# DIRECT DECOMPOSITIONS OF DUALLY RESIDUATED LATTICE ORDERED MONOIDS 

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#### Abstract

The class of dually residuated lattice ordered monoids ( $D R \ell$-monoids) contains, in an appropriate signature, all $\ell$-groups, Brouwerian algebras, $M V$ - and $G M V$-algebras, $B L$ - and pseudo $B L$-algebras, etc. In the paper we study direct products and decompositions of $D R \ell$ monoids in general and we characterize ideals of $D R \ell$-monoids which are direct factors. The results are then applicable to all above mentioned special classes of $D R \ell$-monoids.


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## 1. Introduction

Commutative dually residuated lattice-ordered monoids (in short: DRQmonoids) were introduced and studied by K.L.N. Swamy in [20], [21], [22] as a common generalization of commutative lattice-ordered groups ( $\ell$-groups) and Brouwerian algebras. The papers [23], [24], [9]-[15], [4] and the part of the thesis [5] engaged the further research of structure properties of commutative $D R \ell$-monoids. It was shown that $M V$-algebras (see [13]) and $B L$ algebras (see [14]) which are an algebraic counterpart of the Lukasiewicz infinite valued logic and Hájek basic fuzzy logic, respectively, can be understood as special cases of commutative $D R \ell$-monoids. General $D R \ell$-monoids (i.e., not necessarily commutative), the special case of which are also all $\ell$-groups, were introduced by Kovář in [5]. $G M V$-algebras were defined as a non-commutative generalization of $M V$-algebras in [16] and it was shown there that they are special cases of $D R \ell$-monoids. This fact was then used when studying $G M V$-algebras in [17] and [18]. Similarly, it was proved in [6] that pseudo $B L$-algebras (defined in [2] as a non-commutative generalization of $B L$-algebras) are also a special case of $D R \ell$-monoids. $D R \ell$-monoids were further studied in [8], [7] and [19].

In the paper we shall study direct products and direct decompositions of $D R \ell$-monoids. The general results are then applicable for all mentioned special cases of $D R \ell$-monoids.

## 2. Basic notions and notation

Definition. An algebra $M=(\mathrm{M} ;+, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ of signature $\langle 2,0,2,2,2,2\rangle$ is called a dually residuated (non-commutative) lattice-ordered monoid (a DRe-monoid) if
(M1) $(M ;+, 0, \vee, \wedge)$ is a lattice-ordered monoid $(\ell$-monoid), that is, $(M ;+, 0)$ is a (non-commutative) monoid, $(M, \vee, \wedge)$ is a lattice, and for any $x, y, u, v \in M$, the following identities are satisfied:

$$
\begin{aligned}
& u+(x \vee y)+v=(u+x+v) \vee(u+y+v), \\
& u+(x \wedge y)+v=(u+x+v) \wedge(u+y+v) ;
\end{aligned}
$$

(M2) if $\leq$ denotes the order on $M$ induced by the lattice $(M ; \vee, \wedge)$, then, for any $x, y \in M$, we have
$x \rightharpoonup y$ is the least element $s \in M$ such that $s+y \geq x$,
$x \leftharpoondown y$ is the least element $t \in M$ such that $y+t \geq x ;$
(M3) $M$ fulfils the identities

$$
\begin{aligned}
((x \rightharpoonup y) \vee 0)+y & \leq x \vee y, y+((x \leftharpoondown y) \vee 0) \leq x \vee y, \\
x & \rightharpoonup x \geq 0, x \leftharpoondown x \geq 0 .
\end{aligned}
$$

Commutative $D R \ell$-monoids (called $D R \ell$-semigroups) were introduced by K.L.N. Swamy in [20] as a common generalization of commutative $\ell$-groups and Brouwerian algebras. The present definition of a non-commutative extension of $D R \ell$-monoids is due to [5]. Also, for basic properties of noncommutative $D R \ell$-monoids see [5].

