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ON INTERVAL DECOMPOSITION LATTICES

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Abstract

Intervals in binary or n-ary relations or other discrete structures generalize the concept of interval in an ordered set. They are defined abstractly as closed sets of a closure system on a set V, satisfying certain axioms. Decompositions are partitions of V whose blocks are intervals, and they form an algebraic semimodular lattice. Lattice-theoretical properties of decompositions are explored, and connections with particular types of intervals are established.

Keywords: interval, closure system, modular decomposition, semimodular lattice, partition lattice, strong set, lexicographic sum.

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1. Preliminaries

The concept of interval has its origin in the theory of ordered sets. For totally ordered sets a theory of interval decompositions, and of the corresponding lexicographic sum construction, was developed by Hausdorff (see [17] and [18]), followed by Gleyzal [14]. Decompositions into intervals and lexicographic sum, as applied to well-ordered sets, are the fundamental operations of the arithmetic of ordinal numbers (see e.g. Bachmann [1]). For partially ordered sets, the basic theory of such decompositions is presented e.g. in the texts of Bastiani [2], Trotter [28] and very recently Schröder [26]. In graph theory the analogous concepts of lexicographic joins and modules appear in the articles of Sabidussi [25] and Gallai [12]; very recent work on the structural and algorithmic aspects of graph decompositions include the articles of McConnell and Spinrad [22] and Zverovich [29]. Still in the context of graphs, decompositions into special types of modules were investigated by Habib and Maurer [16]. The articles of Möhring and Radermacher [24] and Möhring [23], while oriented towards optimizational and algorithmic issues, also propose unifying concepts of abstract decomposition theory and contain numerous references, in particular to early work regarding decompositions of Boolean functions and set systems. Other early studies of lexicographic sums of hypergraphs, and of directed graphs which include both ordered sets and undirected graphs, are due to Dörfler and Imrich [5] and Dörfler [4]. In the general context of n-ary relations and relational systems, an interval concept rooted in mathematical logic and based on the extensibility of local automorphisms was proposed by Fraissé in [9] - [11] and studied, e.g., by Gillam [13], one of the present authors ([6] and [7]), and Ille [19] - [21] with some variations in definitions and terminology. An attempt to transfer the lexicographic construction from the relational to the universal-algebraic context was made in [8].

Given any concept of interval in some relational structure, or other type of structure, the set of all decompositions into intervals of the structure's underlying set is a subset of the set of all partitions, naturally ordered by refinement. We propose to study this ordered set of decompositions, where following [23], intervals are defined abstractly as closed sets of some closure system satisfying certain axioms.

Let V be a nonempty set and $\mathcal{Q}$ a subset of its power-set $\mathcal{P}(V)$. If $\bigcap_{k\in K} A_k \in \mathcal{Q}$ holds for any system $A_k \in \mathcal{Q}$, $k \in K$ $(K \neq \varnothing)$ and $V \in \mathcal{Q}$, then $(V, \mathcal{Q})$ is called a *closure system* on V. A family $\mathcal{H}$ of subsets of V is called *updirected,* if for any $A, B \in \mathcal{H}$ there exists a set $C \in \mathcal{H}$ such that

$A \cup B \subseteqq C$. A closure system $(V, \mathcal{Q})$ is called *algebraic*, if for any updirected family $\mathcal{H} = \{A_k \mid k \in K\} \subseteqq \mathcal{Q}$, $\bigcup_{k \in K} A_k \in \mathcal{Q}$ holds.

Definition 1.1. An *interval system* $(V, \mathcal{I})$ is an algebraic closure system with the following properties:

(I_0) $\{x\} \in \mathcal{I}$ for all $x \in V$ and $\varnothing \in \mathcal{I}$,

(I_1) $A, B \in \mathcal{I}$ and $A \cap B \neq \varnothing$ imply $A \cup B \in \mathcal{I}$,

(I_2) For any $A, B \in \mathcal{I}$ the relations $A \cap B \neq \varnothing$, $A \subsetneqq B$ and $B \subsetneqq A$ imply $A \setminus B \in \mathcal{I}$ (and $B \setminus A \in \mathcal{I}$).

Here we list some remarkable examples:

A) Let $G = (V, E)$ be an undirected graph. For any $x \in V$ we define $N(x) = \{v \in V \mid (x, v) \in E\}$. A subset $A \subseteqq V$ is called a *module* of G if $N(x) \setminus A = N(y) \setminus A$ for all $x, y \in A$. Let $\mathcal{M}$ stand for the modules of the graph G. Then $(V, \mathcal{M})$ is an interval system.

B) Let $n \geq 2$ and $R \subseteqq V^n$ be an n-ary relation on the set V. An *interval* of the structure (V, R) is a subset $I \subseteqq V$ with the property that for every $\mathbf{x} = (x_1, ..., x_n) \in V^n \setminus I^n$ with $x_i \in I$ for some $i \in \{1, ..., n\}$, if $\mathbf{y}$ is an n-tuple obtained from $\mathbf{x}$ by replacing x_i by some $y_i \in I$, then $\mathbf{x} \in R \Leftrightarrow \mathbf{y} \in R$. Denoting by $\mathcal{I}$ the set of the intervals of (V, R), we obtain an interval system $(V, \mathcal{I})$.

C) If $G = (V, E)$ is a finite tree, then the vertex sets of its subtrees form a closure system $(V, \mathcal{Q})$ which satisfies conditions (I_0) and (I_1).

A particular case of **B)** is given by a linearly ordered set $(V; \leq)$. In this case the intervals of the relational structure are the usual intervals including singletons, V and $\varnothing$.

Definition 1.2. A *decomposition* in a closure system $(V, \mathcal{Q})$ is a partition $\pi = \{A_i \mid i \in I\}$ of the set V such that $A_i \in \mathcal{Q}$, for all $i \in I$. If $(V, \mathcal{Q})$ is an interval system, then π is called an *interval decomposition*. The set of all decompositions in $(V, \mathcal{Q})$ is denoted by $\mathcal{D}(V, \mathcal{Q})$.

Let $\mathrm{Part}(V)$ denote the lattice of all partitions of V. Restricting the partial order $\leq$ in $\mathrm{Part}(V)$ to $\mathcal{D}(V, \mathcal{Q}) \subseteqq \mathrm{Part}(V)$, we obtain again a partially ordered set $(\mathcal{D}(V, \mathcal{Q}); \leq)$. Moreover, the following is true:

Lemma 1.3. *Let* $(V, \mathcal{Q})$ *be a closure system. Then* $(\mathcal{D}(V, \mathcal{Q}); \leq)$ *is a complete lattice with the greatest element* $\nabla = \{V\}$.

