

## DISTRIBUTIVE LATTICES WITH A GIVEN SKELETON

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### Abstract

We present a construction of finite distributive lattices with a given skeleton. In the case of an  $H$ -irreducible skeleton  $\mathcal{K}$  the construction provides all finite distributive lattices based on  $\mathcal{K}$ , in particular the minimal one.

**Keywords:** distributive lattice, skeleton, gluing, tolerance relation, skeleton tolerance,  $\mathcal{K}$ -atlas,  $H$ -irreducibility.

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Decomposition of a lattice into a system of overlapping intervals indexed by another lattice has been a powerful tool for structural analysis of lattices. This tool was provided by Herrmann in [10] as a generalization of the original Hall-Dilworth gluing of a filter and an ideal (see [9]). This was further extended in one direction by means of tolerance relations (see [5] and [1]) and in the other, which was widely applied in the concept analysis, by the notion of  $\mathcal{K}$ -atlases (see [12] and [6]).

In this paper we want to reverse the operation of decomposition into the operation of reconstruction of all distributive lattices based on the same skeleton. Some tools for that reconstruction were provided by Herrmann (in [10]) and Day and Herrmann in [5]. There were also some exemplary constructions in [10], [12] and [2]. Our aim is to generalize these ideas and describe a construction which may lead to a minimal distributive lattice with a given skeleton.

Let us start with recalling some basic notions and theorems and establishing the notation.

A *tolerance relation*  $\Theta$  on a lattice  $\mathcal{L}$  is a reflexive and symmetric binary relation compatible with lattice operations, i.e.  $x_1\Theta x_2$  and  $y_1\Theta y_2$  imply

$(x_1 \wedge y_1) \Theta (x_2 \wedge y_2)$  and  $(x_1 \vee y_1) \Theta (x_2 \vee y_2)$  for every elements  $x_1, x_2, y_1, y_2$  from  $L$ . Thus, the transitive closure of a tolerance relation is a lattice congruence.

It was proved by Chajda and Zelinka in [3] that if  $\Theta$  is a tolerance relation on a lattice  $\mathcal{L}$  and  $a \Theta b$  for some  $a, b \in L$  then  $x \Theta y$  for every  $x, y \in [a \wedge b, a \vee b]$ . It means that every tolerance relation  $\Theta$  on a lattice is uniquely determined by the set of quotients contained in  $\Theta$ .

A block of a tolerance relation  $\Theta$  on a lattice  $\mathcal{L}$  is a maximal subset  $A \subseteq L^2$  such that  $x \Theta y$  for every  $x, y \in A$ . Let us denote by  $\mathcal{L}/\Theta$  the set of all blocks of  $\Theta$ . We can introduce on  $\mathcal{L}/\Theta$  a partial order by

$$A \leq B \iff 0_A \leq 0_B$$

(which is equivalent to the fact that  $1_A \leq 1_B$ ), where  $A = [0_A, 1_A]$ ,  $B = [0_B, 1_B]$  are blocks of  $\Theta$ .

It was proved by Czedli in [4] that  $\mathcal{L}/\Theta$  with the partial order described above is a lattice, called the *factor lattice* of  $\mathcal{L}$  by  $\Theta$ .

A tolerance relation  $\Theta$  on  $\mathcal{L}$  is said to be *glued* iff  $A \prec B$  implies  $A \cap B \neq \emptyset$  for every blocks  $A, B \in \mathcal{L}/\Theta$ .

The smallest glued tolerance relation on  $\mathcal{L}$  will be denoted by  $\Sigma(\mathcal{L})$ . It can be proved that  $\Sigma(\mathcal{L})$  is generated by the set of all *prime quotients* of the lattice  $\mathcal{L}$ .

The tolerance relation  $\Sigma = \Sigma(\mathcal{L})$  is called the *skeleton tolerance* of  $\mathcal{L}$  and the factor lattice of  $\mathcal{L}$  by  $\Sigma(\mathcal{L})$ , which will be denoted by  $S(\mathcal{L})$ , is said to be the *skeleton* of the lattice  $\mathcal{L}$ .

In our further considerations the following simple observation will be useful:

**Lemma 1.** *Let  $\Sigma$  be the skeleton tolerance of a finite lattice  $\mathcal{K}$ . If  $K_i = [0_i, 1_i]$  is a block of  $\Sigma$ , then  $0_i = p_i(x_1, \dots, x_n)$  and  $1_i = p_i(y_1, \dots, y_n)$  for some lattice polynomial  $p_i$  and a system  $x_1, \dots, x_n, y_1, \dots, y_n$  of elements of  $\mathcal{K}$  such that  $x_j \preceq y_j$  for every  $j = 1, \dots, n$*

**Proof.** Since the relation  $\Sigma$  is generated by the set of all prime quotients of  $\mathcal{K}^2$  and  $0_i \Sigma 1_i$  for every block  $K_i$ , there is a lattice polynomial  $p_i$  such that  $0_i = p_i(x_1, \dots, x_n)$  and  $1_i = p_i(y_1, \dots, y_n)$  for some  $x_1, \dots, x_n, y_1, \dots, y_n$  such that  $x_j \preceq y_j$  or  $y_j \preceq x_j$  for  $j = 1, \dots, n$ .

However, if  $y_j \prec x_j$ , then

$$1_i = p_i(y_1, \dots, y_n) \leq p_i(y_1, \dots, y_{j-1}, x_j, y_{j+1}, \dots, y_n).$$

Since  $1_i$  is the maximal element  $x$  of  $\mathcal{K}$  such that  $0_i \Sigma x$ , we have

$$1_i = p_i(y_1, \dots, y_{j-1}, x_j, y_{j+1}, \dots, y_n)$$

and hence we can replace in the system  $x_1, \dots, x_n, y_1, \dots, y_n$  the element  $y_j$  by  $x_j$ . Thus, we can conclude that  $0_i = p(x_1, \dots, x_n)$  and  $1_i = p(y_1, \dots, y_n)$ , where  $x_j \preceq y_j$  for  $j = 1, \dots, n$ . ■

A lattice  $\mathcal{K}$  is said to be *H-irreducible* iff its skeleton is the trivial lattice. It can be proved that a finite modular lattice is *H-irreducible* iff it is atomistic. Similarly, a finite distributive lattice is *H-irreducible* iff it is boolean.

As it was shown in the beginning, a glued tolerance relation on a finite lattice provides a decomposition of the lattice into blocks, which are intervals of the lattice and which themselves form a lattice called the factor lattice. It is convenient to describe the construction in abstract terms as a *K-atlas* with overlapping neighborhood.

Let  $(\mathcal{L}_x)_{x \in K}$  be a family of finite lattices and let the index set  $K$  be also a finite lattice. We call the family  $(\mathcal{L}_x)_{x \in K}$  a *K-atlas* with overlapping neighborhood if the following conditions hold for every  $x, y \in K$ :

1. If  $L_x \subseteq L_y$ , then  $x = y$ .
2. If  $x \prec y$ , then  $L_x \cap L_y \neq \emptyset$ .
3. If  $x \leq y$  and  $L_x \cap L_y \neq \emptyset$ , then the orders of  $\mathcal{L}_x$  and  $\mathcal{L}_y$  coincide on the intersection  $L_x \cap L_y$  and the interval  $L_x \cap L_y$  is at the same time a filter of  $\mathcal{L}_x$  and an ideal of  $\mathcal{L}_y$ .
4.  $L_x \cap L_y = L_{x \wedge y} \cap L_{x \vee y}$ .