Let us denote by $M^{+}=\{x \in \mathrm{M}: 0 \leq x\}$ the set of all positive elements in $M$.

## Examples.

a) Let $G=(G ;+, 0,-(\cdot), \vee, \wedge)$ be an $\ell$-group. Set $x \rightharpoonup y=x-y$ and $x \leftharpoondown y=-y+x$ for any $x, y \in G$. Then $(G ;+, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ is a $D R \ell$-monoid.
b) Let $G$ be an $\ell$-group and $G^{+}$be its positive cone, i.e.: $G^{+}=\{x \in$ $\mathrm{G}: 0 \leq x\}$. Set $x \rightharpoonup y=(x-y) \vee 0$ and $x \leftharpoondown y=(-y+x) \vee 0$ for any elements $x, y \in G^{+}$. Then $\left(G^{+} ;+, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown\right)$ is a $D R \ell$-monoid.
c) Let $B=(B ; \vee, \wedge)$ be a Brouwerian algebra, i.e. a dually relative pseudo-complemented lattice with the largest element (that means, for any $a, b \in B$, there exists the smallest element $x \in B$ such that $b \vee x \geq$ $a)$. Let us denote by $a-b$ this relative pseudocomplement $x$ of the element $b$ with respect to the element $a$. The lattice $(B ; \vee, \wedge)$ has the smallest element 0 and if we set $a+b=a \vee b$ and $a \rightharpoonup b=a \leftharpoondown b=a-b$ for every $a, b \in B$, then $(B ;+, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ is a commutative $D R \ell-$ monoid.
d) Let $A=(A ; \oplus, \neg, \sim, 0,1)$ be a $G M V$-algebra (see, e.g., [16]), i.e. a non-commutative generalization of an $M V$-algebra. For any $x, y \in A$, put $x \odot y=\sim(\neg x \oplus \neg y), x \rightharpoonup y=\neg y \odot x$ and $x \leftharpoondown y=x \odot \sim y$. If we denote $x \vee y=x \oplus(y \odot \sim x)$ and $x \wedge y=x \odot(y \oplus \sim x)$, then $(A ; \vee, \wedge)$ is a bounded distributive lattice and the algebra $(A ; \oplus, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ is
a (bounded) $D R \ell$-monoid. If the addition $\oplus$ is commutative, then the negations $\neg$ and $\sim$ coincide, $A$ is an $M V$-algebra, and the induced $D R \ell$-monoid is commutative.

Let $M$ be a $D R \ell$-monoid and $x \in M$. Then the absolute value of an element $x$ is $|x|=x \vee(0 \rightharpoonup x)$.

## Definitions.

a) If $M$ is a $D R \ell$-monoid and $\emptyset \neq I \subseteq M$, then $I$ is called an ideal of $M$ if the following conditions are satisfied:
(1) $x, y \in I \Longrightarrow x+y \in I$;
(2) $x \in I, y \in M,|y| \leq|x| \Longrightarrow y \in I$.
b) An ideal $I$ is said to be normal if for each $x, y \in M$ the equivalence:

$$
x \rightharpoonup y \in I \Longleftrightarrow x \leftharpoondown y \in I
$$

is satisfied.
Remark. By [8], normal ideals are just kernels of $D R \ell$-homomorphisms.
It is proved in [8] that the set $\mathcal{C}(M)$ of all ideals of an arbitrary $D R \ell$ $\operatorname{monoid} M$, ordered by set inclusion, is an algebraic Brouwerian lattice in which infima coincide with set intersections. Further, by Lemma 21 of [8], if $I$ and $J$ are normal ideals of a $D R \ell$-monoid $M$, then their join $I \vee J$ in $\mathcal{C}(M)$ is the following set:

$$
I \vee J=\left\{x \in \mathrm{M}:|x| \leq a+b, \text { for some } a \in I^{+}, b \in J^{+}\right\}
$$