Proof. As $\nabla = \{V\}$ is the greatest element of $(\mathcal{D}(V, \mathcal{Q}); \leq)$, to prove that $(\mathcal{D}(V, \mathcal{Q}); \leq)$ is a complete lattice, it is enough to show that for any system $\pi_k \in \mathcal{D}(V, \mathcal{Q})$, $k \in K$, its infimum exists in $\mathcal{D}(V, \mathcal{Q})$. Since $\mathcal{D}(V, \mathcal{Q}) \subseteqq \mathrm{Part}(V)$, and $(\mathrm{Part}(V); \wedge, \vee)$ is a complete lattice, it is enough to prove $\bigwedge_{k \in K} \pi_k \in \mathcal{D}(V, \mathcal{Q})$. However, this is obvious, as all the blocks of the partition $\bigwedge_{k \in K} \pi_k$ being intersections of some blocks of the decompositions π_k, $k \in K$, belong to $\mathcal{Q}$. ■

In graph theory, interval decompositions are closely related to the transitive orientation problem (see [12] and [22]). It is also important that, using decompositions we can define "quotients" of various discrete structures. For instance, if $\pi = \{A_i \mid i \in I\}$ is an interval decomposition of a graph $G = (V, E)$ (or of a relational structure (V, R)), then a graph $G^* = (V/\pi, E^*)$ (a relational structure $(V/\pi, R^*)$) is induced naturally on the factor set $V/\pi = \{A_i \mid i \in I\}$ which in some sense is a "quotient" of G (of (V, R)). (See, e.g., [7] and [23].) In general, if π is a decomposition in a closure system $(V, \mathcal{Q})$, then the corresponding *quotient* $(V/\pi, \mathcal{Q}/\pi)$ is defined on the set $V/\pi = \{A_i \mid i \in I\}$ as follows: For $J \subseteqq I$, we have

$$B = \{A_i \mid i \in J\} \in \mathcal{Q}/\pi \Leftrightarrow \bigcup_{i \in J} A_i \in \mathcal{Q}.$$

It is easy to see that the quotient $(V/\pi, \mathcal{Q}/\pi)$ is a closure system; if $(V, \mathcal{Q})$ is an interval system then $(V/\pi, \mathcal{Q}/\pi)$ is an interval system, too.

Remark 1.4. If $(V, \mathcal{Q})$ is a closure system satisfying condition (I_0), then clearly $\triangle = \{\{x\} \mid x \in V\}$ is the least element of $\mathcal{D}(V, \mathcal{Q})$, and to any $A \in \mathcal{Q} \setminus \{\varnothing\}$ corresponds the decomposition

$$\pi_A = \{A\} \cup \{\{x\} \mid x \in V \setminus A\}.$$

An interval system $(V, \mathcal{I})$ is called *degenerate* if $\mathcal{I} = \mathcal{P}(V)$. It is called *prime* if $\mid V \mid \geq 2$ and $\mathcal{I}$ contains only *nonproper* intervals, i.e., $\varnothing$, V and singletons $\{a\}$, $a \in V$. Remark 1.4 also implies:

Lemma 1.5. *Let $(V, \mathcal{I})$ be an interval system. Then*

(i) *$(V, \mathcal{I})$ is degenerate if and only if $\mathcal{D}(V, \mathcal{I}) = \mathrm{Part}(V)$;*

(ii) *$(V, \mathcal{I})$ is prime if and only if $\mathcal{D}(V, \mathcal{I}) = \{\triangle, \nabla\}$ and $| V | \geq 2$.*

∎

A family of sets $A_j \subseteqq V$, $j \in J$ is called *connected* if for each $k, l \in J$ there exists a finite subset $\{j_1, ..., j_n\} \subseteqq J$ ($n \in \mathbb{N}$) such that $j_1 = k$, $j_n = l$ and $A_j \cap A_{j+1} \neq \varnothing$, for all $1 \leq j \leq n-1$. If $(V, \mathcal{Q})$ is a closure system satisfying condition (I_1) and $\{A_1, ..., A_k\} \subseteqq \mathcal{Q}$ is a connected family, then it can be proved by an easy induction on $k \in \mathbb{N}$ that $\bigcup_{j=1}^{k} A_j \in \mathcal{Q}$. Moreover, as an extension of Proposition 2 in [7], we get:

Proposition 1.6. *If $(V, \mathcal{Q})$ is an algebraic closure system satisfying condition (I_1) and $\{A_j \mid j \in J\} \subseteqq \mathcal{Q}$ is a connected family, then $\bigcup_{j \in J} A_j \in \mathcal{Q}$.*

Proof. Let $\mathcal{F}$ denote the family of all those finite subsets $F \subseteqq J$ which have the property that $\{A_j \mid j \in F\}$ is a connected family and define $B_F = \bigcup_{j \in F} A_j$, $F \in \mathcal{F}$. Then $B_F \in \mathcal{Q}$ and obviously $\{B_F \mid F \in \mathcal{F}\}$ is an updirected family. As $(V, \mathcal{Q})$ is an algebraic closure system, we obtain $\bigcup\{B_F \mid F \in \mathcal{F}\} \in \mathcal{Q}$. Clearly, $B_F \subseteqq \bigcup_{j \in J} A_j$ for all $F \in \mathcal{F}$. Since $\{A_j\}$ itself is a connected family for each $j \in J$, we get $\{A_j \mid j \in J\} \subseteqq \{B_F \mid F \in \mathcal{F}\}$, whence $\bigcup_{j \in J} A_j \subseteqq \bigcup\{B_F \mid F \in \mathcal{F}\}$. Thus we conclude $\bigcup_{j \in J} \bigcup A_j = \bigcup\{B_F \mid F \in \mathcal{F}\} \in \mathcal{Q}$. ∎

2. Algebraic closure systems satisfying condition (I_1)

The purpose of this section is to describe algebraic properties of the lattice $\mathcal{D}(V, \mathcal{Q})$ corresponding to a closure system which satisfies condition (I_1).

For a binary relation $\rho \subseteqq V \times V$ and $a \in V$ let $\rho[a] = \{x \in V \mid (a, x) \in \rho\}$. Clearly, if ρ is an equivalence relation, then $\rho[a]$ stands for the equivalence class of a.

Proposition 2.1. *Let $(V, \mathcal{Q})$ be a closure system. If $(V, \mathcal{Q})$ is algebraic and satisfies condition (I_1), then $\mathcal{D}(V, \mathcal{Q})$ is a complete sublattice of $\mathrm{Part}(V)$. If $(V, \mathcal{Q})$ satisfies condition (I_0) and $\mathcal{D}(V, \mathcal{Q})$ is a sublattice of $\mathrm{Part}(V)$, then $(V, \mathcal{Q})$ satisfies condition (I_1).*

Proof. Let $(V,\mathcal{Q})$ be an algebraic closure system which satisfies condition (I_1) and take any $\pi_i \in \mathcal{D}(V,\mathcal{Q})$, $i \in I$. Since $\bigwedge_{i\in I}\pi_i \in \mathcal{D}(V,\mathcal{Q})$, to prove that $\mathcal{D}(V,\mathcal{Q})$ is a complete sublattice of $\mathrm{Part}(V)$, we have to show that their join $\bigvee_{i\in I}\pi_i$ in $\mathrm{Part}(V)$ is also a decomposition in $(V,\mathcal{Q})$.

Let ρ_i, $i \in I$ denote the equivalence relations induced by the partitions π_i, $i \in I$ on V. As $\bigvee_{i\in I}\rho_i$ is the equivalence induced by $\bigvee_{i\in I}\pi_i$, to prove $\bigvee_{i\in I}\pi_i \in \mathcal{D}(V,\mathcal{Q})$ it is enough to show that $(\bigvee_{i\in I}\rho_i)[a] \in \mathcal{Q}$, for any $a \in V$. Denote $K = (\bigvee_{i\in I}\rho_i)[a]$. Clearly, $K = \bigcup\{\rho_i[x] \mid x \in K,\, i \in I\}$. We claim that $\{\rho_i[x] \mid x \in K,\, i \in I\}$ is a connected family of sets.

Indeed, take any $\rho_k[x], \rho_l[y] \subseteqq K$, i.e. any $k,l \in I$ and $x,y \in K$. Then there exist $\{z_0, z_1, ..., z_n\} \subseteqq K$ and $\{i_1, i_2, ..., i_n\} \subseteqq I$ with $z_0 = x$, $i_1 = k$ and $z_n = y$, $i_n = l$ and such that $(z_{j-1}, z_j) \in \rho_{i_j}$, $1 \le j \le n$. As $z_j \in \rho_{i_j}[z_{j-1}] \cap \rho_{i_{j+1}}[z_j]$, the family $\{\rho_i[x] \mid x \in K, i \in I\}$ is connected. Since $\rho_i[x] \in \mathcal{Q}$, for all $x \in K$ and $i \in I$ and since $(V,\mathcal{Q})$ is an algebraic closure system, Proposition 1.6 gives that $\bigcup\{\rho_i[x] \mid x \in K,\, i \in I\} \in \mathcal{Q}$. Hence $(\bigvee_{i\in I}\rho_i)[a] \in \mathcal{Q}$, for each $a \in V$.