The structure  $\mathcal{L} = \langle \bigcup_{x \in K} L_x, \leq \rangle$ , where  $\leq$  is the transitive closure of the union of partial orders of the lattices  $\mathcal{L}_x$  for  $x \in K$ , is called the *sum of K-atlas with overlapping neighborhood* (or simply a *K-gluing of the family*  $(\mathcal{L}_x)_{x \in K}$ ). It was proved by Day and Herrmann that:

**Theorem 2** (see [5]). *The sum of a K-atlas with overlapping neighborhood is a lattice  $\mathcal{L}$  for which the lattices  $\mathcal{L}_x$ , where  $x \in K$ , are the blocks of some glued tolerance relation  $\Theta$  and the mapping  $x \mapsto L_x$  is an isomorphism of  $K$  onto the factor lattice  $\mathcal{L}/\Theta$ .*

*Conversely, if  $\Theta$  is a glued tolerance relation on a lattice  $\mathcal{L}$ , then the blocks of the relation  $\Theta$  together with the factor lattice  $K = \mathcal{L}/\Theta$  form a K-atlas with overlapping neighborhood which K-gluing is the lattice  $\mathcal{L}$ .* ■

If  $(\mathcal{L}_x)_{x \in K}$  is a  $\mathcal{K}$ -atlas with overlapping neighborhood, then the lattices  $\mathcal{L}_x$ , being the blocks of a tolerance relation on  $L = \bigcup_{x \in K} L_x$ , are intervals of  $\mathcal{L}$ . We shall write  $L_x = [0_x, 1_x]$ . It is quite natural to consider two mappings  $\sigma, \pi : K \rightarrow L$  such that  $\sigma(x) = 0_x$  and  $\pi(x) = 1_x$ .

**Lemma 3** (see [10]). *The mappings  $\sigma$  and  $\pi$  are (strictly) monotone,  $\sigma$  is join-preserving and  $\pi$  is meet-preserving. ■*

Thus, the boundaries of the blocks fulfill the following conditions for every  $x, y \in K$ :

- (1)  $0_{x \vee y} = 0_x \vee 0_y;$
- (2)  $0_{x \wedge y} \leq 0_x \wedge 0_y;$
- (3)  $1_x \vee 1_y \leq 1_{x \vee y};$
- (4)  $1_{x \wedge y} = 1_x \wedge 1_y.$

What is more, we can prove that:

**Theorem 4.** *Let  $(\mathcal{L}_x)_{x \in K}$  be a  $\mathcal{K}$ -atlas with overlapping neighborhood, where  $\mathcal{K}$  and  $\mathcal{L}_x$ , for every  $x \in K$ , are finite lattices. Let  $\Sigma$  be the skeleton tolerance on  $\mathcal{K}$ . Then, for every elements  $x, y \in K$  such that  $x \leq y$  and  $x, y$  belong to the same block of  $\Sigma$  we have  $L_x \cap L_y \neq \emptyset$ .*

**Proof.** Let  $x \leq y$  and  $x, y \in K_i = [0_i, 1_i]$ , where  $K_i$  is a block of  $\Sigma$ .

Let  $p$  be a lattice polynomial and let us denote by  $l(p)$  the length of  $p$ , i.e. the number of lattice operation appearing in it.

We shall prove by the induction on the length of the polynomial  $p$  that for every  $x_1, \dots, x_n, y_1, \dots, y_n \in K$  such that  $x_j \preceq y_j$  for  $j = 1, \dots, n$  we have

$$(5) \quad 0_{p(y_1, \dots, y_n)} \leq 1_{p(x_1, \dots, x_n)}.$$

If  $l(p) = 0$ , then  $0_{y_1} \leq 1_{x_1}$ , by the definition of  $\mathcal{K}$ -atlas. Let us suppose that (5) holds for all polynomials shorter than  $n$  and let  $l(p) = n > 1$ . Then  $p = q \vee r$  or  $p = q \wedge r$ , where the polynomials  $q$  and  $r$  fulfill the inductive assumption.

In the first case

$$0_{p(y_1, \dots, y_n)} = 0_{q(y_1, \dots, y_n)} \vee 0_{r(y_1, \dots, y_n)} \leq 1_{q(x_1, \dots, x_n)} \vee 1_{r(x_1, \dots, x_n)} \leq 1_{p(x_1, \dots, x_n)}$$

and in the second case

$$0_{p(y_1, \dots, y_n)} \leq 0_{q(y_1, \dots, y_n)} \wedge 0_{r(y_1, \dots, y_n)} \leq 1_{q(x_1, \dots, x_n)} \wedge 1_{r(x_1, \dots, x_n)} = 1_{p(x_1, \dots, x_n)},$$

which proves (5). By Lemma 1, there is a polynomial  $p_i$  and the system  $x_1, \dots, x_n, y_1, \dots, y_n \in K$  such that  $0_i = p_i(x_1, \dots, x_n)$ ,  $1_i = p_i(y_1, \dots, y_n)$  and  $x_j \preceq y_j$  for  $j = 1, \dots, n$ . Thus, by (5),  $0_{1_i} \leq 1_{0_i}$  and hence

$$L_{0_i} \cap L_{1_i} \neq \emptyset.$$

Now, by the assumption,

$$0_i \leq x \leq y \leq 1_i.$$

Therefore,

$$0_{0_i} \leq 0_x \leq 0_y \leq 0_{1_i} \leq 1_{0_i} \leq 1_x \leq 1_y \leq 1_{1_i},$$

so  $L_x \cap L_y \neq \emptyset$ . ■

Applying Lemma 3 to the reducts of the mappings  $\sigma$  and  $\pi$  to the blocks of the skeleton tolerance  $\Sigma(\mathcal{K})$  and combining it with Theorem 4, one can notice that:

**Corollary 5.** *If  $(\mathcal{L}_x)_{x \in K}$  is a  $\mathcal{K}$ -atlas with overlapping neighborhood, where  $\mathcal{K}$  and  $\mathcal{L}_x$  for every  $x \in K$  are finite lattices and  $K_i = [0_i, 1_i]$  for  $i \in S(\mathcal{K})$  are the blocks of the skeleton tolerance  $\Sigma(\mathcal{K})$ , then  $x \mapsto 0_x$  and  $x \mapsto 1_x$  are, respectively, the join-embedding of  $\mathcal{K}_i$  into  $\mathcal{L}_{0_i}$  and meet-embedding of  $\mathcal{K}_i$  into  $\mathcal{L}_{1_i}$ .* ■

Herrmann proved in [10] that if  $(\mathcal{M}_x)_{x \in K}$  is a  $\mathcal{K}$ -atlas with overlapping neighborhood and every  $\mathcal{M}_x$  is a finite modular lattice for every  $x \in K$ , then the  $\mathcal{K}$ -gluing  $\mathcal{M}$  of the atlas is a modular lattice, as well. Similarly, if every lattice  $\mathcal{M}_x$  for  $x \in K$  is a distributive one, then the  $\mathcal{K}$ -gluing  $\mathcal{M}$  is also distributive. Moreover, there is a dependence between the skeleton tolerance of a modular lattice and its maximal atomistic intervals.

**Theorem 6** (see [11]). *Let  $\mathcal{M}$  be a finite modular lattice,  $S(\mathcal{M}) = \mathcal{M}/\Sigma$ , where  $\Sigma$  is the skeleton tolerance on  $\mathcal{M}$ . Then the blocks  $(M_x)_{x \in S(\mathcal{M})}$  of  $\Sigma$  are the maximal atomistic intervals of  $\mathcal{M}$ .* ■

Thus, let us observe that every finite modular lattice can be decomposed into its maximal atomistic intervals, and then glued from them again according to the skeleton of the lattice, which can be regarded as the pattern of this gluing.

