

## DIRECT DECOMPOSITIONS OF DUALY RESIDUATED LATTICE ORDERED MONOIDS

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### Abstract

The class of dually residuated lattice ordered monoids ( $DR\ell$ -monoids) contains, in an appropriate signature, all  $\ell$ -groups, Brouwerian algebras,  $MV$ - and  $GMV$ -algebras,  $BL$ - and pseudo  $BL$ -algebras, etc. In the paper we study direct products and decompositions of  $DR\ell$ -monoids in general and we characterize ideals of  $DR\ell$ -monoids which are direct factors. The results are then applicable to all above mentioned special classes of  $DR\ell$ -monoids.

**Keywords:**  $DR\ell$ -monoid, lattice-ordered monoid, ideal, normal ideal, polar, direct factor.

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## 1. Introduction

Commutative dually residuated lattice-ordered monoids (in short: *DRℓ-monoids*) were introduced and studied by K.L.N. Swamy in [20], [21], [22] as a common generalization of commutative lattice-ordered groups (*ℓ-groups*) and Brouwerian algebras. The papers [23], [24], [9]–[15], [4] and the part of the thesis [5] engaged the further research of structure properties of commutative *DRℓ-monoids*. It was shown that *MV*-algebras (see [13]) and *BL*-algebras (see [14]) which are an algebraic counterpart of the Łukasiewicz infinite valued logic and Hájek basic fuzzy logic, respectively, can be understood as special cases of commutative *DRℓ-monoids*. General *DRℓ-monoids* (i.e., not necessarily commutative), the special case of which are also all *ℓ-groups*, were introduced by Kovář in [5]. *GMV*-algebras were defined as a non-commutative generalization of *MV*-algebras in [16] and it was shown there that they are special cases of *DRℓ-monoids*. This fact was then used when studying *GMV*-algebras in [17] and [18]. Similarly, it was proved in [6] that pseudo *BL*-algebras (defined in [2] as a non-commutative generalization of *BL*-algebras) are also a special case of *DRℓ-monoids*. *DRℓ-monoids* were further studied in [8], [7] and [19].

In the paper we shall study direct products and direct decompositions of *DRℓ-monoids*. The general results are then applicable for all mentioned special cases of *DRℓ-monoids*.

## 2. Basic notions and notation

**Definition.** An algebra  $M = (M; +, 0, \vee, \wedge, \rightarrow, \leftarrow)$  of signature  $\langle 2, 0, 2, 2, 2, 2 \rangle$  is called a *dually residuated (non-commutative) lattice-ordered monoid* (a *DRℓ-monoid*) if

- (M1)  $(M; +, 0, \vee, \wedge)$  is a lattice-ordered monoid (*ℓ-monoid*), that is,  $(M; +, 0)$  is a (non-commutative) monoid,  $(M, \vee, \wedge)$  is a lattice, and for any  $x, y, u, v \in M$ , the following identities are satisfied:

$$u + (x \vee y) + v = (u + x + v) \vee (u + y + v),$$

$$u + (x \wedge y) + v = (u + x + v) \wedge (u + y + v);$$

- (M2) if  $\leq$  denotes the order on  $M$  induced by the lattice  $(M; \vee, \wedge)$ , then, for any  $x, y \in M$ , we have

$x \rightarrow y$  is the least element  $s \in M$  such that  $s + y \geq x$ ,

$x \leftarrow y$  is the least element  $t \in M$  such that  $y + t \geq x$ ;

(M3)  $M$  fulfils the identities

$$((x \rightarrow y) \vee 0) + y \leq x \vee y, \quad y + ((x \leftarrow y) \vee 0) \leq x \vee y,$$

$$x \rightarrow x \geq 0, \quad x \leftarrow x \geq 0.$$

Commutative  $DR\ell$ -monoids (called  $DR\ell$ -semigroups) were introduced by K.L.N. Swamy in [20] as a common generalization of commutative  $\ell$ -groups and Brouwerian algebras. The present definition of a non-commutative extension of  $DR\ell$ -monoids is due to [5]. Also, for basic properties of non-commutative  $DR\ell$ -monoids see [5].

Let us denote by  $M^+ = \{x \in M : 0 \leq x\}$  the set of all positive elements in  $M$ .

