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BOUNDED LATTICES WITH ANTITONE INVOLUTIONS AND PROPERTIES OF *MV*-ALGEBRAS

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Abstract

We introduce a bounded lattice $\mathcal{L} = (L; \vee, \wedge, 0, 1)$, where for each $p \in L$ there exists an antitone involution on the interval $[p, 1]$. We show that there exists a binary operation $\cdot$ on L such that $\mathcal{L}$ is term equivalent to an algebra $\mathcal{A}(\mathcal{L}) = (L; \cdot, 0)$ (the assigned algebra to $\mathcal{L}$) and we characterize $\mathcal{A}(\mathcal{L})$ by simple axioms similar to that of Abbott's implication algebra. We define new operations $\oplus$ and $\neg$ on $\mathcal{A}(\mathcal{L})$ which satisfy some of the axioms of MV-algebra. Finally we show what properties must be satisfied by $\mathcal{L}$ or $\mathcal{A}(\mathcal{L})$ to obtain all axioms of MV-algebra.

Keywords: antitone involution, distributive lattice, implication algebra, MV-algebra.

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1. Bounded lattice with antitone involutions and assigned operation

A mapping $f : A \to A$ is called an *involution* if $f(f(x)) = x$ for each $x \in A$. If $(A; \leq)$ is an ordered set, then $f : A \to A$ is called *antitone* provided $x \leq y$ implies $f(y) \leq f(x)$ for all $x, y \in A$.

Let $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ be a bounded lattice, where 0 or 1 denotes the least or greatest element of $\mathcal{L}$, respectively. $\mathcal{L}$ is said to have *sectionally antitone involutions* if for every $x \in L$ there is an antitone involution on the interval $[x, 1]$; i.e. a mapping which assignes to each $a \in [x, 1]$ an element $a^x \in [x, 1]$ with $a^{xx} = a$ and $a \leq b$ entails $b^x \leq a^x$. The interval $[x, 1]$ is called a *section*.

Remarks. (a) Since $\mathcal{L}$ is a bounded lattice, $L = [0, 1]$ and hence if $\mathcal{L}$ has sectionally antitone involutions there is also an antitone involution on $\mathcal{L}$ (assigning to each $a \in L$ an element a^0) satisfying $0^0 = 1$ and $1^0 = 0$.

(b) It is evident that for each $x \in L$, the corresponding antitone involution on $[x, 1]$ is a dual automorphism of $([x, 1]; \vee, \wedge)$ and hence the De Morgan laws are satisfied, i.e. $(a \vee b)^x = a^x \wedge b^x$ and $(a \wedge b)^x = a^x \vee b^x$ hold for all $a, b \in [x, 1]$.

Analogously as in [3], we can introduce a new binary operation on L as follows:

$$x \cdot y = (x \vee y)^y \tag{A}$$

One can easily see that $\cdot$ is correctly defined since for any $x, y \in L$, we have $x \vee y \in [y, 1]$, thus the antitone involution of $x \vee y$ in the section $[y, 1]$, i.e. $(x \vee y)^y$, exists.

Lemma 1. *Let $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ be a bounded lattice with sectionally antitone involutions. Then the operation $\cdot$ defined by (A) satisfies the following identities:*

(1) $1 \cdot x = x$, $x \cdot 1 = 1$, $0 \cdot x = 1$;

(2) $(x \cdot y) \cdot y = (y \cdot x) \cdot x$;

(3) $(((x \cdot y) \cdot y) \cdot z) \cdot (x \cdot z) = 1$.

Proof.
Ad (1):

$$1 \cdot x = (1 \vee x)^x = 1^x = x,$$

$$x \cdot 1 = (x \vee 1)^1 = 1^1 = 1,$$

$$0 \cdot x = (0 \vee x)^x = x^x = 1$$

Ad (2): $(x \cdot y) \cdot y = ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y$ since $x \vee y \in [y, 1]$, thus also $(x \vee y)^y \in [y, 1]$, i.e. $(x \vee y)^y \geq y$ giving $(x \vee y)^y \vee y = (x \vee y)^y$. Analogously, $(y \cdot x) \cdot x = y \vee x = x \vee y$ proving (2).

For (3), we use the just proved $(x \cdot y) \cdot y = x \vee y$ and rewrite (3) in the form

$$((x \vee y) \cdot z) \cdot (x \cdot z) = 1. \tag{3'}$$

Since $x \leq x \vee y$, $(x \vee y) \cdot z = (x \vee y \vee z)^z$ and $x \cdot z = (x \vee z)^z$, it yields

$$((x \vee y) \cdot z) \leq x \cdot z, \tag{3''}$$

due to the antitony of involutions.

However, if $a \leq b$ in $\mathcal{L}$, then $a \cdot b = (a \vee b)^b = b^b = 1$, i.e. $(3'')$ gives

$$((x \vee y) \cdot z) \cdot (x \cdot z) = 1.$$

This is $(3')$, so we have shown the identity (3). ■

Remark. Of course, $0 \cdot 0 = 1$ and to every bounded lattice $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ with sectionally antitone involutions there can be assigned an algebra $\mathcal{A}(\mathcal{L}) = (L; \cdot, 0)$ of type $(2, 0)$, where the operation $\cdot$ is defined by (A), satisfiying the identities (1), (2), (3) of Lemma 1. Call $\mathcal{A}(\mathcal{L})$ the *assigned algebra* to $\mathcal{L}$.