In particular, for every modular lattice  $\mathcal{M}$ , the filter  $\nabla(\mathcal{M})$  generated by all coatoms of  $\mathcal{M}$  and the ideal  $\Delta(\mathcal{M})$  generated by all atoms of  $\mathcal{M}$  are the maximal atomistic intervals of  $\mathcal{M}$ , so they have to appear in the  $S(\mathcal{M})$ -gluing as the "top" and "bottom" intervals.

In the case of distributive lattices, atomistic intervals coincide with boolean ones.

Let  $\mathcal{K}$  be a finite lattice and  $\{\mathcal{L}_x\}_{x \in K}$  be a family of finite lattices. To construct a family  $\{\mathcal{L}'_x\}_{x \in K}$  of lattices forming a  $\mathcal{K}$ -atlas with overlapping neighborhood such that  $\mathcal{L}_x$  is isomorphic to  $\mathcal{L}'_x$  for every  $x \in K$  we need a family of lattice isomorphisms which identify the parts of lattices  $\mathcal{L}_x$  that counterpart to overlapping parts of lattices  $\mathcal{L}'_x$  in the  $\mathcal{K}$ -atlas. In fact, it is enough to construct the family of isomorphisms  $\phi_{yx}$  identifying, for each  $x \leq y$  in  $\mathcal{K}$  a filter of the lattice  $\mathcal{L}_x$  with an ideal of the lattice  $\mathcal{L}_y$  unless  $\mathcal{L}'_x \cap \mathcal{L}'_y$  is empty.

Strictly speaking, it was proved in [5] that if there is given a finite lattice  $\mathcal{K}$  and a family  $\{\mathcal{L}_x\}_{x \in K}$  of disjoint finite lattices together with a family of lattice isomorphisms  $\phi_{yx}$  for every  $x \leq y$  in  $\mathcal{K}$  satisfying the following conditions:

(6) every  $\phi_{yx}$  maps a (possibly empty) filter  $F_{y,x}$

of the lattice  $\mathcal{L}_x$  onto a (possibly empty) ideal  $I_{x,y}$  of the lattice  $\mathcal{L}_y$ ;

(7)  $\phi_{xx} = id_{\mathcal{L}_x}$ , for every  $x \in K$ ;

(8) if  $x \leq y \leq z$  in  $\mathcal{K}$ , then  $\phi_{yz} \circ \phi_{xy} = \phi_{xz}$ ;

(9)  $F_{x,x \wedge y} \cap F_{y,x \wedge y} \subseteq F_{x \vee y, x \wedge y}$ , for all  $x, y \in K$ ;

(10)  $I_{x,x \vee y} \cap I_{y,x \vee y} \subseteq I_{x \wedge y, x \vee y}$ , for all  $x, y \in K$ ;

(11) for every  $x \leq y$  in  $\mathcal{K}$  there exists a sequence

$x = z_0 \leq \dots \leq z_n = y$  such that  $F_{z_{i+1}, z_i} \neq \emptyset$  for each  $i = 0, \dots, n-1$ ,

then there is a  $\mathcal{K}$ -atlas  $\{\mathcal{L}'_x\}_{x \in K}$  such that  $\mathcal{L}'_x$  is isomorphic to  $\mathcal{L}_x$  for every  $x \in K$ .

Moreover, a relation  $\sim$  given by

$$a \sim b \iff \phi_{(x \vee y)x}(a) = \phi_{(x \vee y)y}(b)$$

for  $a \in L_x$  and  $b \in L_y$  is an equivalence relation on  $\bigcup_{x \in K} L_x$  and  $(\bigcup_{x \in K} L_x)/\sim$  is a lattice isomorphic to the  $\mathcal{K}$ -gluing  $\mathcal{L}$  of the  $\mathcal{K}$ -atlas with overlapping neighborhood  $\{\mathcal{L}'_x\}_{x \in K}$ .

The above construction can be regarded as an alternative and more general definition of the  $\mathcal{K}$ -atlas with overlapping neighborhood and the  $\mathcal{K}$ -sum of the  $\mathcal{K}$ -atlas.

In particular, since every finite boolean lattice is, up to isomorphism, uniquely given by its dimension (which can be understood, for example, as the number of the atoms of the boolean lattice), we can describe a family of boolean lattices  $\{\mathcal{B}_x\}_{x \in K}$  indexed by elements of a lattice  $\mathcal{K}$  by the family  $W = \{n_x\}_{x \in K}$  of natural numbers such that  $\dim B_x = n_x$  for every  $x \in K$ .

Let  $\mathcal{K}$  be a finite lattice. The question is for which families  $\{n_x\}_{x \in K}$  there exist distributive lattices with the skeleton  $\mathcal{K}$  and the maximal boolean intervals of given dimensions.

Herrmann proved in [10]:

**Theorem 7.** *Every finite lattice is a skeleton of a finite distributive lattice.* ■

It means that for every finite lattice  $\mathcal{K}$  there is at least one family of numbers described above. In fact, there are infinitely many such families as there are infinitely many non-isomorphic distributive lattices with the given skeleton  $\mathcal{K}$ .

Let us consider the set  $\mathcal{W}(\mathcal{K})$  of all families  $W = \{n_x\}_{x \in K}$  such that there exists a finite distributive lattice  $\mathcal{D}$  with the skeleton  $\mathcal{K}$  in which  $\dim B_x = n_x$  for every  $x \in K$ . In  $\mathcal{W}(\mathcal{K})$  we can introduce a partial order by

$$W \leq W' \text{ iff } n_x \leq n'_x \text{ for every } x \in K.$$

The distributive lattice  $\mathcal{D}$  corresponding to a minimal element of the family  $\mathcal{W}(\mathcal{K})$  will be called a minimal distributive lattice with the skeleton  $\mathcal{K}$ . The question arises how to construct a minimal distributive lattice with the skeleton  $\mathcal{K}$ .

Herrmann constructed in his proof (see [10]) one of the spectrum of distributive lattices with the given skeleton  $\mathcal{K}$ .

Let  $\mathcal{K}$  be a finite lattice. we shall denote by  $\mathcal{P}(\mathcal{K})^d$  the lattice dual to the lattice  $\mathcal{P}(\mathcal{K})$  of all subsets of  $K$  partially ordered by the inclusion. If  $x \in K$  then  $[x]$  denotes the filter of  $\mathcal{K}$  and  $(x)$  – the ideal of  $\mathcal{K}$  generated by  $x$ .

Herrmann considered a family of intervals  $L_x$  of the product lattice  $\mathcal{P}(K) \times \mathcal{P}(K)^d$  defined by

$$L_x = [(\emptyset, [x]), ((x), \emptyset)]$$

for every  $x \in K$ . He proved that the family  $\{\mathcal{L}_x\}_{x \in K}$  forms a  $\mathcal{K}$ -atlas with overlapping neighborhood. Since all  $\mathcal{L}_x$  are boolean lattices, the  $\mathcal{K}$ -gluing of the family  $\{\mathcal{L}_x\}_{x \in K}$  is a distributive lattice with the skeleton  $\mathcal{K}$ .