## Definitions.

a) Let $M$ be a $D R \ell$-monoid and $X \subseteq M$. Then the set

$$
X^{\perp}=\{y \in \mathrm{M}:|x| \wedge|y|=0, \text { for each } x \in X\}
$$

is called the polar of $X$ in $M$.
b) A subset $X \subseteq M$ is a polar in $M$ if there exists $Y \subseteq M$ such that $X=Y^{\perp}$.
By [7], every polar in $M$ belongs to $\mathcal{C}(M)$ and it is a polar of some ideal of $M$. The polar of any ideal $I \in \mathcal{C}(M)$ is its pseudocomplement in the lattice $\mathcal{C}(M)$ and therefore the set $\mathcal{P}(M)$ of all polars in $M$ is a complete Boolean algebra with respect to set inclusion.

## 3. Direct products and decompositions

In this section we will study properties of direct products of $D R \ell$-monoids, in particular with respect to possibilities of introduction of inner direct products.

Lemma 1. Let $M$ be a DRt-monoid. Then for any $v, w \in M$ we have $v \rightharpoonup w=0$ if and only if $v \leftharpoondown w=0$.

Proof. If $v \rightharpoonup w=0$ and $x \in M$, then $x+v \geq w$ if and only if $x \geq 0$. Hence $w=0+w \geq v$. Then also $w+0 \geq v$, thus $0 \geq v \leftharpoondown w$. At the same time $w \geq v$ implies $v \leftharpoondown w \geq 0$; therefore, $v \leftharpoondown w=0$.

Let $B$ and $C$ be $D R \ell$-monoids and let $M=B \times C$ be their direct product. Denote $\widetilde{B}, \widetilde{C} \subseteq M$ such that

$$
\begin{aligned}
& \widetilde{B}=\left\{\left(x_{1}, 0\right): x_{1} \in B\right\} \\
& \widetilde{C}=\left\{\left(0, x_{2}\right): x_{2} \in C\right\}
\end{aligned}
$$

The following proposition seems to be well-known as a folklore:
Proposition 2. If $B$ and $C$ are $D R \ell$-monoids and $M=B \times C$ then $\widetilde{B}$ and $\widetilde{C}$ are normal ideals of $D R \ell$-monoid $M$ and it holds:
a) $\widetilde{B}+\widetilde{C}=M, \quad \widetilde{B} \cap \widetilde{C}=\{0\} ;$
b) $x+y=x^{\prime}+y^{\prime}$ implies $x=x^{\prime}, y=y^{\prime}$ for each $x, x^{\prime} \in \widetilde{B}$ and $y, y^{\prime} \in \widetilde{C}$.

Proposition 3. If $M=B \times C$, then

$$
\widetilde{B}=\widetilde{C}^{\perp} \quad \text { and } \quad \widetilde{C}=\widetilde{B}^{\perp}
$$

Proof. For any elements $x_{1} \in B$ and $y_{2} \in C$ it is satisfied

$$
\left|\left(x_{1}, 0\right)\right| \wedge\left|\left(0, y_{2}\right)\right| \in \widetilde{B} \cap \widetilde{C}=\{(0,0)\}
$$

Therefore, $\widetilde{B} \subseteq \widetilde{C}^{\perp}$ and $\widetilde{C} \subseteq \widetilde{B}^{\perp}$.

Conversely, let $\left(z_{1}, z_{2}\right) \in\left(\widetilde{B}^{\perp}\right)^{+}$. Then

$$
\left(z_{1}, z_{2}\right)=\left(z_{1}, 0\right)+\left(0, z_{2}\right) \text { and }\left(z_{1}, 0\right)=\left(z_{1}, 0\right) \wedge\left(z_{1}, z_{2}\right)=(0,0) .
$$

Thus $\left(\widetilde{B}^{\perp}\right)^{+} \subseteq \widetilde{C}$, therefore also $\widetilde{B}^{\perp} \subseteq \widetilde{C}$, it means $\widetilde{B}^{\perp}=\widetilde{C}$.
Analogously, $\widetilde{C}^{\perp} \subseteq \widetilde{B}$.
Now we will deal with possibility of introduction of an inner direct decomposition of $D R \ell$-monoids.