Conversely, suppose that $(V,\mathcal{Q})$ satisfies condition (I_0) and $\mathcal{D}(V,\mathcal{Q})$ is a sublattice of $\mathrm{Part}(V)$. Take any $A, B \in \mathcal{Q}$ with $A \cap B \neq \varnothing$. Then, in view of Remark 1.4, $\pi_A = \{A\} \cup \{\{x\} \mid x \in V \setminus A\} \in \mathcal{D}(V,\mathcal{Q})$ and $\pi_B = \{B\} \cup \{\{x\} \mid x \in V \setminus B\} \in \mathcal{D}(V,\mathcal{Q})$. Thus $\pi_A \vee \pi_B = \{A \cup B\} \cup \{\{x\} \mid x \in V \setminus (A \cup B)\} \in \mathcal{D}(V,\mathcal{Q})$, whence we get $A \cup B \in \mathcal{Q}$. Thus $(V,\mathcal{Q})$ satisfies condition (I_1). ■

As $\mathrm{Part}(V)$ is an algebraic lattice and since any complete sublattice of an algebraic lattice is an algebraic lattice, too (see, e.g., Crawley and Dilworth [3]), we deduce:

Corollary 2.2. *If an algebraic closure system $(V,\mathcal{Q})$ satisfies condition (I_1), then $\mathcal{D}(V,\mathcal{Q})$ is an algebraic lattice.* ■

Obviously, for any $\pi_1, \pi_2 \in \mathcal{D}(V,\mathcal{Q})$, $\pi_1 \le \pi_2$ holds if and only if any block of π_2 is the union of some blocks of π_1.

Let $\prec$ denote the covering relation in a lattice L, i.e. for any $a, b \in L$ we write $a \prec b$ iff $a < b$ and there is no $c \in L$ with $a < c < b$.

Lemma 2.3. *Let $\pi_1 = \{A_i \mid i \in I\}$ and $\pi_2 = \{B_j \mid j \in J\}$ be two decompositions in a closure system $(V,\mathcal{Q})$. If $\pi_1 \prec \pi_2$ holds in $\mathcal{D}(V,\mathcal{Q})$, then there exists a single $j_0 \in J$ and $I_{j_0} \subseteqq I$ with at least two elements such that $B_{j_0} = \bigcup_{i\in I_{j_0}} A_i$ and $B_j \in \pi_1$, for all $j \in J \setminus \{j_0\}$.*

Proof. As $\pi_1 < \pi_2$, for each $j \in J$ there exists a nonempty $I_j \subseteqq I$ such that $B_j = \bigcup_{i \in I_j} A_i$ and at least one set I_j has at least two elements. Assume $| I_k | \geq 2$ and $| I_l | \geq 2$, for some $k, l \in J$, $k \neq l$. Then $\pi' = \{B_k\} \cup \{A_i \mid i \in I \setminus I_k\}$ and $\pi'' = \{B_k, B_l\} \cup \{A_i \mid i \in I \setminus (I_k \cup I_l)\}$ are also decompositions in $(V, \mathcal{Q})$ and we get $\pi_1 < \pi' < \pi'' < \pi_2$ - contradicting $\pi_1 \prec \pi_2$. Hence, there is a single $j_0 \in J$ and $I_{j_0} \subseteqq I$ with $| I_{j_0} | \geq 2$ and $B_{j_0} = \bigcup_{i \in I_{j_0}} A_i$. This gives $B_j \in \pi_1$, for all $j \in J \setminus \{j_0\}$. ■

The lattice of decompositions is clearly not modular in general. In the context of a somewhat different but related abstract axiomatization of decompositions, semimodularity, stated in the form of Birkhoff's condition in the case of decomposition lattices of finite length, was shown by Möhring and Radermacher [24]. The following proposition is in the same spirit, but finite length is not required and therefore the condition to be established is the stronger version of the upper semimodularity condition. (For the various forms of semimodularity, see Stern [27].) Infinite length lattices to which the proposition applies include the lattices of decompositions, into usual order intervals, of all infinite linearly ordered sets, such as the set of integers, rational numbers, or real numbers.

Proposition 2.4. *Let $(V, \mathcal{Q})$ be an algebraic closure system satisfying condition (I_1). Then $\mathcal{D}(V, \mathcal{Q})$ is an algebraic semimodular lattice.*

Proof. In view of Corollary 2.2 we have to show only that $\mathcal{D}(V, \mathcal{Q})$ is a semimodular lattice. Take two decompositions $\pi_1 = \{B_j \mid j \in J\}$ and $\pi_2 = \{C_k \mid k \in K\}$ with $\pi_1 \wedge \pi_2 \prec \pi_1$ in $\mathcal{D}(V, \mathcal{Q})$. Then $\pi_2 < \pi_1 \vee \pi_2$. We have to prove that $\pi_2 \prec \pi_1 \vee \pi_2$ also holds in $\mathcal{D}(V, \mathcal{Q})$.

Clearly, all the blocks of $\pi_1 \wedge \pi_2$ are nonempty intersections $B_j \cap C_k$ $(j \in J,\ k \in K)$. As $\pi_1 \wedge \pi_2 \prec \pi_1$, in view of Lemma 2.3, there is a single $j_0 \in J$ and $K_0 \subseteqq K$ with $| K_0 | \geq 2$ such that $B_{j_0} = \bigcup_{k \in K_0} (B_{j_0} \cap C_k)$ and the sets $B_{j_0} \cap C_k$, $k \in K_0$ are nonempty. Lemma 2.3 also gives $B_j \in \pi_1 \wedge \pi_2$, for all $j \neq j_0$ and this implies

$$\pi_1 \wedge \pi_2 = \{B_j \mid j \in J \setminus \{j_0\}\} \cup \{B_{j_0} \cap C_k \mid k \in K_0\}.$$

As any C_k, $k \in K_0$ has a nonempty intersection with B_{j_0}, $\{B_{j_0}\} \cup \{C_k \mid k \in K_0\}$ is a connected subfamily of $\mathcal{Q}$ and hence

$$M = B_{j_0} \bigcup \left(\bigcup_{k \in K_0} C_k \right) \in \mathcal{Q}.$$

As $B_{j_0} \subsetneqq \bigcup_{k\in K_0} C_k$, we get $M = \bigcup_{k\in K_0} C_k$. Clearly, $\pi = \{C_k \mid k \in K \backslash K_0\} \cup \{M\}$ is a decomposition in $(V, \mathcal{Q})$ and $\pi_1, \pi_2 \leq \pi$. Since $\pi_2 < \pi_1 \vee \pi_2 \leq \pi$, to prove $\pi_2 \prec \pi_1 \vee \pi_2$ it is sufficient to show $\pi_2 \prec \pi$.