**Example.** Let  $\mathcal{K}$  be the two-element boolean lattice in Figure 1 (a). Then

$$L_0 = [(\emptyset, K), (\{0\}, \emptyset)],$$

$$L_1 = [(\emptyset, \{1\}), (K, \emptyset)],$$

hence  $\dim L_0 = \dim L_1 = |K| + 1 = 3$ . Moreover

$$L_0 \cap L_1 = [(\emptyset, \{1\}), (\{0\}, \emptyset)],$$

so  $\dim(L_0 \cap L_1) = 2$ .

It yields the distributive lattice  $\mathcal{D}$  shown in Figure 1 (b).

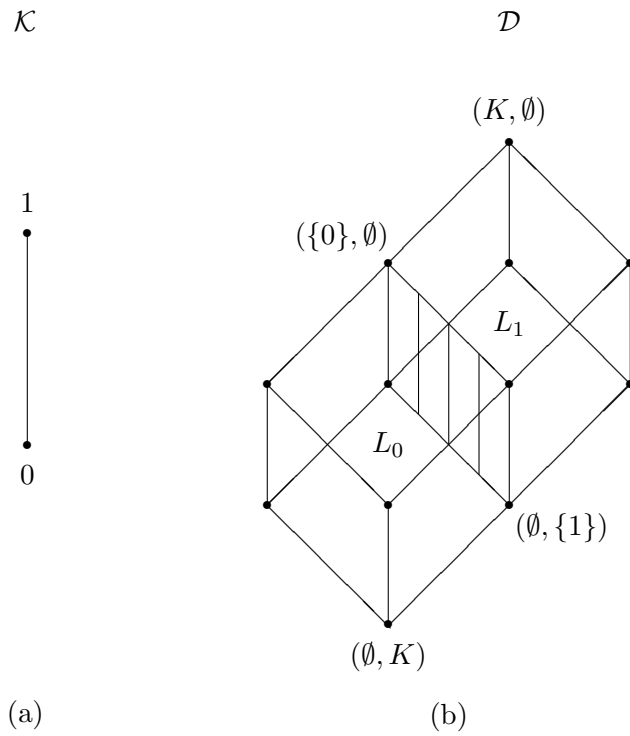


Figure 1



It is easy to notice that the lattice  $\mathcal{D}$  is not the minimal distributive lattice with the skeleton  $\mathcal{K}$ . For the minimal one, we have  $\dim B_0 = \dim B_1 = 1$ .

In general, in the Herrmann construction for every finite lattice  $\mathcal{K}$  we always obtain

$$\dim L_0 = \dim L_1 = |\mathcal{K}| + 1,$$

which means that the resulting distributive lattice  $\mathcal{D}$  is comparatively big. Moreover,

$$L_0 \cap L_1 = [(\emptyset, \{1\}), (\{0\}, \emptyset)],$$

hence  $\dim(L_0 \cap L_1) = 2$ , which means that the bottom and the top boolean cubes of the lattice  $\mathcal{D}$  overlap.

The similar situation occurs if we consider the construction provided by Wille.

Wille and Bartenschlager (in [12] and [2], resp.) dealt with distributive lattices  $F_{\mathcal{D}}(n)$  freely generated by an  $n$ -element antichain. They considered the iterated skeletons of  $F_{\mathcal{D}}(n)$  and then tried to reconstruct, step by step, the lattice itself. In their construction they applied, apart from the Herrmann gluings, the tools of the theory of concept lattices. Skipping the properties of free distributive lattices we can, in general, translate the idea into the language of abstract algebra as follows.

For every finite lattice  $\mathcal{K}$ , we consider the set  $P = \mathcal{K} \times \{0, 1\}$  with a partial order defined by

$$(a, b) > (c, d) \iff (a \not\leq c \text{ and } b > d).$$

Wille proved (see [12]) that the lattice  $\mathcal{I}(P)$  of order ideals of the poset  $P$  (where the lattice order is the inclusion) is a finite distributive lattice with the skeleton  $\mathcal{K}$ .

**Example.** Let, again,  $\mathcal{K}$  be the two-element boolean lattice. The poset  $P$  described above corresponding to  $\mathcal{K}$  is depicted in Figure 2 (b). The lattice  $\mathcal{D} = \mathcal{I}(P)$  consists of two three-dimensional boolean cubes (Figure 2 (c)) which are glued along a two-dimensional boolean lattice being a filter of the bottom cube and an ideal of the top one. For clarity, we denoted in Figure 2 (c)

$$\begin{aligned} a &= \{(1, 0)\}; \\ b &= \{(0, 0), (1, 0)\}; \\ c &= \{(1, 1), (1, 0)\}; \\ d &= \{(0, 0), (1, 0), (1, 1)\}. \end{aligned}$$

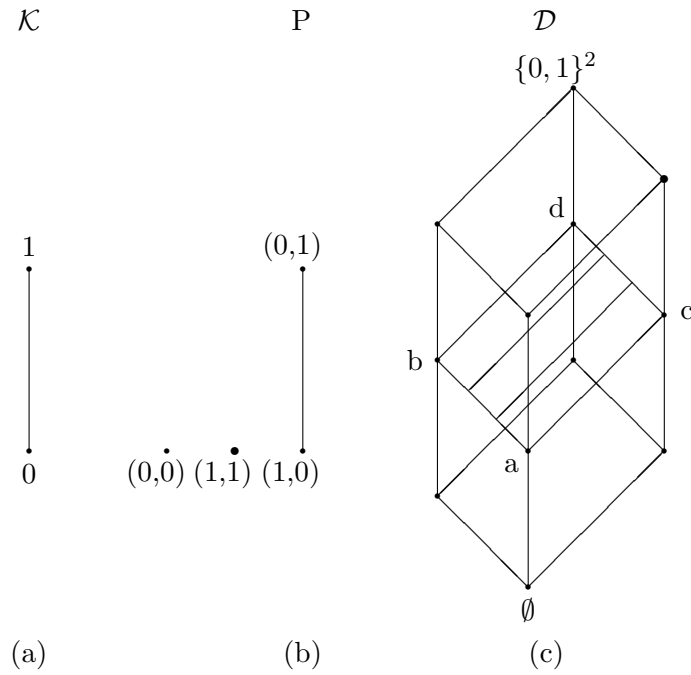


Figure 2

In general, for every finite lattice  $\mathcal{K}$ , if  $|K| = n$ , then the lattice  $\mathcal{I}(P)$  contains  $n + 1$  atoms, namely all singletons  $\{(a, 0)\}$  for  $a \in K$  and, additionally, the singleton  $\{(1, 1)\}$ . Since the bottom cube of the lattice  $\mathcal{I}(P)$  is  $\Delta(\mathcal{I}(P))$ ,

$$\dim B_0 = \dim \Delta(\mathcal{I}(P)) = n + 1.$$

Similarly, all the ideals  $P \setminus \{(a, 1)\}$ , where  $a \in K$ , together with the ideal  $P \setminus \{(0, 0)\}$  are coatoms of  $\mathcal{I}(P)$ , so

$$\dim B_1 = \dim \nabla(\mathcal{I}(P)) = n + 1.$$

Moreover, if  $x$  is the meet of all coatoms of  $\mathcal{I}(P)$ , then

$$x = \bigcup_{a \neq 0} \{(a, 0)\}$$

and  $x$  is obviously situated in  $\mathcal{I}(P)$  below the join

$$y = \bigcup_{a \in K} \{(a, 0)\} \cup \{(1, 1)\}$$

of all atoms of the lattice then the top and the bottom cubes of  $\mathcal{I}(P)$  overlap and

$$\dim(B_0 \cap B_1) = \dim[x, x \cup \{(0,0), (1,1)\}] = 2.$$

It means that the dimensions of boolean cubes of the resulting distributive lattice  $\mathcal{I}(P)$  are similar as in the case of the Herrmann construction.