### Examples.

- a) Let  $G = (G; +, 0, -(\cdot), \vee, \wedge)$  be an  $\ell$ -group. Set  $x \rightarrow y = x - y$  and  $x \leftarrow y = -y + x$  for any  $x, y \in G$ . Then  $(G; +, 0, \vee, \wedge, \rightarrow, \leftarrow)$  is a  $DR\ell$ -monoid.
- b) Let  $G$  be an  $\ell$ -group and  $G^+$  be its positive cone, i.e.:  $G^+ = \{x \in G : 0 \leq x\}$ . Set  $x \rightarrow y = (x - y) \vee 0$  and  $x \leftarrow y = (-y + x) \vee 0$  for any elements  $x, y \in G^+$ . Then  $(G^+; +, 0, \vee, \wedge, \rightarrow, \leftarrow)$  is a  $DR\ell$ -monoid.
- c) Let  $B = (B; \vee, \wedge)$  be a Brouwerian algebra, i.e. a dually relative pseudo-complemented lattice with the largest element (that means, for any  $a, b \in B$ , there exists the smallest element  $x \in B$  such that  $b \vee x \geq a$ ). Let us denote by  $a - b$  this relative pseudocomplement  $x$  of the element  $b$  with respect to the element  $a$ . The lattice  $(B; \vee, \wedge)$  has the smallest element 0 and if we set  $a + b = a \vee b$  and  $a \rightarrow b = a \leftarrow b = a - b$  for every  $a, b \in B$ , then  $(B; +, 0, \vee, \wedge, \rightarrow, \leftarrow)$  is a commutative  $DR\ell$ -monoid.
- d) Let  $A = (A; \oplus, \neg, \sim, 0, 1)$  be a  $GMV$ -algebra (see, e.g., [16]), i.e. a non-commutative generalization of an  $MV$ -algebra. For any  $x, y \in A$ , put  $x \odot y = \sim(\neg x \oplus \neg y)$ ,  $x \rightarrow y = \neg y \odot x$  and  $x \leftarrow y = x \odot \sim y$ . If we denote  $x \vee y = x \oplus (y \odot \sim x)$  and  $x \wedge y = x \odot (y \oplus \sim x)$ , then  $(A; \vee, \wedge)$  is a bounded distributive lattice and the algebra  $(A; \oplus, 0, \vee, \wedge, \rightarrow, \leftarrow)$  is

a (bounded)  $DR\ell$ -monoid. If the addition  $\oplus$  is commutative, then the negations  $\neg$  and  $\sim$  coincide,  $A$  is an  $MV$ -algebra, and the induced  $DR\ell$ -monoid is commutative.

Let  $M$  be a  $DR\ell$ -monoid and  $x \in M$ . Then the *absolute value of an element*  $x$  is  $|x| = x \vee (0 \rightarrow x)$ .

**Definitions.**

- a) If  $M$  is a  $DR\ell$ -monoid and  $\emptyset \neq I \subseteq M$ , then  $I$  is called an *ideal of  $M$*  if the following conditions are satisfied:

- (1)  $x, y \in I \implies x + y \in I$ ;
- (2)  $x \in I, y \in M, |y| \leq |x| \implies y \in I$ .

- b) An ideal  $I$  is said to be *normal* if for each  $x, y \in M$  the equivalence:

$$x \rightarrow y \in I \iff x \leftarrow y \in I$$

is satisfied.

**Remark.** By [8], normal ideals are just kernels of  $DR\ell$ -homomorphisms.

It is proved in [8] that the set  $\mathcal{C}(M)$  of all ideals of an arbitrary  $DR\ell$ -monoid  $M$ , ordered by set inclusion, is an algebraic Brouwerian lattice in which infima coincide with set intersections. Further, by Lemma 21 of [8], if  $I$  and  $J$  are normal ideals of a  $DR\ell$ -monoid  $M$ , then their join  $I \vee J$  in  $\mathcal{C}(M)$  is the following set:

$$I \vee J = \{x \in M : |x| \leq a + b, \text{ for some } a \in I^+, b \in J^+\}.$$

**Definitions.**

- a) Let  $M$  be a  $DR\ell$ -monoid and  $X \subseteq M$ . Then the set

$$X^\perp = \{y \in M : |x| \wedge |y| = 0, \text{ for each } x \in X\}$$

is called the *polar of  $X$  in  $M$* .

- b) A subset  $X \subseteq M$  is a *polar in  $M$*  if there exists  $Y \subseteq M$  such that  $X = Y^\perp$ .