At first, we will prove the following lemma.
Lemma 4. Let $M$ be a DRt-monoid and let $I, J \in \mathcal{C}(M)$ be such that $I+J=M$ and $I \cap J=\{0\}$. If $a \in M$ and $a_{1} \in I, a_{2} \in J$ are such that $a=a_{1}+a_{2}$, then $a \geq 0$ if and only if $a_{1} \geq 0$ and $a_{2} \geq 0$.

Proof. Suppose $0 \leq a=a_{1}+a_{2}$. Then $0 \rightharpoonup a_{2} \leq\left(a_{1}+a_{2}\right) \rightharpoonup a_{2}$. Since, by Lemma 1.1.19 of [5], it holds $(p+q) \rightharpoonup r \leq p+(q \rightharpoonup r)$ for any $p, q, r \in M$, in our case we obtain $\left(a_{1}+a_{2}\right) \rightharpoonup a_{2} \leq a_{1}+\left(a_{2} \rightharpoonup a_{2}\right)=a_{1}$. So $0 \rightharpoonup a_{2} \leq a_{1}$. Therefore, $0 \leq\left(0 \rightharpoonup a_{2}\right) \vee 0 \leq a_{1} \vee 0 \in I$. Hence $\left(0 \rightharpoonup a_{2}\right) \vee 0 \in I \cap J$, that means $\left(0 \rightharpoonup a_{2}\right) \vee 0=0$. Thus $0 \rightharpoonup a_{2} \leq 0$. By Lemma 1.1.16 of [5], $p \geq q$ if and only if $q \rightharpoonup p \leq 0$, for any $p, q \in M$. Thus we have $a_{2} \geq 0$. Similarly, $a_{1} \geq 0$.

The converse implication is obvious.

## Definitions.

a) An element $y$ of a $D R \ell$-monoid $M$ is called singular if $0 \rightharpoonup y=0$ (or equivalently, by Lemma $1,0 \leftharpoondown y=0$ ).
b) An element $x \in M$ is called invertible if there exists an inverse element for it in the monoid ( $M ;+, 0$ ).

Denote by $\operatorname{Sing}(M)$ the set of all singular elements in $M$ and by $\operatorname{Inv}(M)$ the set of all invertible elements in $M$.

Remarks. Kovář proved in [5] (see Theorem 1.2.16 and Lemma 1.2.11) that $\operatorname{Sing}(M) \in \mathcal{C}(M), \quad \operatorname{Sing}(M) \subseteq M^{+}$and 0 is the least element in $\operatorname{Sing}(M)$. Further, by Theorems 1.2.1 and 1.2.4 of [5], $\operatorname{Inv}(M)$ is also an ideal of $M$ which is, moreover, an $\ell$-group. The ideals $\operatorname{Sing}(M)$ and $\operatorname{Inv}(M)$ play an important role in the study of structure properties of $D R \ell$-monoids
because, by Theorem 1.3.6 of [5], each $D R \ell$-monoid $M$ is isomorphic to the direct product of the $D R \ell$-monoids $\operatorname{Sing}(M)$ and $\operatorname{Inv}(M)$.

At the same time, extreme case can arise, because if $M$ is an $\ell$-group, then $\operatorname{Sing}(M)=\{0\}$ and $\operatorname{Inv}(M)=M$. If $M$ is a Brouwerian algebra, then, conversely, $\operatorname{Sing}(M)=M$ and $\operatorname{Inv}(M)=\{0\}$. Consequently, the cardinality of Sing $(M)$ determines the degree of dissimilarity of properties of a given $D R \ell$-monoid from properties of an $\ell$-group.

Proposition 5. If $M$ is a $D R \ell$-monoid and $a, b \in M$ are orthogonal (i.e. $|a| \wedge|b|=0)$, then $a+b=b+a$.