We can refine the Wille construction by dropping the pairs  $(0,0)$  and  $(1,1)$  out of the set  $P$ , i.e., repeating the construction for the set  $P' = P \setminus \{(0,0), (1,1)\}$  but we still get a comparatively big lattice  $\mathcal{I}(P')$ . In that case

$$\dim B_0 = |K| - 1 = \dim B_1$$

and  $(B_0 \cap B_1) \neq \emptyset$  (in fact,  $B_0 \cap B_1$  contains exactly one element).

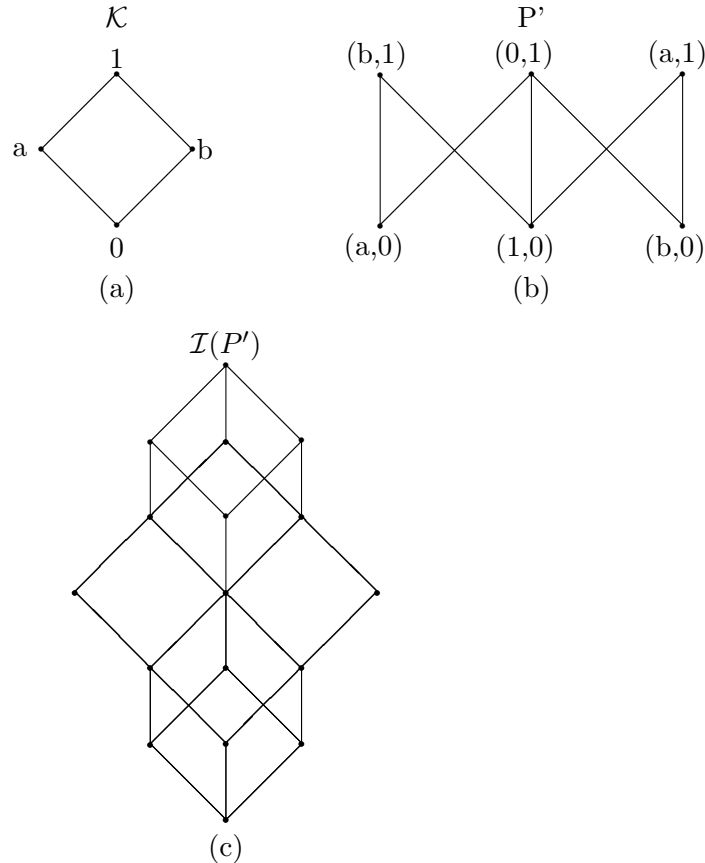


Figure 3

**Example.** If we consider as the lattice  $\mathcal{K}$  the two-dimensional boolean lattice (see Figure 3 (a)), then the corresponding set  $P'$  is shown in Figure 3 (b). The lattice  $\mathcal{I}(P')$  depicted in Figure 3 (c) has three-dimensional boolean cubes as the top and bottom ones and hence it is not the minimal distributive lattice with the skeleton  $\mathcal{K}$  (the minimal one has two-dimensional boolean cubes  $\mathcal{B}_0$  and  $\mathcal{B}_1$ ).

Now, we shall present a construction which is a generalization of, both, Herrmann and Wille constructions and which in the case of a  $H$ -irreducible lattice  $\mathcal{K}$  yields the minimal distributive lattice with the skeleton  $\mathcal{K}$ .

The idea of the construction is the following one. Take two boolean lattices  $\mathcal{B}_0$  and  $\mathcal{B}_1$  big enough to join-embed a given lattice  $\mathcal{K}$  into the first of them and meet-embed  $\mathcal{K}$  in the second one and we can "glue" the lattices  $\mathcal{B}_0$  and  $\mathcal{B}_1$  in a such way that the image of the unit of  $\mathcal{K}$  is in  $\mathcal{B}_0$  in the same distance to the unit of  $\mathcal{B}_0$  as the zero of  $\mathcal{B}_1$  to the image of the zero of  $\mathcal{K}$  in  $\mathcal{B}_1$ . Thus we can construct a  $\mathcal{K}$ -atlas  $\{\mathcal{B}_x\}_{x \in K}$ , where every  $\mathcal{B}_x$ , except from  $\mathcal{B}_0$  and  $\mathcal{B}_1$ , are just boolean lattices generated by maximal chains between the images of  $x$  in  $\mathcal{B}_0$  and  $\mathcal{B}_1$ .

If  $\mathcal{K}$  is a  $H$ -irreducible lattice, then every finite distributive lattice with the skeleton  $\mathcal{K}$  can be obtained in that way.

**Theorem 8.** *Let  $\mathcal{K}$  be a finite lattice. For every boolean lattices  $\mathcal{B}$  and  $\mathcal{C}$  together with two strictly monotone mappings  $\sigma : K \rightarrow B$  and  $\pi : K \rightarrow C$  fulfilling the conditions:*

1.  $\sigma$  is a join-homomorphism and  $\pi$  is a meet-homomorphism;
2.  $l(\sigma(1), 1_{\mathcal{B}}) = l(0_{\mathcal{C}}, \pi(0))$ ,

*there exists a distributive lattice  $\mathcal{D}$  with the skeleton  $\mathcal{K}$  and maximal boolean intervals  $\mathcal{B}_x$  such that  $\dim B_x = l(\sigma(x), 1_{\mathcal{B}}) + l(\pi(0), \pi(x))$ .*

**Proof.** Let us suppose that lattices  $\mathcal{K}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$  together with mappings  $\sigma$  and  $\pi$  fulfill the assumptions of the theorem. We can assume that  $B$  and  $C$  are disjoint.

Our goal is to construct a  $\mathcal{K}$ -atlas with overlapping neighborhood  $\{\mathcal{B}_x\}_{x \in K}$  where  $\mathcal{B}_x$  are boolean lattices such that  $\dim B_x = l(\sigma(x), 1_{\mathcal{B}}) + l(\pi(0), \pi(x))$ .

Let  $\mathcal{B}'_0$  and  $\mathcal{B}'_1$  be isomorphic copies of the intervals  $A_0 = [\sigma(0), 1_{\mathcal{B}}] \subseteq B$  and  $A_1 = [0_{\mathcal{C}}, \pi(1)] \subseteq C$ , respectively, such that  $B'_0 \cap B'_1$  is a filter of  $\mathcal{B}'_0$

and an ideal of  $\mathcal{B}'_1$  and, moreover,

$$(12) \quad \dim(B'_0 \cap B'_1) = l(\sigma(1), 1_{\mathcal{B}}).$$

Then  $A_0$  is a filter of  $\mathcal{B}$  and  $A_1$  is an ideal of  $\mathcal{C}$ . Therefore,  $\mathcal{A}_0$  and  $\mathcal{A}_1$  are boolean lattices and hence  $\mathcal{B}'_0$  and  $\mathcal{B}'_1$  are also boolean lattices which, by the assumptions, form an  $\mathcal{S}$ -atlas, where  $\mathcal{S}$  is a two-element boolean lattice. Let us denote  $\mathcal{B}'_0 \oplus \mathcal{B}'_1$  the  $\mathcal{S}$ -gluing of these lattices.