By [7], every polar in  $M$  belongs to  $\mathcal{C}(M)$  and it is a polar of some ideal of  $M$ . The polar of any ideal  $I \in \mathcal{C}(M)$  is its pseudocomplement in the lattice  $\mathcal{C}(M)$  and therefore the set  $\mathcal{P}(M)$  of all polars in  $M$  is a complete Boolean algebra with respect to set inclusion.

### 3. Direct products and decompositions

In this section we will study properties of direct products of  $DR\ell$ -monoids, in particular with respect to possibilities of introduction of inner direct products.

**Lemma 1.** *Let  $M$  be a  $DR\ell$ -monoid. Then for any  $v, w \in M$  we have  $v \rightarrow w = 0$  if and only if  $v \leftarrow w = 0$ .*

**Proof.** If  $v \rightarrow w = 0$  and  $x \in M$ , then  $x + v \geq w$  if and only if  $x \geq 0$ . Hence  $w = 0 + w \geq v$ . Then also  $w + 0 \geq v$ , thus  $0 \geq v \leftarrow w$ . At the same time  $w \geq v$  implies  $v \leftarrow w \geq 0$ ; therefore,  $v \leftarrow w = 0$ . ■

Let  $B$  and  $C$  be  $DR\ell$ -monoids and let  $M = B \times C$  be their direct product. Denote  $\tilde{B}, \tilde{C} \subseteq M$  such that

$$\tilde{B} = \{(x_1, 0) : x_1 \in B\},$$

$$\tilde{C} = \{(0, x_2) : x_2 \in C\}.$$

The following proposition seems to be well-known as a folklore:

**Proposition 2.** *If  $B$  and  $C$  are  $DR\ell$ -monoids and  $M = B \times C$  then  $\tilde{B}$  and  $\tilde{C}$  are normal ideals of  $DR\ell$ -monoid  $M$  and it holds:*

- a)  $\tilde{B} + \tilde{C} = M, \quad \tilde{B} \cap \tilde{C} = \{0\};$
- b)  $x + y = x' + y' \text{ implies } x = x', y = y' \text{ for each } x, x' \in \tilde{B} \text{ and } y, y' \in \tilde{C}.$

■

**Proposition 3.** *If  $M = B \times C$ , then*

$$\tilde{B} = \tilde{C}^\perp \quad \text{and} \quad \tilde{C} = \tilde{B}^\perp.$$

**Proof.** For any elements  $x_1 \in B$  and  $y_2 \in C$  it is satisfied

$$|(x_1, 0)| \wedge |(0, y_2)| \in \tilde{B} \cap \tilde{C} = \{(0, 0)\}.$$

Therefore,  $\tilde{B} \subseteq \tilde{C}^\perp$  and  $\tilde{C} \subseteq \tilde{B}^\perp$ .

Conversely, let  $(z_1, z_2) \in (\tilde{B}^\perp)^+$ . Then

$$(z_1, z_2) = (z_1, 0) + (0, z_2) \text{ and } (z_1, 0) = (z_1, 0) \wedge (z_1, z_2) = (0, 0).$$

Thus  $(\tilde{B}^\perp)^+ \subseteq \tilde{C}$ , therefore also  $\tilde{B}^\perp \subseteq \tilde{C}$ , it means  $\tilde{B}^\perp = \tilde{C}$ .

Analogously,  $\tilde{C}^\perp \subseteq \tilde{B}$ . ■

Now we will deal with possibility of introduction of an inner direct decomposition of  $DR\ell$ -monoids.

At first, we will prove the following lemma.

**Lemma 4.** *Let  $M$  be a  $DR\ell$ -monoid and let  $I, J \in \mathcal{C}(M)$  be such that  $I + J = M$  and  $I \cap J = \{0\}$ . If  $a \in M$  and  $a_1 \in I$ ,  $a_2 \in J$  are such that  $a = a_1 + a_2$ , then  $a \geq 0$  if and only if  $a_1 \geq 0$  and  $a_2 \geq 0$ .*

**Proof.** Suppose  $0 \leq a = a_1 + a_2$ . Then  $0 \rightarrow a_2 \leq (a_1 + a_2) \rightarrow a_2$ . Since, by Lemma 1.1.19 of [5], it holds  $(p + q) \rightarrow r \leq p + (q \rightarrow r)$  for any  $p, q, r \in M$ , in our case we obtain  $(a_1 + a_2) \rightarrow a_2 \leq a_1 + (a_2 \rightarrow a_2) = a_1$ . So  $0 \rightarrow a_2 \leq a_1$ . Therefore,  $0 \leq (0 \rightarrow a_2) \vee 0 \leq a_1 \vee 0 \in I$ . Hence  $(0 \rightarrow a_2) \vee 0 \in I \cap J$ , that means  $(0 \rightarrow a_2) \vee 0 = 0$ . Thus  $0 \rightarrow a_2 \leq 0$ . By Lemma 1.1.16 of [5],  $p \geq q$  if and only if  $q \rightarrow p \leq 0$ , for any  $p, q \in M$ . Thus we have  $a_2 \geq 0$ . Similarly,  $a_1 \geq 0$ .

