $\mathcal{V}$ has a so-called *subtractive term*, i.e. a binary term s satisfying $s(x,0)=x$ and $s(x,x)=0$, then, conversely, every ideal is a congruence kernel (cf. Theorems 6.6.11 and 10.1.10 of [4]). This is the case with orthorings, because the term $s(x,y):=x+y$ serves as a subtractive term.

Ideals in orthorings can now be characterized as follows:

Theorem 3.2. *A subset I of the base set R of an orthoring $\mathcal{R}$ containing* 0 *is an ideal of $\mathcal{R}$ if and only if $x,y,z,u\in R$ and $xy+x, xy+y, zu+z, zu+u\in I$ together imply $(x+z)(y+u)+(x+z), xyzu+xz\in I$.*

Proof. This follows from Theorem 10.3.1 of [4] by using the terms introduced in the proof of Theorem 3.1. ■

Corollary 3.1. *Every ideal I of an orthoring $\mathcal{R}=(R;+,\cdot,0)$ is the kernel of the congruence $\Theta_I:=\{(x,y)\in R\times R\mid xy+x\in I$ and $xy+y\in I\}$ on $\mathcal{R}$.*

Proof. It is almost evident that I is the kernel of Θ_I where Θ_I is reflexive and compatible. However, the variety of orthorings is congruence permutable according to Theorem 3.1, and by [19], every compatible reflexive relation on $\mathcal{R}$ is a congruence on $\mathcal{R}$ (see also Corollary 3.1.13 in [4]). ■

Finally, we present a finite basis of ideal terms for orthorings:

Theorem 3.3. *The following terms form a basis of ideal terms for orthorings:*

$0,$

$(((x+y_1)+y_2)+((z+y_3)+y_4))(x+z)+(((x+y_1)+y_2)+((z+y_3)+y_4)),$

$(((x+y_1)+y_2)+((z+y_3)+y_4))(x+z)+(x+z),$

$((x+y_1)+y_2)((z+y_3)+y_4)xz+((x+y_1)+y_2)((z+y_3)+y_4),$

$((x+y_1)+y_2)((z+y_3)+y_4)xz+xz,$

and

$y_1+y_2.$

Proof. This follows from Theorem 10.3.4 of [4] by using the terms introduced in the proof of Theorem 3.1. ■

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