Similarly, since  $[\sigma(1), 1_{\mathcal{B}}]$  is a filter of  $\mathcal{B}$ ,  $[0_{\mathcal{C}}, \pi(0)]$  is an ideal of  $\mathcal{C}$ , both are boolean lattices and hence, by the assumption and (12) they are both isomorphic to the boolean lattice  $B'_0 \cap B'_1$ .

Let  $\phi'_0 : [\sigma(1), 1_{\mathcal{B}}] \rightarrow B'_0 \cap B'_1$  and  $\phi'_1 : [0_{\mathcal{C}}, \pi(0)] \rightarrow B'_0 \cap B'_1$  be the mappings establishing the isomorphisms between the lattices. Let  $\phi_0 : A_0 \rightarrow B'_0$  and  $\phi_1 : A_1 \rightarrow B'_1$  be extensions of  $\phi'_0$  and  $\phi'_1$  into isomorphisms on lattices  $\mathcal{B}'_0$  and  $\mathcal{B}'_1$ , respectively.

Then  $\sigma' = \phi_0 \circ \sigma$  and  $\pi' = \phi_1 \circ \pi$  are strictly monotone mappings such that

- $\sigma' : K \rightarrow B'_0 \subseteq B'_0 \oplus B'_1$  is a join-homomorphism;
- $\pi' : K \rightarrow B'_1 \subseteq B'_0 \oplus B'_1$  is a meet-homomorphism;
- $\sigma'(0) = \phi_0(\sigma(0)) = 0_{0'}$ ;
- $\sigma'(1) = \phi_0(\sigma(1)) = 0_{1'}$ ;
- $\pi'(0) = \phi_1(\pi(0)) = 1_{0'}$ ;
- $\pi'(1) = \phi_1(\pi(1)) = 1_{1'}$ ;

where  $B'_0 = [0_{0'}, 1_{0'}]$ ,  $B'_1 = [0_{1'}, 1_{1'}]$ .

Let  $x \in K$ . We can consider an interval  $[\sigma'(x), \pi'(x)]$  in the distributive lattice  $B'_0 \oplus B'_1$ . Then, there is a lattice homomorphism  $g_x$  embedding the interval  $[\sigma'(x), \pi'(x)]$  into a boolean lattice  $A_x = [0_x, 1_x]$  such that

$$\dim A_x = l(\sigma'(x), \pi'(x)).$$

Therefore, the image by  $g_x$  of every maximal chain of the interval  $[\sigma'(x), \pi'(x)]$  is a maximal chain in  $\mathcal{A}_x$  and hence it generates the boolean lattice  $\mathcal{A}_x$ . Moreover,  $g_x(\sigma'(x)) = 0_x$  and  $g_x(\pi'(x)) = 1_x$ . In particular, we can take  $g_0 = \phi_0^{-1}$  and  $g_1 = \phi_1^{-1}$ . We can also assume that  $A_x$  and  $A_y$  are disjoint for  $x \neq y$ .

Let  $x, y \in K$ ,  $x \leq y$ . Then

$$(13) \quad \sigma'(x) \leq \sigma'(y) \leq \sigma'(1) = 0_{1'} \leq 1_{0'} = \pi'(0) \leq \pi'(x) \leq \pi'(y)$$

and hence

$$0_x = g_x(\sigma'(x)) \leq g_x(\sigma'(y)) \leq g_x(\pi'(x)) = 1_x,$$

which means that

$$g_x([\sigma'(y), \pi'(x)]) \subseteq [g_x(\sigma'(y)), 1_x].$$

Since  $[g_x(\sigma'(y)), 1_x]$  is a filter of  $\mathcal{A}_x$ ,

$$\dim[g_x(\sigma'(y)), 1_x] = l[\sigma'(y), \pi'(x)]$$

and we can conclude that the sublattice  $[g_x(\sigma'(y)), 1_x]$  of  $\mathcal{A}_x$  is generated by the image of the interval  $[\sigma'(y), \pi'(x)]$  by the function  $g_x$ .

Similarly,

$$0_y = g_y(\sigma'(y)) \leq g_y(\pi'(x)) \leq g_y(\pi'(y)) = 1_y,$$

thus  $g_y([\sigma'(y), \pi'(x)])$  generates the boolean sublattice  $[0_y, g_y(\pi'(x))]$  of  $\mathcal{A}_y$  which is its ideal.

Therefore, a mapping

$$\phi_{yx} : [g_x(\sigma'(y)), 1_x] \longrightarrow [0_y, g_y(\pi'(x))]$$

given by  $\phi_{yx}(g_x(a)) = g_y(a)$  for  $a \in [\sigma'(y), \pi'(x)]$  can be extended to a lattice isomorphism. In particular,  $\phi_{xx} = \text{id}_{\mathcal{A}_x}$  for every  $x \in K$ .

Now, we have constructed the family  $(\mathcal{A}_x)_{x \in K}$  of boolean lattices together with the family of lattice isomorphisms  $\phi_{yx}$  for  $x \leq y$ . If they fulfill the conditions (6)–(11), then we can "glue" them together into a distributive lattice with the skeleton  $\mathcal{K}$  identifying some elements of lattices  $\mathcal{A}_x$  due to the congruence relation  $\sim$ .

Let us observe that the conditions (6),(7) and (11) hold just by the definition of  $\phi_{yx}$ .

Even more, as we have  $F_{y,x} = [g_x(\sigma'(y)), 1_x]$  and  $I_{x,y} = [0_y, g_y(\pi'(x))]$  for arbitrary  $x, y \in K$ , we obtain

$$\begin{aligned} F_{x,x\wedge y} \cap F_{y,x\wedge y} &= [g_{x\wedge y}(\sigma'(x)), 1_{x\wedge y}] \cap [g_{x\wedge y}(\sigma'(y)), 1_{x\wedge y}] \\ &= [g_{x\wedge y}(\sigma'(x)) \vee g_{x\wedge y}(\sigma'(y)), 1_{x\wedge y}] = [g_{x\wedge y}(\sigma'(x \vee y)), 1_{x\wedge y}] = F_{x\vee y, x\wedge y}. \end{aligned}$$

Similarly, we can show

$$I_{x,x\vee y} \cap I_{y,x\vee y} \subseteq I_{x\wedge y, x\vee y},$$

which proves the conditions (9) and (10).

Now, let us suppose that  $x \leq y \leq z$ .

If  $a \in [\sigma'(z), \pi'(x)]$ , then

$$\phi_{zx}(g_x(a)) = g_z(a).$$

Furthermore,

$$[\sigma'(z), \pi'(x)] \subseteq [\sigma'(y), \pi'(y)]$$

and

$$\phi_{yx}(g_x(a)) = g_y(a) \in [g_y(\sigma'(z)), g_y(\pi'(x))] \subseteq [g_y(\sigma'(z)), 1_y],$$

which proves the existence of  $(\phi_{zy} \circ \phi_{yx})(g_x(a))$  for every  $a \in [\sigma'(z), \pi'(x)]$ .

Moreover,

$$\phi_{zy}(\phi_{yx}(g_x(a))) = \phi_{zy}(g_y(a)) = g_z(a) = \phi_{zx}(g_x(a)).$$

Thus, we proved that

$$(\phi_{zy} \circ \phi_{yx})(b) = \phi_{zx}(b)$$

for every generator  $b$  of the lattice  $F_{zx} = [g_x(\sigma'(z)), 1_x]$ , which means that

$$(\phi_{zy} \circ \phi_{yx})(b) = \phi_{zx}(b)$$

for every  $b \in F_{zx}$ , i.e. the condition (8) is satisfied.