The converse implication is obvious. ■

### Definitions.

- a) An element  $y$  of a  $DR\ell$ -monoid  $M$  is called *singular* if  $0 \rightarrow y = 0$  (or equivalently, by Lemma 1,  $0 \leftarrow y = 0$ ).
- b) An element  $x \in M$  is called *invertible* if there exists an inverse element for it in the monoid  $(M; +, 0)$ .

Denote by  $\text{Sing}(M)$  the set of all singular elements in  $M$  and by  $\text{Inv}(M)$  the set of all invertible elements in  $M$ .

**Remarks.** Kovář proved in [5] (see Theorem 1.2.16 and Lemma 1.2.11) that  $\text{Sing}(M) \in \mathcal{C}(M)$ ,  $\text{Sing}(M) \subseteq M^+$  and  $0$  is the least element in  $\text{Sing}(M)$ . Further, by Theorems 1.2.1 and 1.2.4 of [5],  $\text{Inv}(M)$  is also an ideal of  $M$  which is, moreover, an  $\ell$ -group. The ideals  $\text{Sing}(M)$  and  $\text{Inv}(M)$  play an important role in the study of structure properties of  $DR\ell$ -monoids

because, by Theorem 1.3.6 of [5], each  $DR\ell$ -monoid  $M$  is isomorphic to the direct product of the  $DR\ell$ -monoids  $\text{Sing}(M)$  and  $\text{Inv}(M)$ .

At the same time, extreme case can arise, because if  $M$  is an  $\ell$ -group, then  $\text{Sing}(M) = \{0\}$  and  $\text{Inv}(M) = M$ . If  $M$  is a Brouwerian algebra, then, conversely,  $\text{Sing}(M) = M$  and  $\text{Inv}(M) = \{0\}$ . Consequently, the cardinality of  $\text{Sing}(M)$  determines the degree of dissimilarity of properties of a given  $DR\ell$ -monoid from properties of an  $\ell$ -group.

**Proposition 5.** *If  $M$  is a  $DR\ell$ -monoid and  $a, b \in M$  are orthogonal (i.e.  $|a| \wedge |b| = 0$ ), then  $a + b = b + a$ .*

**Proof.** a) Assume  $a, b \in M^+$  and  $a \wedge b = 0$ . By Lemmas 1.1.5 and 1.1.9 of [5], for any  $x, y, z \in M$  it holds  $x \rightarrow x = 0$  and  $x \rightarrow (y \wedge z) = (x \rightarrow y) \vee (x \rightarrow z)$ , hence in our case we have

$$(a \rightarrow (a \wedge b)) + b = ((a \rightarrow a) \vee (a \rightarrow b)) + b = (0 \vee (a \rightarrow b)) + b = a \vee b,$$

therefore  $a + b = a \vee b = b + a$ , in our case.

b) Now, let  $a, b$  be arbitrary elements in  $M$  such that  $|a| \wedge |b| = 0$ . As mentioned in the previous remark, by Theorem 1.3.6 of [5],  $M$  is the direct product of its ideals  $\text{Sing}(M)$  and  $\text{Inv}(M)$ . Hence there are  $a', b' \in \text{Sing}(M)$  and  $x_a, x_b \in \text{Inv}(M)$  such that  $a = a' + x_a$ ,  $b = b' + x_b$ . By [5],  $|a| = a' + |x_a|$ ,  $|b| = b' + |x_b|$ . Therefore, the assumption  $|a| \wedge |b| = 0$  implies  $a' \wedge b' = 0$  and  $|x_a| \wedge |x_b| = 0$ .