By contradiction, assume that there exists a $\pi_3 = \{D_i \mid i \in I\} \in \mathcal{D}(V, \mathcal{Q})$ such that $\pi_2 < \pi_3 < \pi$. Then, there exists a set $I_0 \subsetneqq I$ with $\mid I_0 \mid \geq 2$ and such that $M = \bigcup_{i\in I_0} D_i$ and $D_i \in \{C_k \mid k \in K \setminus K_0\}$, for all $i \in I \setminus I_0$. Hence π_3 has the form $\pi_3 = \{C_k \mid k \in K \setminus K_0\} \cup \{D_i \mid i \in I_0\}$. As $\pi_2 < \pi_3$, there is at least one $i^* \in I_0$ and $K^* \subsetneqq K_0$ such that $D_{i^*} = \bigcup_{k\in K^*} C_k$ and $\mid K^* \mid \geq 2$. Then $D_{i^*} \cap B_{j_0} = \bigcup_{k\in K^*}(C_k \cap B_{j_0})$, and since $K^* \subsetneqq K_0$, all the members of this union are nonempty, by the definition of K_0. As $\mid I_0 \mid \geq 2$ gives $M \neq D_{i^*}$, we get $K^* \neq K_0$.

Finally, let us consider the decomposition

$$\pi^* = \{B_j \mid j \in J \setminus \{j_0\}\} \cup \{C_k \cap B_{j_0} \mid k \in K_0 \setminus K^*\} \cup \{D_{i^*} \cap B_{j_0}\}.$$

Since $D_{i^*} \cap B_{j_0}$ and $C_k \cap B_{j_0}$, $k \in K_0 \backslash K^*$, are disjoint and nonempty subsets of B_{j_0}, we have $\pi^* < \pi_1$. As $\mid K^* \mid \geq 2$, the union $\bigcup_{k\in K^*}(C_k \cap B_{j_0}) = D_{i^*} \cap B_{j_0}$ has at least two (nonempty and disjoint) members and this implies that

$$\pi_1 \wedge \pi_2 = \{B_j \mid j \in J \setminus \{j_0\}\} \cup \{B_{j_0} \cap C_k \mid k \in K_0\}$$

$$= \{B_j \mid j \in J \setminus \{j_0\}\} \cup \{B_{j_0} \cap C_k \mid k \in K_0 \setminus K^*\} \cup \{B_{j_0} \cap C_k \mid k \in K^*\}$$

is strictly less than π^*. This gives $\pi_1 \wedge \pi_2 < \pi^* < \pi_1$, a contradiction. ■

Remark 2.5. If $(V, \mathcal{I})$ is an interval system, then Theorem 2.4 gives that $\mathcal{D}(V, \mathcal{I})$ is an algebraic semimodular lattice.

Let L be a complete lattice. An element $m \in L \setminus \{1\}$ is called *completely meet-irreducible*, if for any nonempty system $x_i \in L$, $i \in I$ the equality $m = \bigwedge\{x_i \mid i \in I\}$ implies $m = x_{i_0}$, for some $i_0 \in I$. Let $m^{\#} = \bigwedge\{x \in L \mid$

$m < x\}$. It is easy to see that m is completely meet-irreducible if and only if $m^{\#}$ is the unique element of L satisfying $m \prec m^{\#}$. If $m^{\#} = 1$, then m is called a *dual atom* of L.

Theorem 2.6. *Let $(V, \mathcal{Q})$ be an algebraic closure system which satisfies condition (I_1) and μ a completely meet-irreducible element of $\mathcal{D}(V, \mathcal{Q})$. Then the principal filter $[\mu)$ is a distributive lattice.*

Proof. Let $\mu = \{A_i \mid i \in I\} \in \mathcal{D}(V, \mathcal{Q})$ be completely meet-irreducible. As $\mu < \mu^{\#}$, there is an $I_0 \subseteqq I$ such that $\mu^{\#} = \{B\} \cup \{A_i \mid i \in I \setminus I_0\}$ with $B = \bigcup_{i \in I_0} A_i$ and $| I_0 | \geq 2$. First, we prove that any decomposition $\pi > \mu$ has the form $\pi = \{C\} \cup \{A_i \mid i \in I \setminus J_0\}$, where $J_0 \subseteqq I$ and $C = \bigcup_{i \in J_0} A_i \supseteqq B$.

Indeed, $\mu^{\#} \leq \pi$ implies that there exists a block C of π such that $B \subseteqq C$. Then $C = \bigcup_{i \in J_0} A_i$, for some $J_0 \subseteqq I$ with $I_0 \subseteqq J_0$. We claim that any block $D \neq C$ of π coincides with some block A_i, $i \in I \setminus J_0$.

By contradiction, assume $D \notin \{A_i \mid i \in I \setminus J_0\}$. As $C \cap D = \varnothing$, we also get $D \notin \{A_i \mid i \in J_0\}$. Therefore, $D = \bigcup_{i \in K} A_i$ with $| K | \geqslant 2$. Then $\pi^* = \{D\} \cup \{A_i \mid i \in I \setminus K\}$ is also a decomposition with $\mu < \pi^*$. Hence $\mu^{\#} \leq \pi^*$, and this implies $B \subseteqq D$. Thus we get $C \cap D \supseteqq B \neq \varnothing$ – a contradiction. This proves $\pi = \{C\} \cup \{A_i \mid i \in I \setminus J_0\}$.

Now, take $\pi_1, \pi_2 \in [\mu)$, i.e. $\pi_1, \pi_2 \geq \mu$. Then there are $C_1, C_2 \in \mathcal{Q}$ such that $\pi_1 = \{C_1\} \cup \{A_i \mid i \in I \setminus J_1\}$ and $\pi_2 = \{C_2\} \cup \{A_i \mid i \in I \setminus J_2\}$, where $J_1, J_2 \subseteqq I$, $C_1 = \bigcup_{i \in J_1} A_i, C_2 = \bigcup_{i \in J_2} A_i$ and $C_1 \cap C_2 \supseteqq B \neq \varnothing$. As by Proposition 2.1 $\mathcal{D}(V, \mathcal{Q})$ is a sublattice of $\mathrm{Part}(V)$, we obtain:

$$\pi_1 \wedge \pi_2 = \{C_1 \cap C_2\} \cup \{A_i \mid i \in I \setminus (J_1 \cap J_2)\}, \quad \text{and}$$

$$\pi_1 \vee \pi_2 = \{C_1 \cup C_2\} \cup \{A_i \mid i \in I \setminus (J_1 \cup J_2)\}.$$

Obviously, if $C_1 = C_2$, then it follows $\pi_1 = \pi_2$. Now, let $\nu \geq \mu$ be arbitrary and denote by C_ν that unique block of ν containing B. The above argument gives that the map $f : [\mu) \to \{X \in \mathcal{P}(V) \mid B \subseteq X\}$, $\nu \longmapsto C_\nu$ is a lattice embedding. As all the subsets X of V with $B \subseteq X$ form a distributive lattice, the filter $[\mu)$ is distributive, too. ■

3. Strong sets in a closure system

A set $A \in \mathcal{Q}$ is called a *strong set* in the closure system $(V, \mathcal{Q})$, if for any $B \in \mathcal{Q}$, $A \cap B \neq \varnothing$ implies $A \subseteqq B$ or $B \subseteqq A$. If $(V, \mathcal{Q})$ is an interval system, then A is called a *strong interval.*

Clearly, $\varnothing, V$ and any singleton $\{a\}$ are strong sets in any closure system which satisfies condition (I_0). They are called *improper strong sets* in $(V, \mathcal{Q})$. The strong sets have an important role in applications of the interval systems. For instance, if $G = (V, E)$ is a graph, then its connected components are strong intervals in G (see [16], [21] and [22]). In this section we also show that the completely meet-irreducible elements of the lattice of interval decompositions can be characterized by using strong intervals. Let $\mathcal{S}$ stand for the set of the all strong sets in $(V, \mathcal{Q})$. It is easy to see that whenever $\mathcal{S}$ is finite, then the transitive reduction of $(\mathcal{S} \setminus \{\emptyset\}, \subseteqq)$ (i.e. its Hasse diagram) is an updirected tree.

