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POWER INDICES OF TRACE ZERO SYMMETRIC BOOLEAN MATRICES *

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Abstract

The power index of a square Boolean matrix A is the least integer d such that A^d is a linear combination of previous nonnegative powers of A. We determine the maximum power indices for the class of $n \times n$ primitive symmetric Boolean matrices of trace zero, the class of $n \times n$ irreducible nonprimitive symmetric Boolean matrices, and the class of $n \times n$ reducible symmetric Boolean matrices of trace zero, and characterize the extreme matrices respectively.

Keywords: power index, index of convergence, period, Boolean matrix.

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1. Introduction and preliminaries

Let B_n be the set of all $n \times n$ *Boolean matrices*; that is, all $(0,1)$-matrices with the usual arithmetic except that $1+1=1$. For $A, B \in B_n$, we say A dominates B, written $A \geq B$, if $a_{ij} \geq b_{ij}$, for all i, j. If A dominates B, but $A \neq B$, we write $A > B$. The matrix A is said to be *permutationally similar* to B, written $A \cong B$, if $A = PBP^T$ for some permutation matrix P.

To each matrix $A \in B_n$ an *adjacency digraph* $G(A) = (V, E)$ is associated, where V is a set of vertices $1, 2, \ldots, n$ and the arc (i, j) from vertex i

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to vertex j belongs to E if and only if $a_{ij} = 1$. Then $A \cong B$ if and only if $G(A)$ is isomorphic to $G(B)$ for $A, B \in B_n$. If $A \in B_n$ is irreducible, then $G(A)$ is strongly connected, in this case we use $D(A)$ to denote the *diameter* of $G(A)$. The adjacency digraph of a symmetric matrix is a symmetric digraph, that is, for all $u, v \in V$, (u, v) is an arc if and only if (v, u) is. Note that a symmetric digraph G naturally corresponds to an (undirected) graph $\widetilde{G}$ by replacing each pair of arcs (u, v) and (v, u) by an edge uv, for $u \neq v$.

For any $A \in B_n$, the sequence of powers $A^0 = I, A, A^2, \ldots$ forms a finite subsemigroup $<A>$ of B_n. Thus there is a least nonnegative integer $k = k(A)$ such that $A^k = A^{k+t}$ for some $t \geq 1$, and a least positive integer $p = p(A)$ such that $A^k = A^{k+p}$. The parameters $k = k(A)$ and $p = p(A)$ are called the *index of convergerce* (or *cycle depth*) and *period* of A, respectively. A matrix $A \in B_n$ is called *primitive* if $A^{k(A)} = J_n$, the all 1's matrix in B_n. The index of convergence of a primitive matrix $A \in B_n$ is also called the *exponent* of A. It is known that A is primitive if and only if A is irreducible and $p(A) = 1$, and that J_1 is the only primitive matrix with exponent 0. In [3], Liu, McKay, Wormald, and Zhang studied the exponent of primitive symmetric matrices of trace zero.

It is well known that if $A \in B_n$ is irreducible, then $p(A)$ is the greatest common divisor of all the cycle lengths in $G(A)$, and if A is reducible and permutationally similar to a block triangular matrix with irreducible diagonal blocks, then $p(A)$ is the least common multiple of the periods of these irreducible diagonal blocks. Periods of general matrices are considered in [1], where also an efficient algorithm for computing the matrix period is described.

For $A \in B_n$, the *power index* of A, $d(A)$, is defined to be the first integer such that A^d is a linear combination of previous nonnegative powers. Then clearly, $d(A) \leq k(A)+p(A)$ for any $A \in B_n$. Gregory, Pullman and Kirkland (in [2]) proved that $d(A)$ equals the the dimension of the algebra generated by A, they determined the maximum power indices for the class of $n \times n$ symmetric matrices, the class of $n \times n$ irreducible nonprimitive symmetric matrices, and the class of $n \times n$ reducible symmetric matrices, and they characterized the extreme matrices (matrices whose power indices achieve the corresponding maximum value) in these classes.

In this paper, we first determine the maximum power indices for the class of $n \times n$ primitive symmetric matrices of trace zero, and we characterize the extreme matrices in this class. Then we consider analogous problems for irreducible nonprimitive symmetric matrices and reducible symmetric matrices of trace 0 with techniques different from those in [2]. For the case of irreducible nonprimitive symmetric matrices, which has been considered in [2], we provide a much simpler proof.