By the part a), we obtain  $a' + b' = b' + a'$ . As  $\text{Inv}(M)$  is an  $\ell$ -group, it holds that  $|x_a| \wedge |x_b| = 0$  entails  $x_a + x_b = x_b + x_a$ . Moreover, since  $M$  is isomorphic to the direct product of  $\text{Sing}(M)$  and  $\text{Inv}(M)$ , elements in  $\text{Sing}(M)$  commute with those in  $\text{Inv}(M)$ . Thus

$$\begin{aligned} a + b &= (a' + x_a) + (b' + x_b) = a' + b' + x_a + x_b = \\ &= b' + a' + x_b + x_a = (b' + x_b) + (a' + x_a) = b + a. \end{aligned}$$

■

**Theorem 6.** *Let  $M$  be a  $DR\ell$ -monoid and  $I, J \in \mathcal{C}(M)$ . Let the following conditions be satisfied:*

1.  $I + J = M$ ,  $I \cap J = \{0\}$ ;
2.  $\forall x, x' \in I, y, y' \in J; x + y = x' + y' \implies x = x', y = y'$ .

If  $\overline{M} = I \times J$  is the direct product of the  $DR\ell$ -monoids  $I$  and  $J$ , then  $M$  and  $\overline{M}$  are isomorphic.

**Proof.** The conditions 1 and 2 obviously yield that for every element  $a \in M$  there exist unique elements  $a_1 \in I$  and  $a_2 \in J$  such that  $a = a_1 + a_2$ . Hence the mapping  $f : a \mapsto (a_1, a_2)$  is a bijection of  $M$  onto  $\overline{M}$ .

Let us suppose  $x \in I$  and  $y \in J$ . Then  $|x| \in I$ ,  $|y| \in J$  and  $|x| \wedge |y| \in I \cap J = \{0\}$ . It follows that  $x$  and  $y$  are orthogonal. Therefore,  $x + y = y + x$  by Proposition 5. For this reason it holds for any elements  $a, b \in M$

$$a + b = (a_1 + a_2) + (b_1 + b_2) = (a_1 + b_1) + (a_2 + b_2),$$

therefore

$$f(a + b) = (a_1 + b_1, a_2 + b_2) = f(a) + f(b).$$

Assume again  $a = a_1 + a_2$ ,  $b = b_1 + b_2 \in M$ ,  $a_1, b_1 \in I$ ,  $a_2, b_2 \in J$  and let  $a \leq b$ . By Lemma 1.1.14 of [5], there exists  $x \in M^+$  such that  $a + x = b$ . Let  $x = x_1 + x_2$ , where  $x_1 \in I$ ,  $x_2 \in J$ . By Lemma 4, it holds  $x_1 \in I^+$  and  $x_2 \in J^+$ . From this we have  $(a_1 + x_1) + (a_2 + x_2) = b_1 + b_2$ , i.e.  $a_1 + x_1 = b_1$ ,  $a_2 + x_2 = b_2$ , where  $0 \leq x_1$ ,  $0 \leq x_2$ . As  $0 \leq x_1$ , it holds  $a_1 \leq a_1 + x_1 = b_1$ . Similarly,  $a_2 \leq b_2$ .

Hence, for any  $a, b \in M$ ,  $a \leq b$  if and only if  $f(a) \leq f(b)$ .

We have proved that  $f$  is an isomorphism of lattice-ordered monoids  $(M; +, 0, \vee, \wedge)$  and  $(\overline{M}; +, 0, \vee, \wedge)$ . Since the values of the operations  $\rightarrow$  and  $\leftarrow$  are uniquely determined in both the  $DR\ell$ -monoids  $M = (M; +, 0, \vee, \wedge, \rightarrow, \leftarrow)$  and  $\overline{M} = (\overline{M}; +, 0, \vee, \wedge, \rightarrow, \leftarrow)$  in the same manner by means of the operation  $+$  and the order relation  $\leq$ ,  $DR\ell$ -monoids  $M = I + J$  and  $\overline{M} = I \times J$  are also isomorphic. ■

#### Remarks.

- a) Let  $\tilde{I} = \{(x, 0); x \in I\}$  and  $\tilde{J} = \{(0, y); y \in J\}$ . Since  $I \cong \tilde{I}$  and  $J \cong \tilde{J}$ , the ideals  $I$  and  $J$  are (by Proposition 2 and Theorem 6) normal in  $M$ .
- b) By Theorem 6 and Proposition 3, the set of all direct factors of a  $DR\ell$ -monoid  $M$  is a subset of the set of all polars in  $M$ . In particular,  $\text{Sing}(M)$  and  $\text{Inv}(M)$  are polars in  $M$ . It holds

$$(\text{Sing}(M))^\perp = \text{Inv}(M) \quad \text{and} \quad (\text{Inv}(M))^\perp = \text{Sing}(M).$$

If  $M$  is a  $DR\ell$ -monoid and  $I \in \mathcal{C}(M)$ , let us denote by  $D(I)$  the join of ideals  $I$  and  $I^\perp$  in the lattice  $\mathcal{C}(M)$ .