Proposition 3.1. *If $(V, \mathcal{Q})$ is a closure system, then $(V, \mathcal{S})$ is a closure system which satisfies conditions (I_1) and (I_2).*

Proof. Take $C_i \in \mathcal{S}$, $i \in I$. As $V \in \mathcal{S}$, to prove that $(V, \mathcal{S})$ is a closure system we have to show only $\bigcap_{i \in I} C_i \in \mathcal{S}$. Since $\mathcal{S} \subseteqq \mathcal{Q}$, $\bigcap_{i \in I} C_i \in \mathcal{Q}$ is clear. If there exist some $k, l \in I$ with $C_k \cap C_l = \varnothing$, then $\bigcap_{i \in I} C_i = \varnothing \in \mathcal{S}$. Suppose $C_k \cap C_l \neq \varnothing$, for all $k, l \in I$. Then $\{C_i \in \mathcal{S} \mid i \in I\}$ is a chain, since all the sets C_i are strong. In order to prove $\bigcap_{i \in I} C_i \in \mathcal{S}$, suppose $\left(\bigcap_{i \in I} C_i\right) \cap A \neq \varnothing$, for some $A \in \mathcal{Q}$. Then clearly, $C_i \cap A \neq \varnothing$, for all $i \in I$. Hence, we have by definition either $C_i \subseteqq A$ or $A \subseteqq C_i$, for each $i \in I$. If each C_i satisfies $A \subseteqq C_i$, we get $A \subseteqq \bigcap_{i \in I} C_i$. If $C_{i_0} \subseteqq A$ holds for some $i_0 \in I$, we obtain $\bigcap_{i \in I} C_i \subseteqq A$. These inclusions show that $\bigcap_{i \in I} C_i$ is a strong set. The remaining part is obvious. ∎

A decomposition $\pi = \{A_i \mid i \in I\}$ in a closure system $(V, \mathcal{Q})$ is called a *strong decomposition* if every A_i, $i \in I$ is a strong set in $(V, \mathcal{Q})$. Since the strong decompositions in $(V, \mathcal{Q})$ can be considered also as decompositions in the closure system $(V, \mathcal{S})$, they form a complete lattice $\mathcal{D}(V, \mathcal{S})$ whose greatest element is $\nabla = \{V\}$.

Proposition 3.2. *Let $(V, \mathcal{Q})$ be a closure system which satisfies condition (I_0). If $(V, \mathcal{Q})$ has a greatest decomposition $\mu_{\max} \neq \nabla$, then $\mu_{\max}$ is a strong decomposition.*

Proof. Let $\mu_{\max} = \{B_j \mid j \in J\}$ be the greatest decomposition of $(V, \mathcal{Q})$ with $\mu_{\max} \neq \nabla$. Suppose $B \cap B_j \neq \varnothing$, for some $B \in \mathcal{Q} \setminus \{V\}$ and $B_j \in \mu_{\max}$ and consider the decomposition $\pi_B = \{B\} \cup \{\{x\} \mid x \in V \setminus B\}$. As $\pi_B \neq \nabla$, we have by assumption $\pi_B \leq \mu_{\max}$. This implies $B \subseteqq B_j$, proving that B_j is a strong set in $(V, \mathcal{Q})$. ∎

Lemma 3.3. *Let $(V, \mathcal{Q})$ be a closure system, $\pi_1 \in \mathcal{D}(V, \mathcal{S})$, $\pi_2 \in \mathcal{D}(V, \mathcal{Q})$ and let ρ_1 and ρ_2 be the equivalences induced by π_1 and π_2 on V. Then:*

(i) *for each $a \in V$ either $\rho_1[a] \subseteqq \rho_2[a]$ or $\rho_2[a] \subseteqq \rho_1[a]$ holds;*

(ii) *we have $\rho_1 \cup \rho_2 = \rho_1 \vee \rho_2$, where $\vee$ denotes the join of equivalence relations.*

Proof. Ad (i): Since $\rho_1[a]$ is a strong set in $(V, \mathcal{Q})$ for each $a \in V$, $a \in \rho_1[a] \cap \rho_2[a]$ implies (i).

Ad (ii): As the equivalence $\rho_1 \vee \rho_2$ is the transitive closure of the relation $\rho_1 \cup \rho_2$, to show (ii), it is enough to prove $\rho_1 \cup \rho_2$ is transitive. Take $a, b \in V$ with $(a, b), (b, c) \in \rho_1 \cup \rho_2$. Due to (i), $(\rho_1 \cup \rho_2)[b] = \rho_1[b] \cup \rho_2[b] = \rho_i[b]$, for some $i \in \{1, 2\}$, thus we get $a, c \in \rho_i[b]$. Now, $(a, b), (b, c) \in \rho_i$ implies $(a, c) \in \rho_i \subseteqq \rho_1 \cup \rho_2$, proving that $\rho_1 \cup \rho_2$ is transitive. ∎

An element a of a lattice L is called *standard* (see, e.g., Grätzer [15]), if

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y) \text{ holds for all } x, y \in L.$$

The standard elements of L form a distributive sublattice of L denoted by $\mathrm{S}(L)$. If L is a bounded lattice, then $0, 1 \in S(L)$.

Let $\sqcup$, $\veebar$ and $\vee$ stand for the join operation in lattices $\mathcal{D}(V, \mathcal{S})$, $\mathcal{D}(V, \mathcal{Q})$ and $\mathrm{Part}(V)$, respectively.

Theorem 3.4. *Let $(V, \mathcal{Q})$ be a closure system. Then the strong decompositions in $(V, \mathcal{Q})$ are standard elements of $\mathcal{D}(V, \mathcal{Q})$ and $\mathcal{D}(V, \mathcal{S})$ is a distributive sublattice of $\mathcal{D}(V, \mathcal{Q})$ and of $\mathrm{Part}(V)$. Moreover, if $\pi_1 \in \mathcal{D}(V, \mathcal{S})$ and $\pi_2 \in \mathcal{D}(V, \mathcal{Q})$, then $\pi_1 \veebar \pi_2 = \pi_1 \vee \pi_2$.*

Proof. Clearly, the "meet" operation is the same in the lattices $\mathcal{D}(V, \mathcal{S})$, $\mathcal{D}(V, \mathcal{Q})$ and $\mathrm{Part}(V)$. As $\mathcal{D}(V, \mathcal{S}) \subseteqq \mathcal{D}(V, \mathcal{Q}) \subseteqq \mathrm{Part}(V)$, we have $\pi_1 \vee \pi_2 \leq \pi_1 \veebar \pi_2 \leq \pi_1 \sqcup \pi_2$, for all $\pi_1, \pi_2 \in \mathcal{D}(V, \mathcal{S})$.

Let $\rho_1, \rho_2 \subseteqq V \times V$ be the equivalences induced by π_1 and π_2. By Lemma 3.3 we have either $(\rho_1 \vee \rho_2)[x] = (\rho_1 \cup \rho_2)[x] = \rho_1[x] \cup \rho_2[x] = \rho_1[x]$ or $(\rho_1 \vee \rho_2)[x] = (\rho_1 \cup \rho_2)[x] = \rho_1[x] \cup \rho_2[x] = \rho_2[x]$, for all $x \in V$. As in the both cases $(\rho_1 \vee \rho_2)[x] \in \mathcal{S}$, the partition $\pi_1 \vee \pi_2$ induced by $\rho_1 \vee \rho_2$ is a strong decomposition, and hence $\pi_1 \vee \pi_2 = \pi_1 \sqcup \pi_2$. This result also gives $\pi_1 \veebar \pi_2 = \pi_1 \sqcup \pi_2$. Thus $\mathcal{D}(V, \mathcal{S})$ is a sublattice of Part(V) and $\mathcal{D}(V, \mathcal{Q})$.