Let SB_n be the set of symmetric matrices in B_n. Let

$$C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

let $F_n \in B_n$ be the matrix with $(i, i+1)$-th and $(i+1, i)$-th entries 1 for all $i = 1, 2, \ldots, n-1$ and all other entries 0, $M_3 = C$ and for $n \geq 4$,

$$M_n = \begin{bmatrix} C & E \\ E^T & F_{n-3} \end{bmatrix},$$

where E is the $3 \times (n-3)$ matrix with only the $(3, 1)$-th entry 1 and all other entries 0. We provide two propositions that we will use later.

Proposition 1. *If $A \in B_n$ is irreducible, then $d(A) \geq D(A) + 1$; If $A \in B_n$ is primitive, then $d(A) \leq \max\{D(A) + 1, k(A)\}$.*

Proof. Suppose $A \in B_n$ is irreducible and the distance from u to v in $G(A)$ is $D(A)$. If $n = 1$, then clearly $d(A) = 1$ or 2 and $D(A) = 0$ and hence $d(A) \geq D(A) + 1$. Suppose $n \geq 2$. Then clearly $u \neq v$. Since the (u, v)-th entry of $A^{D(A)}$ is 1 and the (u, v)-th entry of A^i is 0 for each i with $0 \leq i < D(A)$, we have $d(A) \geq D(A) + 1$.

Suppose A is primitive. Then $p(A) = 1$ and hence $d(A) \leq k(A) + p(A) = k(A) + 1$. Hence $d(A) = D(A) + 1$ if $k(A) = D(A)$. If $k(A) \geq D(A) + 1$, then $A^{k(A)} = J_n = I + A + \cdots + A^{D(A)}$ and hence $d(A) \leq k(A)$. Note that $k(A) \geq D(A)$. We have $d(A) \leq \max\{D(A) + 1, k(A)\}$. ∎

Proposition 2. *Suppose $A \in SB_n$ is a primitive matrix. If there is a positive odd integer h such that for some i, the (i, i)-th entry of A^h is 0, then $d(A) \geq h + 1$.*

Proof. If $h = 1$, then clearly $A \neq I$ and hence $d(A) \geq h + 1$. Suppose $h \geq 3$. Since $A \in SB_n$ is primitive and the (i, i)-th entry of A^h is 0, we have $I < A^2$ and $A^h < J_n$. Then $I < A^2 < \ldots < A^{h-1} < J_n$ and hence $A < A^3 < \ldots < A^h < J_n$. Note that the (i, i)-th entry of A^j is 0 if the index

j, $j \leq h$, is odd, and 1 if j is even. Thus if we could express A^h as a linear combination of lower powers, the coefficients of the even powers would then all have to be 0. But $A < A^3 < \ldots < A^h$. Thus, A^h can not be expressed as a linear combination of lower powers and hence $d(A) \geq h + 1$. ■

2. Symmetric primitive matrices

In this section we consider the power indices of primitive matrices $A \in SB_n$ of trace zero.

Lemma 1 (see [3] and [4]). *Suppose $A \in SB_n$ is a primitive matrix of trace zero, $n \geq 3$. Then*

(1) *$k(A) \leq 2n - 4$, and equality holds if and only if $A \cong M_n$;*

(2) *$k(A) \neq 2n - 5$.*

■

Theorem 1 . *Suppose $A \in SB_n$ is a primitive matrix of trace zero.*

(1) *If $n = 3$, then $A = C$ and $d(A) = 2$;*

(2) *If $n \geq 4$, then $d(A) \leq 2n-4$, and equality holds if and only if $A \cong M_n$.*

Proof. The case $n = 3$ is trivial. Suppose $n \geq 4$. Note that $D(A) \leq n - 2$, since $\widetilde{G(A)}$ has no loops. By Proposition 1 and Lemma 1,

$$d(A) \leq \max\{D(A) + 1, k(A)\} \leq \max\{n - 1, 2n - 4\} = 2n - 4.$$

Suppose $d(A) = 2n - 4$. Then it follows immediately from the above inequality that $k(A) = 2n - 4$ and hence $A \cong M_n$ by Lemma 1.

On the other hand, for $n \geq 4$, the (n, n)-th entry of M_n^{2n-5} is 0. By Proposition 2, we have $d(M_n) \geq 2n - 4$ and hence $d(M_n) = 2n - 4$. ■

Note that $A \in SB_n$ is primitive if and only if $\widetilde{G(A)}$ is connected and contains at least one odd cycle, where an odd cycle is a cycle of odd length. When $A \in SB_n$ is primitive, the length of a shortest odd cycle in $\widetilde{G(A)}$ is called the *odd girth* of $\widetilde{G(A)}$. For a positive odd integer $r \leq n$, let $A \in SB_n$ be a primitive matrix such that $\widetilde{G(A)}$ has odd girth r. Then $D(A) \leq n - \frac{r+1}{2}$. Let $C_1 = [1]$ and $C_r = (c_{ij}) \in SB_r$ with $c_{ii+1} = c_{i+1i} = c_{1r} = c_{r1} = 1$ for