**Proposition 7.** *If  $M$  is a  $DR\ell$ -monoid,  $I \in \mathcal{C}(M)$  and  $I$  is a direct factor of  $DR\ell$ -monoid  $D(I)$ , then  $I + I^\perp \in \mathcal{C}(D(I))$  and  $I + I^\perp = D(I) = I \vee I^\perp$  (in the sense of  $\mathcal{C}(M)$ ).*

**Proof.** Since  $I$  and  $I^\perp$  are normal ideals of  $D(I)$ , by Lemma 21 of [8], it holds that

$$I \vee I^\perp = \{x \in D(I); |x| \leq a + b, \text{ where } a \in I, b \in I^\perp\}.$$

in  $\mathcal{C}(D(I))$  (consequently, also in  $\mathcal{C}(M)$ ).

By Proposition 5,  $a + b = b + a = a \vee b$ . By Theorem 1.1.23 of [5], the underlying lattice  $(M; \vee, \wedge)$  is distributive, therefore the lattice  $(D(I)^+; \vee, \wedge)$  is also distributive. For this reason, from the inequality  $|x| \leq a + b$ , where  $a \in I^+$  and  $b \in (I^\perp)^+$ , it follows the existence of elements  $0 \leq a_1 \leq a$ ,  $0 \leq b_1 \leq b$  in  $D(I)$  such that  $|x| = a_1 \vee b_1 = a_1 + b_1$ . At the same time  $a_1 \in I$ ,  $b_1 \in I^\perp$  and hence  $I \vee I^\perp = I + I^\perp$ . ■

**Corollary 8.** *If  $M$  is a  $DR\ell$ -monoid and  $I \in \mathcal{C}(M)$ , then  $DR\ell$ -monoids  $I + I^\perp$  and  $I \times I^\perp$  are isomorphic if and only if  $x + y = x' + y'$  implies  $x = x'$  and  $y = y'$ , for any  $x, x' \in I$  and  $y, y' \in I^\perp$ .* ■

By Proposition 15 of [8], for any ideal  $I$  of a  $DR\ell$ -monoid  $M$  (and hence also for each polar in  $M$ ) it holds that its polar  $I^\perp$  is the pseudocomplement of  $I$  in  $\mathcal{C}(M)$ . We can specify this result for the direct factors of  $M$ .

**Proposition 9.** *If an ideal  $I$  of a  $DR\ell$ -monoid  $M$  is a direct factor in  $M$ , then the polar  $I^\perp$  is the complement of  $I$  in the lattice  $\mathcal{C}(M)$ .* ■

Now we can prove the following proposition:

**Proposition 10.** *If  $M$  is an arbitrary  $DR\ell$ -monoid, then ideals  $I$  of  $M$ , for which there exists an ideal  $J \in \mathcal{C}(M)$  such that  $I$  and  $J$  satisfy condition 1 from Theorem 6, form a Boolean lattice. This lattice is a sublattice of  $\mathcal{C}(M)$ .*

**Proof.** Let  $I$  and  $J$  satisfy the given assumptions. Then from distributivity of the lattice  $\mathcal{C}(M)$  we obtain

$$(I \vee J) \cap (I^\perp \cap J^\perp) = (I \cap I^\perp \cap J^\perp) \vee (J \cap I^\perp \cap J^\perp) = \{0\},$$

$$(I \vee J) \vee (I^\perp \cap J^\perp) = (I \vee J \vee I^\perp) \cap (I \vee J \vee J^\perp) = M,$$

hence  $I \vee J$  and  $I^\perp \cap J^\perp$  satisfy condition 1.