Similarly, for $\pi_1 \in \mathcal{D}(V, \mathcal{S})$ and $\pi_2 \in \mathcal{D}(V, \mathcal{Q})$ by using Lemma 3.3 again, we get $(\rho_1 \vee \rho_2)[x] = (\rho_1 \cup \rho_2)[x] \in \mathcal{Q}$, for all $x \in V$ and this implies $\pi_1 \vee \pi_2 \in \mathcal{D}(V, \mathcal{Q})$. Hence, we obtain $\pi_1 \veebar \pi_2 = \pi_1 \vee \pi_2$.

Now, take any $\pi \in \mathcal{D}(V, \mathcal{S})$ and $\tau, \varphi \in \mathcal{D}(V, \mathcal{Q})$ and denote the corresponding equivalences by ρ_π, ρ_τ and ρ_φ, respectively. Clearly, in order to show that π is a standard element of $(\mathcal{D}(V, \mathcal{Q}), \wedge, \veebar)$, it is enough to prove $\varphi \wedge (\pi \veebar \tau) \leq (\varphi \wedge \pi) \veebar (\varphi \wedge \tau)$. As $\pi \veebar \tau = \pi \vee \tau$, we get

$$\varphi \wedge (\pi \veebar \tau) = \varphi \wedge (\pi \vee \tau).$$

On the other hand, Lemma 3.3 implies

$$\rho_\varphi \wedge (\rho_\pi \vee \rho_\tau) = \rho_\varphi \cap (\rho_\pi \cup \rho_\tau) = (\rho_\varphi \cap \rho_\pi) \cup (\rho_\varphi \cap \rho_\tau) \subseteqq (\rho_\varphi \wedge \rho_\pi) \vee (\rho_\varphi \wedge \rho_\tau).$$

Hence, we obtain $\varphi \wedge (\pi \vee \tau) \leq (\varphi \wedge \pi) \vee (\varphi \wedge \tau)$ and this implies

$$\varphi \wedge (\pi \veebar \tau) \leq (\varphi \wedge \pi) \vee (\varphi \wedge \tau) \leq (\varphi \wedge \pi) \veebar (\varphi \wedge \tau),$$

proving that π is a standard element of $\mathcal{D}(V, \mathcal{Q})$.

Since $\mathcal{D}(V, \mathcal{S})$ is included as a sublattice in the distributive lattice generated by the standard elements of $\mathcal{D}(V, \mathcal{Q})$, it is distributive, too. ■

Lemma 3.5. *Let $(V, \mathcal{Q})$ be a closure system and $\pi \in \mathcal{D}(V, \mathcal{S})$, $\varphi \in \mathcal{D}(V, \mathcal{Q})$. If $\pi \veebar \varphi = \nabla$, then either $\pi = \nabla$ or $\varphi = \nabla$ holds.*

Proof. In view of Theorem 3.4 we have $\pi \vee \varphi = \pi \veebar \varphi = \nabla$. Therefore, there exists an $a \in V$ such that $(\rho_\pi \vee \rho_\varphi)[a] = V$. As, in view of Lemma 3.3(ii), $(\rho_\pi \vee \rho_\varphi)[a] = (\rho_\pi \cup \rho_\varphi)[a]$, we get $\rho_\pi[a] \cup \rho_\varphi[a] = (\rho_\pi \cup \rho_\varphi)[a] = V$. Now, Lemma 3.3(i) implies either $\rho_\pi[a] = V$ or $\rho_\varphi[a] = V$. In the first case we get $\pi = \nabla$ and in the second $\varphi = \nabla$. ■

A lattice L with 1 is called *dually atomistic* if each $x \in L \setminus \{1\}$ is a meet of dual atoms of L. L is called *discrete* if any chain of L is finite.

Proposition 3.6. *Let $(V, \mathcal{Q})$ be a closure system. Then:*

(i) *any strong decomposition $\pi \neq \nabla$ in $(V, \mathcal{Q})$ is less than or equal to any dual atom of $\mathcal{D}(V, \mathcal{Q})$;*

(ii) *if $\mathcal{D}(V, \mathcal{S})$ has a dual atom, then it is unique and it is the greatest element of $\mathcal{D}(V, \mathcal{S}) \setminus \{\nabla\}$.*

Proof. Ad (i): Let ν be a dual atom of $\mathcal{D}(V, \mathcal{Q})$ and $\pi \in \mathcal{D}(V, \mathcal{S})$, $\pi \neq \nabla$. Assume $\pi \not\leq \nu$. Then $\nu < \pi \vee \nu$ and hence $\pi \vee \nu = \nabla$. Now Lemma 3.5 implies that either $\pi = \nabla$ or $\nu = \nabla$ is satisfied - i.e. a contradiction. Thus $\pi \leq \nu$ holds for all $\pi \in \mathcal{D}(V, \mathcal{S}) \setminus \{\nabla\}$.

Ad (ii): Suppose that μ is a dual atom of $\mathcal{D}(V, \mathcal{S})$ and take any $\pi \in \mathcal{D}(V, \mathcal{S})$, $\pi \neq \nabla$. Then $\mu \leq \mu \vee \pi \leq \nabla$ and $\mu \vee \pi \in \mathcal{D}(V, \mathcal{S})$. As $\mu \vee \pi = \nabla$ would imply either $\mu = \nabla$ or $\pi = \nabla$, we must have $\mu = \mu \vee \pi$, i.e. $\pi \leq \mu$. Therefore, μ is the greatest element of $(\mathcal{D}(V, \mathcal{S}) \setminus \{\nabla\}, \leq)$ - and hence it is the unique dual atom in $\mathcal{D}(V, \mathcal{S})$. ∎

Corollary 3.7.

(i) *If a closure system $(V, \mathcal{Q})$ satisfies condition (I_0) and $\mathcal{D}(V, \mathcal{Q})$ is dually atomistic, then $(V, \mathcal{Q})$ has no proper strong sets.*

(ii) *If $(V, \mathcal{Q})$ is a closure system and $\mathcal{D}(V, \mathcal{S})$ is a nontrivial discrete lattice, then $(V, \mathcal{Q})$ has a greatest strong decomposition $\varsigma_{\max} \neq \nabla$.*

Proof. Ad (i): Let $\mathcal{A}$ denote the family of dual atoms of $\mathcal{D}(V, \mathcal{Q})$. As $\mathcal{D}(V, \mathcal{Q})$ is dually atomistic, we have $\bigwedge\{\pi \mid \pi \in \mathcal{A}\} = \triangle$. Now assume that D is a proper strong set and consider the strong decomposition $\pi_D = \{D\} \cup \{\{x\} \mid x \in V \setminus A\}$. Then $\triangle < \pi_D < \nabla$, and Proposition 3.6(i) implies $\pi_D \leq \bigwedge\{\pi \mid \pi \in \mathcal{A}\} = \triangle$, a contradiction.

Ad (ii): As any nontrivial discrete lattice contains at least one dual atom, (ii) is an easy consequence of Proposition 3.6(ii). ∎

Theorem 3.8. *Let $(V, \mathcal{I})$ be an interval system, $\pi = \{A_i \mid i \in I\}$ a completely meet-irreducible element in $\mathcal{D}(V, \mathcal{I})$ and $\pi^{\#}$ the unique upper cover of π. Then there exists a $J \subsetneqq I$, $\mid J \mid \geq 2$ and a strong interval $B = \bigcup_{i \in J} A_i$,* *such that for any $C \in \mathcal{I}$ with the property $C \subsetneqq A_i$, for all $i \in I$, either $B \subseteqq C$ or $C \subseteqq B$ holds, and such that*

$$\pi^{\#} = \{B\} \cup \{A_i \mid i \in I \setminus J\}.$$

Proof. As $\pi \neq \nabla$, $\mid I \mid \geq 2$. If π is a dual atom of $\mathcal{D}(V, \mathcal{I})$, then take $B = V$ and $J = I$. Then $V = \bigcup_{i \in I} A_i$, and our assertion is satisfied for any $C \in \mathcal{I}$. It is also clear that $\pi^{\#} = \nabla = \{V\}$.