Thus, all the conditions (6)–(11) are fulfilled for the family  $(\mathcal{A}_x)_{x \in K}$  together with the family of lattice isomorphisms  $\phi_{yx}$  for  $x \leq y$  in  $\mathcal{K}$  and hence there is a distributive lattice  $\mathcal{D}$  with the skeleton  $\mathcal{K}$  and the maximal boolean intervals  $\mathcal{B}_x$  isomorphic to the boolean lattices  $\mathcal{A}_x$  for  $x \in K$ .

Then

$$\begin{aligned} \dim B_x &= \dim A_x = l[\sigma'(x), \pi'(x)] = \\ &= l[\sigma'(x), 1_{0'}] + l[1_{0'}, \pi'(x)] = l[\sigma(x), 1_{\mathcal{B}}] + l[\pi(0), \pi(x)]. \end{aligned}$$

This completes the proof. ■

Let us observe that, in the above construction, the mappings  $\sigma$  and  $\pi$  determine uniquely (up to isomorphism) the  $\mathcal{K}$ -atlas with overlapping neighborhood  $\{\mathcal{B}_x\}_{x \in K}$  and  $\mathcal{B}_x$  are boolean lattices for all  $x \in K$ . Therefore, distributive lattice  $\mathcal{D}$  in Theorem 8 is determined uniquely (up to isomorphism) by the lattices  $\mathcal{B}, \mathcal{C}$  and the mappings  $\sigma$  and  $\pi$  fulfilling assumptions of the theorem.

If we assume additionally that  $\sigma(0) = 0_{\mathcal{B}}$  and  $\pi(1) = 1_{\mathcal{C}}$ , then  $\dim \Delta(\mathcal{D}) = \dim B = \dim B_0$  and  $\dim \nabla(\mathcal{D}) = \dim C = \dim B_1$ , which means that the given lattices  $\mathcal{B}$  and  $\mathcal{C}$  can be regarded, respectively, as the bottom and top cubes of the obtained distributive lattice  $\mathcal{D}$ .

Moreover, if  $\mathcal{K}$  is  $H$ -irreducible, then all elements of  $\mathcal{K}$  belong to the same block of the skeleton tolerance on  $\mathcal{K}$ , and then, for every finite distributive lattice  $\mathcal{D}$  with the skeleton  $\mathcal{K}$ , by Corollary 5, the mappings  $\sigma(x) = 0_x$  and  $\pi(x) = 1_x$  are, respectively, the join-embedding of  $\mathcal{K}$  into  $\mathcal{B}_0$  and meet-embedding of  $\mathcal{K}$  into  $\mathcal{B}_1$ . It means that  $\mathcal{D}$  is given uniquely by its "top" and "bottom" boolean cubes  $\mathcal{B}_1 = \nabla(\mathcal{D})$  and  $\mathcal{B}_0 = \Delta(\mathcal{D})$  together with the embeddings  $\sigma$  and  $\pi$ .

**Corollary 9.** *If  $\mathcal{K}$  is an  $H$ -irreducible finite lattice, then every finite distributive lattice  $\mathcal{D}$  with the skeleton  $\mathcal{K}$  is uniquely (up to isomorphism) determined by the pair of boolean lattices  $\mathcal{B}$  and  $\mathcal{C}$  together with the mappings  $\sigma$  and  $\pi$  fulfilling the conditions of Theorem 8. ■*

Let us observe that  $\sigma$  is a join-embedding of a lattice  $\mathcal{K}$  into a boolean lattice  $\mathcal{B}$ , hence  $\mathcal{B}$  cannot be too small. In fact,  $\dim B \geq \text{card}(M(\mathcal{K}))$ , where  $M(\mathcal{K})$  denotes the set of all meet-irreducible elements of  $\mathcal{K}$ .

Analogously, since  $\pi$  is a meet-embedding of  $\mathcal{K}$  into a boolean lattice  $\mathcal{C}$ ,  $\dim C \geq \text{card}(J(\mathcal{K}))$ , where  $J(\mathcal{K})$  is the set of all join-irreducible elements of the lattice  $\mathcal{K}$ .

However, it is always possible, for a given finite lattice  $\mathcal{K}$  to construct two boolean lattices  $\mathcal{B}$  and  $\mathcal{C}$  such that  $\dim B = \text{card}(M(\mathcal{K}))$ ,  $\dim C = \text{card}(J(\mathcal{K}))$  and  $B$  and  $C$  have exactly one element in common, namely the unit of  $\mathcal{B}$ , which is at the same time the zero of  $\mathcal{C}$ . In that case, we can always find the mappings  $\sigma$  and  $\pi$  satisfying the assumptions of Theorem 8.

**Example.** Let  $\mathcal{K}$  be a pentagon (Figure 4 (a)). Then the minimal dimension of both lattices  $\mathcal{B}$  and  $\mathcal{C}$  is three. Figure 4 (b) illustrates the lattice  $\mathcal{B} \oplus \mathcal{C}$  with the embeddings  $\sigma$  and  $\pi$  fulfilling the assumptions of Theorem 8. The distributive lattice  $\mathcal{D}$  shown in Figure 4 (c) is the effect of the described construction.