$i = 1, \ldots, r-1$ and all other entries 0 if $r > 1$. Let E_r be the $r \times (n-r)$ matrix with only the $(r,1)$-th entry 1 and all other entries 0. Let

$$M_{n,r} = \begin{bmatrix} C_r & E_r \\ E_r^T & F_{n-r} \end{bmatrix}.$$

(Observe that $M_{n,3} = M_n$, $M_{1,1} = [1]$, $C_3 = C$, and $E_3 = E$.) Then (see [8]):

(1) $k(A) \leq 2n - r - 1$, and equality holds if and only if $A \cong M_{n,r}$;

(2) $k(A) \neq 2n - r - 2$.

Similarly, we have the following.

Theorem 2. *Suppose $A \in SB_n$ is a primitive matrix and the odd girth of $\widetilde{G(A)}$ is r.*

(1) *If $n = r = 1$, then $A = [1]$ and $d(A) = 1$;*

(2) *If $n = r > 1$, then $A \cong C_r$ and $d(A) = r - 1$;*

(3) *If $n = 2$ and $r = 1$, then $A \cong M_{2,1}$ or $A = J_2$, and $d(A) = 2$;*

(4) *If $n \geq \max\{3, r+1\}$, then $d(A) \leq 2n - r - 1$, and equality holds if and only if $A \cong M_{n,r}$.*

∎

This result includes both Theorem 1 and a similar result on primitive symmetric matrices (with no restriction on trace) in [2].

3. Irreducible nonprimitive symmetric matrices

Suppose $A \in SB_n$ is irreducible nonprimitive with $n \geq 2$. Then $p(A) = 2$ and $\widetilde{G(A)}$ is a connected bipartite graph. The following lemma was proved in [7]. To be more self-contained, a proof is reproduced here.

Lemma 2 (see [7]). *If $A \in SB_n$ is an irreducible nonprimitive matrix with $n \geq 2$, then $k(A) = D(A) - 1 \leq n - 2$.*

Proof. Write $D = D(A)$. If $D = 1$, then $A = F_2$, $k(A) = 0$.

Suppose $D \geq 2$.

First suppose the (i,j)-th entry of A^{D+1} is 1. Then since the distance l between vertex i and vertex j in $\widetilde{G(A)}$ is at most D and $l \equiv D+1 \equiv D-1 (\bmod 2)$, we know that $l \leq D-1$, $D-1 = l+2s$ for some nonnegative integer s, and hence the (i,j)-th entry of A^{D-1} is 1. So $A^{D-1} \geq A^{D+1}$. Note that $A^{D+1} \geq A^{D-1}$ is obvious. It follows that $A^{D-1} = A^{D+1}$ and hence $k(A) \leq D-1$.

On the other hand, suppose the distance between u and v in $\widetilde{G(A)}$ is D. Then the (u,v)-th entry of A^{D-2} is 0 while the (u,v)-th entry of A^D is 1, and hence $A^{D-2} \neq A^D$. So, $k(A) \geq D-1$. ■

Theorem 3. *If $A \in SB_n$ is an irreducible nonprimitive matrix with $n \geq 2$, then $d(A) = D(A) + 1 \leq n$.*

Proof. By Lemma 2, $k(A) = D(A) - 1$. By Lemma 3 of [2], $d(A) = k(A) + p(A) = D(A) + 1$, since A is irreducible, $p(A) = 2$ and $A^2 \geq I$. ■

Note that Theorem 3 also follows from Lemma 2 and Proposition 1.

Theorem 4 (Theorem 4 of [2]). *If $A \in SB_n$ is an irreducible nonprimitive matrix with $n \geq 2$, then $d(A) \leq n$, and equality holds if and only if $A \cong F_n$.*

Proof. By Theorem 3, $d(A) = D(A) + 1 \leq n$, and equality holds if and only if $D(A) = n-1$, i.e., $A \cong F_n$. ■

Theorem 5. *The power index set of irreducible nonprimitive matrices in SB_n, $IS(n) = \{d(A) \mid A \in SB_n, A \text{ is irreducible nonprimitive}\}$, is the set $\{3, \ldots, n\}$ for $n \geq 3$, and $IS(2) = \{2\}$.*

Proof. It is obvious that $IS(2) = \{2\}$. Suppose $n \geq 3$. For any irreducible nonprimitive $A \in SB_n$, $D(A) \geq 2$. By Theorem 3, $IS(n) \subseteq \{3, \ldots, n\}$. Conversely, for any $k \in \{3, \ldots, n\}$, it is obvious that there is a matrix $A \in SB_n$ such that $\widetilde{G(A)}$ is a tree with diameter $k-1$. Then $k = d(A) \in IS(n)$. ■

4. Symmetric reducible matrices

In this section we consider the power indices of reducible matrices of trace zero in SB_n.