Proof. a) Assume $a, b \in M^{+}$and $a \wedge b=0$. By Lemmas 1.1.5 and 1.1.9 of [5], for any $x, y, z \in M$ it holds $x \rightharpoonup x=0$ and $x \rightharpoonup(y \wedge z)=$ $(x \rightharpoonup y) \vee(x \rightharpoonup z)$, hence in our case we have

$$
(a \rightharpoonup(a \wedge b))+b=((a \rightharpoonup a) \vee(a \rightharpoonup b))+b=(0 \vee(a \rightharpoonup b))+b=a \vee b,
$$

therefore $a+b=a \vee b=b+a$, in our case.
b) Now, let $a, b$ be arbitrary elements in $M$ such that $|a| \wedge|b|=0$. As mentioned in the previous remark, by Theorem 1.3.6 of [5], $M$ is the direct product of its ideals $\operatorname{Sing}(M)$ and $\operatorname{Inv}(M)$. Hence there are $a^{\prime}, b^{\prime} \in$ $\operatorname{Sing}(M)$ and $x_{a}, x_{b} \in \operatorname{Inv}(M)$ such that $a=a^{\prime}+x_{a}, b=b^{\prime}+x_{b}$. By [5], $|a|=a^{\prime}+\left|x_{a}\right|,|b|=b^{\prime}+\left|x_{b}\right|$. Therefore, the assumption $|a| \wedge|b|=0$ implies $a^{\prime} \wedge b^{\prime}=0$ and $\left|x_{a}\right| \wedge\left|x_{b}\right|=0$.

By the part a), we obtain $a^{\prime}+b^{\prime}=b^{\prime}+a^{\prime}$. As $\operatorname{Inv}(M)$ is an $\ell$-group, it holds that $\left|x_{a}\right| \wedge\left|x_{b}\right|=0$ entails $x_{a}+x_{b}=x_{b}+x_{a}$. Moreover, since $M$ is isomorphic to the direct product of $\operatorname{Sing}(M)$ and $\operatorname{Inv}(M)$, elements in $\operatorname{Sing}(M)$ commute with those in $\operatorname{Inv}(M)$. Thus

$$
\begin{aligned}
a+b & =\left(a^{\prime}+x_{a}\right)+\left(b^{\prime}+x_{b}\right)=a^{\prime}+b^{\prime}+x_{a}+x_{b}= \\
& =b^{\prime}+a^{\prime}+x_{b}+x_{a}=\left(b^{\prime}+x_{b}\right)+\left(a^{\prime}+x_{a}\right)=b+a .
\end{aligned}
$$

Theorem 6. Let $M$ be a $D R \ell$-monoid and $I, J \in \mathcal{C}(M)$. Let the following conditions be satisfied:

1. $\quad I+J=M, \quad I \cap J=\{0\} ;$
2. $\forall x, x^{\prime} \in I, y, y^{\prime} \in J ; \quad x+y=x^{\prime}+y^{\prime} \Longrightarrow x=x^{\prime}, y=y^{\prime}$.

If $\bar{M}=I \times J$ is the direct product of the DR€-monoids $I$ and $J$, then $M$ and $\bar{M}$ are isomorphic.

Proof. The conditions 1 and 2 obviously yield that for every element $a \in M$ there exist unique elements $a_{1} \in I$ and $a_{2} \in J$ such that $a=a_{1}+a_{2}$. Hence the mapping $f: a \longmapsto\left(a_{1}, a_{2}\right)$ is a bijection of $M$ onto $\bar{M}$.