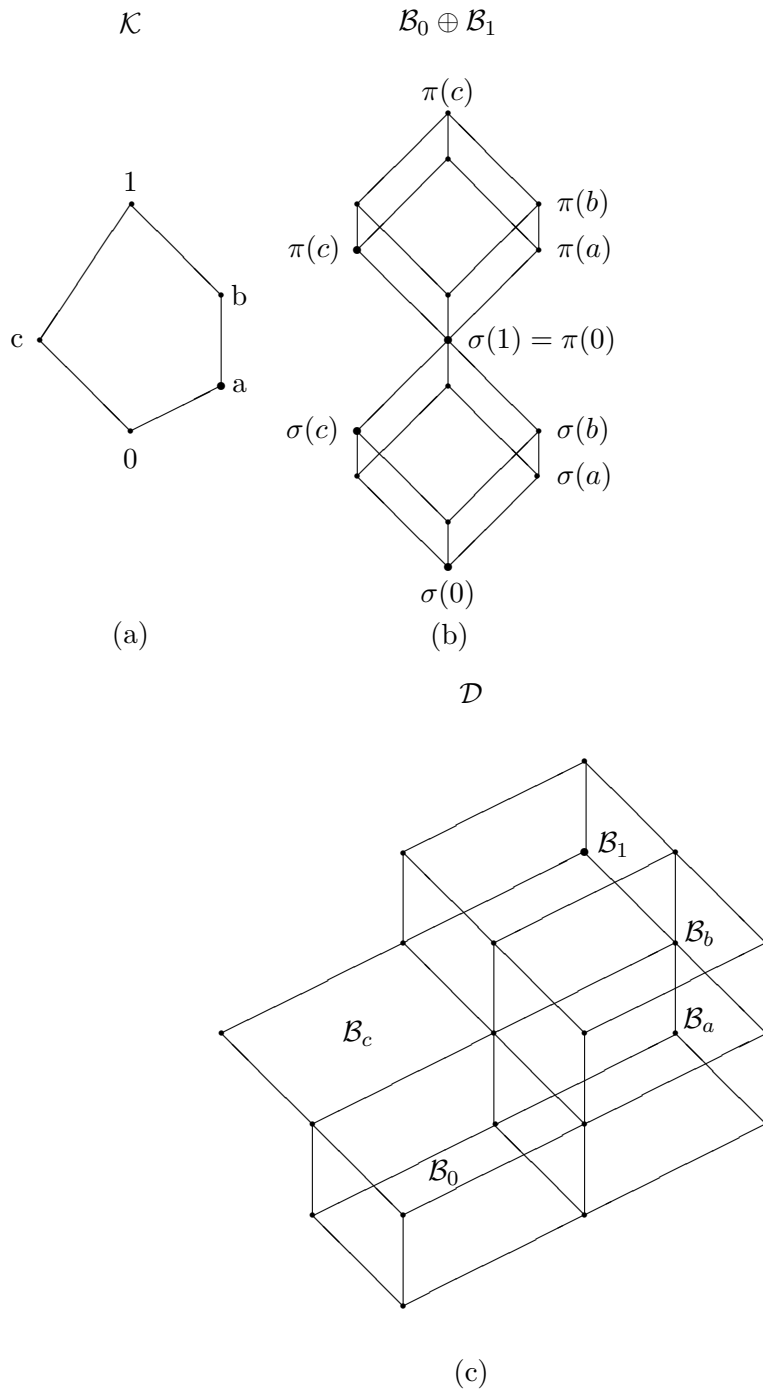


Figure 4

It is easy to observe that  $\mathcal{D}$  is a minimal distributive lattice with the skeleton  $\mathcal{K}$ . Moreover, it is the unique (up to isomorphism) minimal distributive lattice with that skeleton. What is more, since the pentagon is  $H$ -irreducible then the described construction provides all finite distributive lattices corresponding to elements of  $\mathcal{W}(\mathcal{K})$ .

Let us notice that the construction of a finite distributive lattice  $\mathcal{D}$  with a given skeleton  $\mathcal{K}$  used in the proof of Theorem 8 forces the bottom and the top cubes of  $\mathcal{D}$  to overlap. In fact, every finite distributive lattice  $\mathcal{D}$  with that property can be built in that way. It is the reason why the construction can be regarded as the generalization of both, Herrmann's and Wille's, constructions.

In particular, in Herrmann's construction two  $(|K| + 1)$ -dimensional boolean lattices  $L_0, L_1 \subseteq \mathcal{P}(K) \times \mathcal{P}(K)^d$  stand for the lattices  $\mathcal{B}$  and  $\mathcal{C}$  and the mappings  $\sigma(x) = (\emptyset, [x])$  and  $\pi(x) = ([x], \emptyset)$  satisfy the assumptions of Theorem 8.

Similarly, in Wille's construction we have again two  $(|K| + 1)$ -dimensional boolean lattices  $\mathcal{B}, \mathcal{C} \subset \mathcal{P}(\{0, 1\}^2)$  and the mappings

$$\sigma(x) = [(x, 0)], \text{ for } x \neq 0;$$

$$\pi(x) = [(x, 1)], \text{ for } x \neq 1;$$

$$\sigma(0) = \emptyset; \quad \pi(1) = \{0, 1\}^2,$$

which satisfy all assumptions of Theorem 8.

However, let us observe that the construction described in Theorem 8 does not always lead to a minimal distributive lattice with a given skeleton.

**Example.** Let us consider a lattice  $\mathcal{K}$  shown in Figure 5 (a). The smallest boolean lattices  $\mathcal{B}_0$  and  $\mathcal{B}_1$  in which we can, respectively, join- and meet-embed the lattice  $\mathcal{K}$  are two dimensional lattices. Figure 5 (b) illustrates  $\mathcal{B}_0 \oplus \mathcal{B}_1$  together with the embeddings  $\sigma$  and  $\pi$ . In Figure 5 (c), we can see the smallest distributive lattice  $\mathcal{D}$  with the skeleton  $\mathcal{K}$  which can be obtained by our construction. However,  $\mathcal{D}$  is not a minimal distributive lattice with this skeleton. The minimal one is shown in Figure 5 (d).

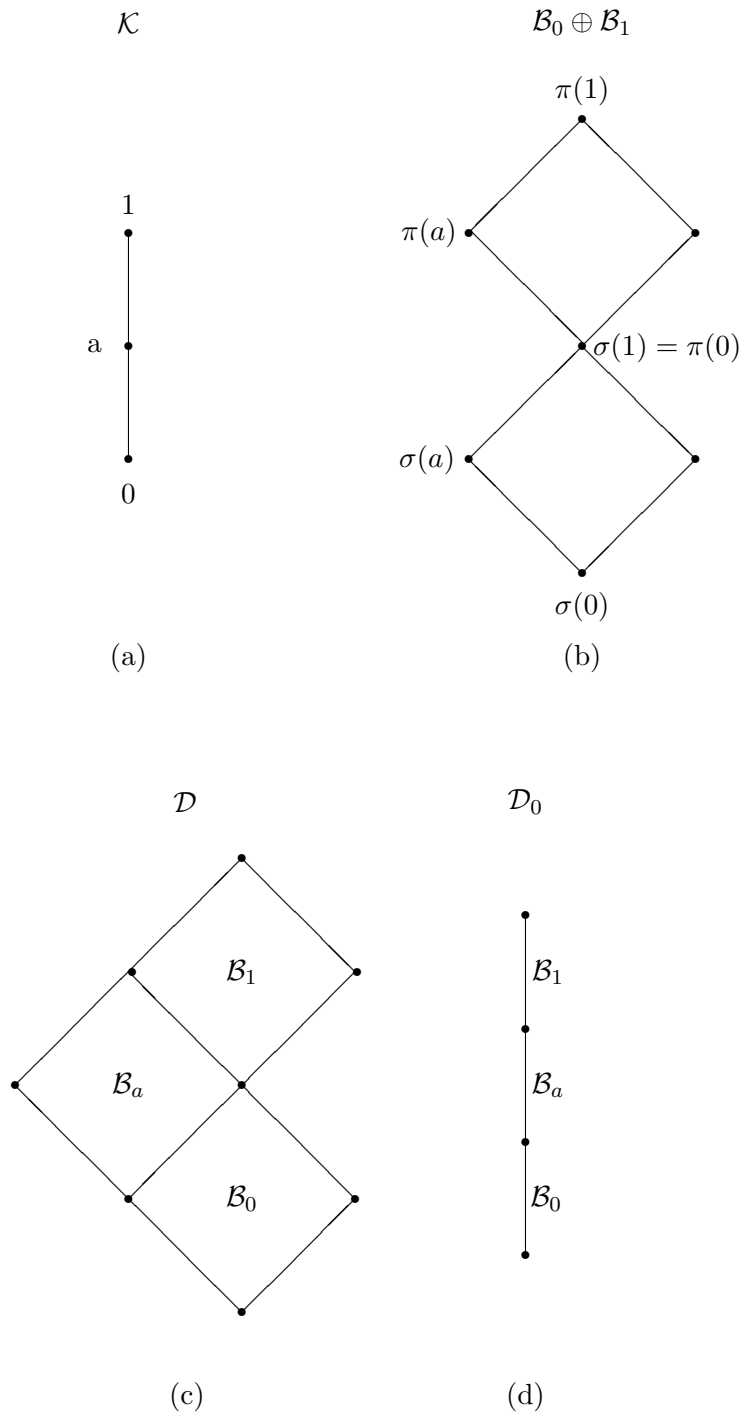


Figure 5

As we mentioned before, our construction gives all possible distributive lattices (and, in particular, the minimal ones) with a skeleton  $\mathcal{K}$  if  $\mathcal{K}$  is a  $H$ -irreducible lattice.

To construct all finite distributive lattices with a  $H$ -reducible skeleton some more sophisticated tools are needed.

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