Assume $\pi^{\#} \neq \nabla$. As $\pi < \pi^{\#}$, there is a set $J \subsetneqq I$ with $\mid J \mid \geq 2$, such that $B = \bigcup_{i \in J} A_i \in \mathcal{I}$ and $\pi^{\#} = \{B\} \cup \{A_i \mid i \in I \setminus J\}$. Take any $C \in \mathcal{I} \setminus \{\varnothing\}$ and set $K = \{i \in I \mid A_i \cap C \neq \varnothing\}$. Clearly, $K \neq \varnothing$ and $V = \bigcup_{i \in I} A_i$ implies $C = \bigcup i \in I (A_i \cap C) = \bigcup_{i \in K} (A_i \cap C) \subseteqq \bigcup_{i \in K} A_i$.

In order to prove that B is a strong interval, suppose $C \subsetneqq B$. Then $K \subsetneqq J$. If $C \subseteqq A_{i_0}$, for some $i_0 \in I$, then $A_{i_0} \subsetneqq B$, whence $C \cap B = \varnothing$. If $K = \{i_0\}$, for some $i_0 \in I$, then $C = A_{i_0} \cap C$ implies $C \subseteqq A_{i_0}$ and hence $C \cap B = \varnothing$. Now, assume $C \subsetneqq A_i$, for all $i \in I$. Then the previous argument gives $\mid K \mid \geq 2$. We prove $B \subseteqq C$.

Let us prove first $J \subseteqq K$. Indeed, $\{A_i \mid i \in K\} \cup \{C\}$ being a connected family of intervals, Proposition 1.6 implies that

$$E = \left(\bigcup_{i \in K} A_i \right) \cup C = \bigcup_{i \in K} A_i$$

is an interval and $\nu_E = \{E\} \cup \{A_i \mid i \in I \setminus K\}$ is a decomposition. Clearly, $\nu_E > \pi$. Then $\nu_E \geq \pi^{\#}$ implies $B \subseteqq E$, whence we get $J \subseteqq K$.

Further, we prove $A_i \subseteqq C$, for all $i \in J$.

By the way of contradiction, assume that $A_{j_0} \subsetneqq C$, for some $j_0 \in J$. Then $\{A_i \mid i \in K \setminus \{j_0\}\} \cup \{C\}$ being also a connected family, we get

$$D = \left(\bigcup_{i \in K \setminus \{j_0\}} A_i \right) \cup C \in \mathcal{I}.$$

As $j_0 \in K$, we have $A_{j_0} \cap D \supseteqq A_{j_0} \cap C \neq \varnothing$. Since $D \subsetneqq A_{j_0}$, $A_{j_0} \subsetneqq D$, the set $F = D \setminus A_{j_0}$ is an interval. Now $F = (D \cup A_{j_0}) \setminus A_{j_0}$ gives

$$F = \left[\left(\bigcup_{i\in K} A_i\right) \cup C\right] \setminus A_{j_0} = \left(\bigcup_{i\in K} A_i\right) \setminus A_{j_0} = \bigcup_{i\in K\setminus\{j_0\}} A_i.$$

As $K \supsetneqq J$ and $J \subseteqq K$, we get $\mid K \mid > \mid J \mid \geq 2$, and hence $K \setminus \{j_0\}$ has at least two elements. Therefore, $\nu_F = \{F\} \cup \{A_{j_0}\} \cup \{A_i \mid i \in I \setminus K\}$ is a decomposition and $\nu_F > \pi$. Hence $\nu_F \geq \pi^{\#}$, and this implies that $B = \bigcup_{i\in J} A_i$ is contained in one of the blocks of ν_F. As $A_{j_0} \subseteqq B$ and $A_{j_0} \neq B$, this is a contradiction. Thus we have $A_i \subseteqq C$, for all $i \in J$. This result implies $B \subseteqq C$, proving that B is a strong interval.

Finally, observe that the above argument also gives that for any $C \in \mathcal{I}$ with $C \nsubseteqq A_i$, for all $i \in I$ either $C \subseteqq B$ or $B \subseteqq C$ is satisfied. ■

Corollary 3.9. *Let $(V, \mathcal{I})$ be an interval system with $\mid V \mid \geq 2$. Then the lattice $\mathcal{D}(V, \mathcal{I})$ is dually atomistic if and only if $(V, \mathcal{I})$ has no proper strong intervals.*

Proof. The "only if part" was proved by Corollary 3.7(i). To prove the "if part" of our assertion, assume that $(V, \mathcal{I})$ has no proper strong intervals. Now let π be a completely meet-irreducible element of $\mathcal{D}(V, \mathcal{I})$. Then, according to Theorem 3.8, there exists a strong interval $B \in \pi^{\#}$ with$\mid B \mid \geq 2$. As our assumption gives $B = V$, we get $\pi^{\#} = \nabla$, and hence π is a dual atom of $\mathcal{D}(V, \mathcal{I})$. As $\mathcal{D}(V, \mathcal{I})$ is an algebraic lattice, any element of it is a meet of its completely-meet irreducible elements, that is, of its dual atoms. Thus $\mathcal{D}(V, \mathcal{I})$ is dually atomistic. ■

4. Fragile and nonfragile intervals

Let $(V, \mathcal{Q})$ be a closure system. A set $A \in \mathcal{Q}$ is called *fragile* if it is the union of two disjoint nonempty members of $\mathcal{Q}$, otherwise A is called *nonfragile.* Also, $(V, \mathcal{Q})$ is called a *fragile (nonfragile) closure system* if the set V is fragile (nonfragile). This generalizes the concept of fragility studied by Habib and Maurer [16] in the context of the module systems of graphs. If $(V, \mathcal{Q})$ is an interval system, then its fragile and nonfragile intervals have remarkable properties. Two sets B, C are said to be *comparable* if either $B \subseteqq C$ or $C \subseteqq B$ holds.

Proposition 4.1. *Let $(V, \mathcal{I})$ be an interval system. Then:*

(i) *any nonfragile interval $A \in \mathcal{I}$ is a strong interval;*

(ii) *if B, C are noncomparable intervals with $B \cap C \neq \varnothing$, then $B \cup C$ is a fragile interval.*

Proof. Ad (i): Assume that $A \cap B \neq \varnothing$, for some $B \in \mathcal{I}$. If neither $A \subseteqq B$ nor $B \subseteqq A$ would be satisfied, then $A \setminus B \in \mathcal{I} \setminus \{\varnothing\}$, $A \cap B \neq \varnothing$ and $A = (A \setminus B) \cup (A \cap B)$ would imply that A is fragile – a contradiction. Thus either $A \subseteqq B$ or $B \subseteqq A$ holds, proving that A is a strong interval.