The remaining part of the assertion follows from Proposition 9. ■

Let us consider the following condition of uniqueness of decomposition for  $DR\ell$ -monoids  $M$ :

$$\begin{aligned} & \text{If } I, J \in \mathcal{C}(M), I \cap J = \{0\}, x, x' \in I, y, y' \in J \\ \text{(UD)} \quad & \text{and } x + y = x' + y', \text{ then } x = x' \text{ and } y = y'. \end{aligned}$$

**Theorem 11.** *If  $DR\ell$ -monoid  $M$  satisfies condition (UD), then the direct factors in  $M$  form a Boolean sublattice of the lattice  $\mathcal{C}(M)$ .*

**Proof.** If condition (UD) holds in  $M$ , then  $I \in \mathcal{C}(M)$  is a direct factor if and only if  $I$  and  $J = I^\perp$  satisfy condition 1. Therefore, the theorem follows from Proposition 10. ■

**Remark.** If  $G$  is an  $\ell$ -group, then  $DR\ell$ -monoids  $G$  and  $G^+$  satisfy condition (UD). Hence their direct factors form a Boolean lattice.

## REFERENCES

- [1] R.L.O. Cignoli, I.M.L. D'Ottaviano and D. Mundici, *Foundations of Many-valued Reasoning*, Kluwer Acad. Publ., Dordrecht 2000.
- [2] A. Di Nola, G. Georgescu and A. Iorgulescu, *Pseudo BL-algebras: Part I*, Multiple-Valued Logic **8** (2002), 673–714.
- [3] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Acad. Publ., Dordrecht 1998.
- [4] M.E. Hansen, *Minimal prime ideals in autometrized algebras*, Czechoslovak Math. J. **44** (119) (1994), 81–90.
- [5] T. Kovář, *A general theory of dually residuated lattice-ordered monoids*, Ph.D. Thesis, Palacký Univ., Olomouc 1996.

- [6] J. Kühr, *Pseudo BL-algebras and DRL-monoids*, Math. Bohemica **128** (2003), 199–208.
- [7] J. Kühr, *Prime ideals and polars in DRL-monoids and pseudo BL-algebras*, Math. Slovaca **53** (2003), 233–246.
- [8] J. Kühr, *Ideals of noncommutative DRL-monoids*, Czechoslovak Math. J. (to appear).
- [9] J. Rachůnek, *Prime ideals in autometrized algebras*, Czechoslovak Math. J. **37** (112) (1987), 65–69.
- [10] J. Rachůnek, *Polars in autometrized algebras*, Czechoslovak Math. J. **39** (114) (1989), 681–685.
- [11] J. Rachůnek, *Regular ideals in autometrized algebras*, Math. Slovaca **40** (1990), 117–122.
- [12] J. Rachůnek, *DRL-semigroups and MV-algebras*, Czechoslovak Math. J. **48** (123) (1998), 365–372.
- [13] J. Rachůnek, *MV-algebras are categorically equivalent to a class of DRL-semigroups*, Math. Bohemica **123** (1998), 437–441.
- [14] J. Rachůnek, *A duality between algebras of basic logic and bounded representable DRL-monoids*, Math. Bohemica **126** (2001), 561–569.
- [15] J. Rachůnek, *Polars and annihilators in representable DRL-monoids and MV-algebras*, Math. Slovaca **51** (2001), 1–12.
- [16] J. Rachůnek, *A non-commutative generalization of MV-algebras*, Czechoslovak Math. J. **52** (127) (2002), 255–273.
- [17] J. Rachůnek, *Prime ideals and polars in generalized MV-algebras*, Multiple-Valued Logic **8** (2002), 127–137.
- [18] J. Rachůnek, *Prime spectra of non-commutative generalizations of MV-algebras*, Algebra Univers. **48** (2002), 151–169.
- [19] D. Šalounová, *Lex-ideals of DRL-monoids and GMV-algebras*, Math. Slovaca **53** (2003), 321–330.
- [20] K.L.N. Swamy, *Dually residuated lattice-ordered semigroups I*, Math. Ann. **159** (1965), 105–114.
- [21] K.L.N. Swamy, *Dually residuated lattice-ordered semigroups II*, Math. Ann. **160** (1965), 64–71.
- [22] K.L.N. Swamy, *Dually residuated lattice-ordered semigroups III*, Math. Ann. **167** (1966), 71–74.

- [23] K.L.N. Swamy and N.P. Rao, *Ideals in autometrized algebras*, J. Austral. Math. Soc. Ser. A **24** (1977), 362–374.
- [24] K.L.N. Swamy, and B.V. Subba Rao, *Isometries in dually residuated lattice-ordered semigroups*, Math. Sem. Notes Kobe Univ. **8** (1980), 369–379.

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