Let us suppose $x \in I$ and $y \in J$. Then $|x| \in I,|y| \in J$ and $|x| \wedge|y| \in I \cap J=\{0\}$. It follows that $x$ and $y$ are orthogonal. Therefore, $x+y=y+x$ by Proposition 5. For this reason it holds for any elements $a, b \in M$

$$
a+b=\left(a_{1}+a_{2}\right)+\left(b_{1}+b_{2}\right)=\left(a_{1}+b_{1}\right)+\left(a_{2}+b_{2}\right),
$$

therefore

$$
f(a+b)=\left(a_{1}+b_{1}, a_{2}+b_{2}\right)=f(a)+f(b) .
$$

Assume again $a=a_{1}+a_{2}, b=b_{1}+b_{2} \in M, a_{1}, b_{1} \in I, a_{2}, b_{2} \in J$ and let $a \leq b$. By Lemma 1.1.14 of [5], there exists $x \in M^{+}$such that $a+x=b$. Let $x=x_{1}+x_{2}$, where $x_{1} \in I, x_{2} \in J$. By Lemma 4, it holds $x_{1} \in I^{+}$and $x_{2} \in J^{+}$. From this we have $\left(a_{1}+x_{1}\right)+\left(a_{2}+x_{2}\right)=b_{1}+b_{2}$, i.e. $a_{1}+x_{1}=b_{1}, a_{2}+x_{2}=b_{2}$, where $0 \leq x_{1}, 0 \leq x_{2}$. As $0 \leq x_{1}$, it holds $a_{1} \leq a_{1}+x_{1}=b_{1}$. Similarly, $a_{2} \leq b_{2}$.

Hence, for any $a, b \in M, a \leq b$ if and only if $f(a) \leq f(b)$.
We have proved that $f$ is an isomorphism of lattice-ordered monoids $(M ;+, 0, \vee, \wedge)$ and $(\bar{M} ;+, 0, \vee, \wedge)$. Since the values of the operations $\rightharpoonup$ and $\leftharpoondown$ are uniquely determined in both the $D R \ell$-monoids $M=(M ;+, 0, \vee, \wedge, \rightharpoonup$ $, \leftharpoondown)$ and $(\bar{M} ;+, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ in the same manner by means of the operation + and the order relation $\leq, D R \ell$-monoids $M=I+J$ and $\bar{M}=I \times J$ are also isomorphic.

## Remarks.

a) Let $\widetilde{I}=\{(x, 0) ; x \in I\}$ and $\widetilde{J}=\{(0, y) ; y \in J\}$. Since $I \cong \widetilde{I}$ and $J \cong \widetilde{J}$, the ideals $I$ and $J$ are (by Proposition 2 and Theorem 6) normal in $M$.
b) By Theorem 6 and Proposition 3, the set of all direct factors of a $D R \ell$ monoid $M$ is a subset of the set of all polars in $M$. In particular, $\operatorname{Sing}(M)$ and $\operatorname{Inv}(M)$ are polars in $M$. It holds

$$
(\operatorname{Sing}(M))^{\perp}=\operatorname{Inv}(M) \quad \text { and } \quad(\operatorname{Inv}(M))^{\perp}=\operatorname{Sing}(M)
$$

If $M$ is a $D R \ell$-monoid and $I \in \mathcal{C}(M)$, let us denote by $D(I)$ the join of ideals $I$ and $I^{\perp}$ in the lattice $\mathcal{C}(M)$.

Proposition 7. If $M$ is a $D R \ell$-monoid, $I \in \mathcal{C}(M)$ and $I$ is a direct factor of DRQ-monoid $D(I)$, then $I+I^{\perp} \in \mathcal{C}(D(I))$ and $I+I^{\perp}=D(I)=I \vee I^{\perp}$ (in the sense of $\mathcal{C}(M)$ ).