Ad (ii): As $B, C \in \mathcal{I}$ and $B \cap C \neq \varnothing$, we have $B \cup C \in \mathcal{I}$. Since $B \nsubseteqq C$ and $C \nsubseteqq B$, the sets $C \setminus B$ and B are nonempty disjoint intervals and hence $B \cup C = B \cup (C \setminus B)$ implies that $B \cup C$ is fragile. ∎

Proposition 4.2. *Let $(V, \mathcal{I})$ be a finite nonfragile interval system with $\mid V \mid \geq 2$. Then:*

(i) *$(V, \mathcal{I})$ has a greatest proper decomposition $\mathcal{M}$, and $\mathcal{M}$ is a strong decomposition;*

(ii) *the maximal proper fragile intervals are pairwise disjoint strong intervals.*

Proof. Ad (i): Denote by $\mathcal{M}$ the set of all maximal proper intervals of V. As for any $a \in A$, $\{a\} \in \mathcal{I}$ is included in some $M \in \mathcal{M}$, the members of $\mathcal{M}$ form a covering of V. In order to prove that $\mathcal{M}$ is a decomposition, suppose $M_1 \cap M_2 \neq \varnothing$, for some $M_1, M_2 \in \mathcal{M}$, $M_1 \neq M_2$. Then $M_1 \cup M_2 \in \mathcal{I}$, and since neither $M_1 \subseteqq M_2$ nor $M_2 \subseteqq M_1$ is satisfied, we get $M_1 \subset M_1 \cup M_2$. Since M_1 is a maximal element in $\mathcal{I} \setminus \{V\}$, we obtain $M_1 \cup M_2 = V$. As M_1 and M_2 are noncomparable intervals and $M_1 \cap M_2 \neq \varnothing$, we get that $V = M_1 \cup M_2$ is fragile, contrary to our assumption. Thus all the members of $\mathcal{M}$ are pairwise disjoint (and nonempty) and hence $\mathcal{M}$ is an interval decomposition.

Now let $\pi = \{A_i \mid i \in I\} \neq \nabla$ be an arbitrary decomposition in $(V, \mathcal{I})$. As any A_i is strictly included in V, there exists an $M^{(i)} \in \mathcal{M}$ with $A_i \subseteqq M^{(i)}$. Hence $\pi \leq \mu$, i.e. $\mathcal{M}$ is the greatest element of $\mathcal{D}(V, \mathcal{I}) \setminus \{\nabla\}$. Now, Proposition 3.1 implies that $\mathcal{M}$ is a strong decomposition.

Ad (ii): Let I be a maximal proper fragile interval and take a $B \in \mathcal{I}$ with $I \cap B \neq \varnothing$. If I and B would be noncomparable, then $I \cup B$ would be also a fragile interval with $I \subset I \cup B$, and hence we would get $V = I \cup B$, i.e., V would be fragile, contrary to our assumption. Hence either $I \subseteqq B$ or $B \subseteqq I$ holds, and this proves that I is a strong interval. If B is also a maximal proper fragile interval, then both $I \subseteqq B$ and $B \subseteqq I$ give $I = B$. Therefore, $I \neq B$ implies $I \cap B = \varnothing$. ■

Proposition 4.3. *Any finite interval system $(V, \mathcal{I})$ contains a greatest decomposition with the property that any block of it is either fragile or a singleton.*

Proof. For a fragile V, $\nabla = \{V\}$ is the greatest decomposition with the above property. If V is nonfragile, then let $\mathcal{F}$ denote the set of maximal proper fragile intervals of $(V, \mathcal{I})$ and F their union. As the members of $\mathcal{F}$ are disjoint, $\mu_F = \mathcal{F} \cup \{\{x\} \mid x \in V \setminus F\}$ is a decomposition with the required property. It is also clear that any decomposition $\mu = \{A_i \mid i \in I\}$, whose blocks A_i are either fragile or singletons, satisfies $\mu \leq \mu_F$. ■

We note that a particular case of Proposition 4.2(i), where $(V, \mathcal{I})$ is the module system of a graph $G = (V, E)$, is implicitly contained in [16].

A nonempty interval $A \in \mathcal{I}$ is called *indecomposable*, if in each disjoint union $A = \bigcup\{B_i \mid i \in I\}$, with $B_i \in \mathcal{I} \setminus \{\varnothing, A\}$ all the intervals B_i, $i \in I$ are singletons. Obviously, any indecomposable interval with at least three elements is nonfragile, and hence it is strong. If A is a fragile or indecomposable interval, then we call it a *good interval.* We call a decomposition $\pi = \{A_i \mid i \in I\}$ a *good decomposition*, if each one of its blocks $A_i, i \in I$ is a good interval. Clearly, $\triangle$ is a good decomposition in any interval system.

Lemma 4.4. *If $\{A_i \mid 1 \leq i \leq n\}$ is a finite connected set of good intervals in $(V, \mathcal{I})$, then $B = \bigcup_{i=1}^{n} A_i$ is also a good interval. If $n \geq 2$ and $B \neq A_i$, for each $1 \leq i \leq n$, then B is fragile.*

Proof. To prove the first assertion, we use induction on $n \in \mathbb{N}$, $n \geq 1$: For $n = 1$ the assertion is obvious. Now, assume that our assertion is true for $n - 1$. Then $C = \bigcup_{i=1}^{n-1} A_i$ is also a good interval. As $\{A_i \mid 1 \leq i \leq n\}$ is a connected family, $C \cap A_n \neq \varnothing$. If A_n and C are comparable, then $C \cup A_n$ is clearly a good interval. If A_n and B are noncomparable, then, using Proposition 4.1(ii), we obtain that $B = \bigcup_{i=1}^{n} A_i = C \cup A_n$ is fragile, therefore it is a good interval. Using induction on $n \in \mathbb{N}$, $n \geq 2$, the same argument implies our second assertion. ■

Proposition 4.5. *If $(V, \mathcal{I})$ is a finite interval system, then the join $\bigvee_{i\in I} \pi_i$ of any good decompositions $\pi_i \in \mathcal{D}(V, \mathcal{I})$, $i \in I$, is also a good decomposition.*

Proof. Let $\rho_i \subseteqq V \times V$, $i \in I$, be the equivalences induced by π_i, $i \in I$. As $\bigvee_{i\in I} \rho_i$ is induced by $\bigvee_{i\in I} \pi_i$, it is sufficient to show that for any $a \in V$, $(\bigvee_{i\in I} \rho_i)[a]$ is a good interval. Denote $K = (\bigvee_{i\in I} \rho_i)[a]$. Using the same argument as in the proof of Proposition 2.1, we get that$\{\rho_i[x] \mid x \in K, i \in I\}$ is a connected family and $K = \bigcup\{\rho_i[x] \mid x \in K, i \in I\}$. Since $(V, \mathcal{I})$ is finite, the family $\{\rho_i[x] \mid x \in K, i \in I\}$ contains only a finite number of different equivalence classes $\rho_i[x]$. As any class $\rho_i[x]$ is a good interval, we obtain that $K = (\bigvee_{i\in I} \rho_i)[a]$ is also a good interval. ∎

As an immediate consequence, we obtain a result which extends of Theorem 2 in [16].

Theorem 4.6. *Any finite interval system $(V, \mathcal{I})$ has a greatest good decomposition.*

Proof. Let $\pi_k \in \mathcal{D}(V, \mathcal{I})$, $k \in K$ denote the good decompositions in $(V, \mathcal{I})$. Since $(V, \mathcal{I})$ is finite, $\mathcal{D}(V, \mathcal{I})$ and K are finite too. Hence, in view of Proposition 4.5, $\bigvee_{k\in K} \pi_k$ is also a good decomposition and clearly, it is the greatest element of the set $\{\pi_k \mid k \in K\}$. ∎

Remark 4.7. $\triangle = \{\{x\} \mid x \in V\}$ being a good decomposition, Proposition 4.5 also implies that the good decompositions of a finite interval system $(V, \mathcal{I})$ form a lattice, however in general this is not a sublattice of $\mathcal{D}(V, \mathcal{I})$.

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