Theorem 6 . *Suppose $A \in SB_n$ is a reducible matrix of trace zero.*

(1) *If $n = 2$, then A is the zero matrix and $d(A) = 2$;*

(2) *If $n = 3$, then $d(A) \leq 3$, and equality holds if and only if $A \cong F_2 \oplus [0]$;*

(3) *If $n = 4$, then $d(A) \leq 3$, and equality holds if and only if $A \cong M_3 \oplus [0]$, $A \cong F_3 \oplus [0]$ or $A \cong F_2 \oplus [0] \oplus [0]$;*

(4) *If $n = 5$, then $d(A) \leq 2n - 6 = 4$, and equality holds if and only if $A \cong M_4 \oplus [0]$, $A \cong M_3 \oplus F_2$ or $A \cong F_4 \oplus [0]$;*

(5) *If $n \geq 6$, then $d(A) \leq 2n - 6$, and equality holds if and only if $A \cong M_{n-1} \oplus [0]$ or $A \cong M_{n-2} \oplus F_2$.*

Proof. The cases $n = 2, 3$ and 4 can be checked easily. Suppose $n \geq 5$. Since A is reducible, $A \cong A_1 \oplus \ldots \oplus A_m$ for irreducible matrix $A_i \in SB_{n_i}$ with $m \geq 2$. Clearly, $k(A) = \max\{k(A_i) : 1 \leq i \leq m\}$. Suppose $k(A) = k(A_{i_0})$.

Case 1. $n_{i_0} = n - 1$. Then, by Lemmas 1 and 2, $k(A_{i_0}) \leq \max\{2(n-1) - 4, n - 3\} = 2n - 6$, and equality holds if and only if $A_{i_0} \cong M_{n-1}$.

Subcase 1.1. $k(A_{i_0}) = 2n - 6$. Then $A \cong M_{n-1} \oplus [0]$. It is easy to see that $d(M_{n-1} \oplus [0]) = 2n - 6$.

Subcase 1.2. $k(A_{i_0}) < 2n - 6$. If A_{i_0} is primitive, then $p(A) = 1$ and, by Lemma 1, $k(A) \leq 2n - 8$, and hence $d(A) = k(A_{i_0}) \leq k(A) + p(A) < 2n - 6$. Now suppose A_{i_0} is nonprimitive. Then $p(A) = 2$ and by, Lemma 2, $k(A) \leq n_{i_0} - 2 = n - 3$, equality holds if and only if $A_{i_0} \cong F_{n-1}$. Hence $d(A) \leq k(A) + p(A) \leq n - 1 \leq 2n - 6$, and if equality holds, then $n = 5$ and $A \cong F_4 \oplus [0]$. It is easy to see that $d(F_4 \oplus [0]) = 4 = 2n - 6$.

Case 2. $n_{i_0} \leq n - 2$. By Lemmas 1 and 2, $k(A_{i_0}) \leq \max\{2(n-2) - 4, n - 4\} = 2n - 8$, and equality holds if and only if $A_{i_0} \cong M_{n-2}$. Thus $d(A) \leq k(A) + p(A) \leq 2n - 6$, and if equality holds, then $A \cong M_{n-2} \oplus X$ for $X \in SB_2$ with $p(X) = 2$, and hence $A \cong M_{n-2} \oplus F_2$. It is easy to see that $d(M_{n-2} \oplus F_2) = 2n - 6$. ∎

5. Closing remark

Combining Theorems 1, 4 and 6, we have the following.

Theorem 7. *Suppose A is a matrix in SB_n of trace zero.*

(1) *If $n = 1$, then $d(A) = 2$, and $A = [0]$;*

(2) *If $n = 2$, then $d(A) = 2$, and either $A = [0] \oplus [0]$ or $A = F_2$;*

(3) *If $n = 3$, then $d(A) \leq 3$, and equality holds if and only if $A \cong F_3$ or $A \cong F_2 \oplus [0]$;*

(4) *If $n = 4$, then $d(A) \leq 4$, and equality holds if and only if $A \cong M_4$ or $A \cong F_4$;*

(5) *If $n \geq 5$, then $d(A) \leq 2n-4$, and equality holds if and only if $A \cong M_n$.*

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