Proof. Since $I$ and $I^{\perp}$ are normal ideals of $D(I)$, by Lemma 21 of [8], it holds that

$$
I \vee I^{\perp}=\left\{x \in D(I) ;|x| \leq a+b, \text { where } a \in I, b \in I^{\perp}\right\} .
$$

in $\mathcal{C}(D(I))$ (consequently, also in $\mathcal{C}(M)$ ).
By Proposition 5, $a+b=b+a=a \vee b$. By Theorem 1.1.23 of [5], the underlying lattice $(M ; \vee, \wedge)$ is distributive, therefore the lattice $\left(D(I)^{+} ; \vee, \wedge\right)$ is also distributive. For this reason, from the inequality $|x| \leq a+b$, where $a \in I^{+}$and $b \in\left(I^{\perp}\right)^{+}$, it follows the existence of elements $0 \leq a_{1} \leq$ $a, 0 \leq b_{1} \leq b$ in $D(I)$ such that $|x|=a_{1} \vee b_{1}=a_{1}+b_{1}$. At the same time $a_{1} \in I, b_{1} \in I^{\perp}$ and hence $I \vee I^{\perp}=I+I^{\perp}$.

Corollary 8. If $M$ is a $D R \ell$-monoid and $I \in \mathcal{C}(M)$, then $D R \ell$-monoids $I+I^{\perp}$ and $I \times I^{\perp}$ are isomorphic if and only if $x+y=x^{\prime}+y^{\prime}$ implies $x=x^{\prime}$ and $y=y^{\prime}$, for any $x, x^{\prime} \in I$ and $y, y^{\prime} \in I^{\perp}$.

By Proposition 15 of [8], for any ideal $I$ of a $D R \ell$-monoid $M$ (and hence also for each polar in $M$ ) it holds that its polar $I^{\perp}$ is the pseudocomplement of $I$ in $\mathcal{C}(M)$. We can specify this result for the direct factors of $M$.

Proposition 9. If an ideal I of a DRধ-monoid $M$ is a direct factor in $M$, then the polar $I^{\perp}$ is the complement of $I$ in the lattice $\mathcal{C}(M)$.

Now we can prove the following proposition:
Proposition 10. If $M$ is an arbitrary $D R \ell$-monoid, then ideals $I$ of $M$, for which there exists an ideal $J \in \mathcal{C}(M)$ such that $I$ and $J$ satisfy condition 1 from Theorem 6, form a Boolean lattice. This lattice is a sublattice of $\mathcal{C}(M)$.

Proof. Let $I$ and $J$ satisfy the given assumptions. Then from distributivity of the lattice $\mathcal{C}(M)$ we obtain

$$
\begin{gathered}
(I \vee J) \cap\left(I^{\perp} \cap J^{\perp}\right)=\left(I \cap I^{\perp} \cap J^{\perp}\right) \vee\left(J \cap I^{\perp} \cap J^{\perp}\right)=\{0\} \\
(I \vee J) \vee\left(I^{\perp} \cap J^{\perp}\right)=\left(I \vee J \vee I^{\perp}\right) \cap\left(I \vee J \vee J^{\perp}\right)=M
\end{gathered}
$$

hence $I \vee J$ and $I^{\perp} \cap J^{\perp}$ satisfy condition 1 .
The remaining part of the assertion follows from Proposition 9.
Let us consider the following condition of uniqueness of decomposition for $D R \ell$-monoids $M$ :

$$
\begin{align*}
& \text { If } I, J \in \mathcal{C}(M), I \cap J=\{0\}, x, x^{\prime} \in I, y, y^{\prime} \in J \\
& \text { and } x+y=x^{\prime}+y^{\prime}, \text { then } x=x^{\prime} \text { and } y=y^{\prime} \tag{UD}
\end{align*}
$$

Theorem 11. If $D R \ell$-monoid $M$ satisfies condition (UD), then the direct factors in $M$ form a Boolean sublattice of the lattice $\mathcal{C}(M)$.

Proof. If condition (UD) holds in $M$, then $I \in \mathcal{C}(M)$ is a direct factor if and only if $I$ and $J=I^{\perp}$ satisfy condition 1 . Therefore, the theorem follows from Proposition 10.

Remark. If $G$ is an $\ell$-group, then $D R \ell$-monoids $G$ and $G^{+}$satisfy condition (UD). Hence their direct factors form a Boolean lattice.

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