We are going to show that this assignment is a one-to-one correspondence:

Theorem 1. *Let* $\mathcal{A} = (A; \cdot, 0)$ *be an algebra of type* $(2, 0)$ *and let* 1 *denote* $0 \cdot 0$. *Assume that identities* (1), (2), *and* (3) *hold. Define a binary relation* $\leq$ *on* A *as follows*

(B) $$x \leq y \quad \text{if and only if} \quad x \cdot y = 1.$$

Then $\mathcal{L}(\mathcal{A}) = (A; \leq)$ is a bounded lattice with respect to $\leq$, where 0 is the least and 1 the greatest element. Moreover, the supremum and infimum with respect to $\leq$ are the following:

$$x \vee y = (x \cdot y) \cdot y,$$

$$x \wedge y = (((x \cdot 0) \cdot (y \cdot 0)) \cdot (y \cdot 0)) \cdot 0.$$

Finally, for each $p \in A$, the mapping $a \mapsto a^p = a \cdot p$ (for $a \in [p, 1]$) is an antitone involution on the section $[p, 1]$.

Proof. Define $1 := 0 \cdot 0$ and $x \leq y$ if and only if $x \cdot y = 1$. By (1) and (2), we have $x \cdot x = (1 \cdot x) \cdot x = (x \cdot 1) \cdot 1 = 1$, thus $\leq$ is reflexive. By (1), $0 \cdot x = 1$, $x \cdot 1 = 1$ thus $0 \leq x \leq 1$ for each $x \in A$.

Suppose $x \leq y$ and $y \leq x$. Then $x \cdot y = 1$ and $y \cdot x = 1$, thus $x = 1 \cdot x = (y \cdot x) \cdot x = (x \cdot y) \cdot y = 1 \cdot y = y$ proving antisymmetry of $\leq$.

Finally, suppose $x \leq y$ and $y \leq z$. Then $x \cdot y = 1$, $y \cdot z = 1$ and $(y \cdot x) \cdot x = (x \cdot y) \cdot y = 1 \cdot y = y$.

Hence

(C) $$y \cdot z = ((y \cdot x) \cdot x) \cdot z.$$

Taking $y = 1$ in (3) and applying (1), we conclude

(D) $$z \cdot (x \cdot z) = 1 \quad \text{for each} \quad x, z \in A.$$

By the assumption $x \leq y$, i.e. $x \cdot y = 1$, and (2), we have shown $(x \cdot y) \cdot y = (y \cdot x) \cdot x = y$. Therefore, applying (3) once more, we obtain

(E) $$(y \cdot z) \cdot (x \cdot z) = (((x \cdot y) \cdot y) \cdot z) \cdot (x \cdot z) = 1.$$

But $x \cdot y = 1$ means $x \leq y$, thus, taking into account (E), we get the implication:

(F) $$x \leq y \Rightarrow y \cdot z \leq x \cdot z.$$

Applying (F) and (1) to the assumption $x \leq y$, we obtain $x \leq (y \cdot x) \cdot x$, From this and the assumption $y \leq z$, applying (C) and (F), we get

$$1 = y \cdot z = ((y \cdot x) \cdot x) \cdot z \leq x \cdot z.$$

Hence, $x \cdot z = 1$ means $x \leq z$, thus $\leq$ is also transitive, i.e. it is an partial order on A.

Define $a \vee b = (a \cdot b) \cdot b$ for $a, b \in A$. By (2) and (D) we have

$$a \cdot ((a \cdot b) \cdot b) = a \cdot ((b \cdot a) \cdot a) = 1$$

and

$$b \cdot ((a \cdot b) \cdot b) = 1,$$

thus $a \leq a \vee b$ and $b \leq a \vee b$. Suppose $a \leq c$ and $b \leq c$. Then $b \cdot c = 1$ and $c = 1 \cdot c = (b \cdot c) \cdot c = (c \cdot b) \cdot b$. This implies

$$((a \cdot b) \cdot b) \cdot c = ((a \cdot b) \cdot b) \cdot ((c \cdot b) \cdot b).$$

By using of (F) we obtain

$$a \leq c \Rightarrow c \cdot b \leq a \cdot b \Rightarrow (a \cdot b) \cdot b \leq (c \cdot b) \cdot b$$

and hence

$$((a \cdot b) \cdot b) \cdot ((c \cdot b) \cdot b) = 1,$$

i.e. $((a \cdot b) \cdot b) \cdot c = 1$ proving $a \vee b \leq c$. We have shown $a \vee b = \sup\{a, b\}$ for each $a, b \in A$.

Now, consider $p \in A$ and $a \in [p, 1]$. Then $a^{pp} = (a \cdot p) \cdot p = a \vee p = a$ thus the mapping $a \mapsto a^p$ is an involution on $[p, 1]$. By (F), $a \leq b \Rightarrow b^p \leq a^p$ for $a, b \in [p, 1]$, i.e. it is also antitone. Thus $0^0 = 1, 1^0 = 0$ and, due to De Morgan laws, $a \wedge b = (a^0 \vee b^0)^0 = (((x \cdot 0) \cdot (y \cdot 0)) \cdot (y \cdot 0)) \cdot 0$ is the infimum of $a, b \in A$. Altogether, $(A; \vee, \wedge, 0, 1)$ is a bounded lattice with sectionally antitone involutions. ■

Remark. As it was shown in the proof of Theorem 1, if $\mathcal{A} = (A; \cdot, 0)$ is an algebra satisfying the identities (1), (2), (3) for $1 = 0 \cdot 0$, then it satisfies also

$$(1') \qquad x \cdot x = 1.$$

due to $x \cdot x = (1 \cdot x) \cdot x = (x \cdot 1) \cdot 1 = 1$. Moreover,

$$(x \cdot 0) \cdot 0 = x \vee 0 = x. \tag{1''}$$

2. Assigned algebras

Let us recall from [2] that by an *MV-algebra* is meant an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2,1,0)$ satisfying the axioms:

(MV1) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,

(MV2) $a \oplus b = b \oplus a$,

(MV3) $a \oplus 0 = a$,

(MV4) $\neg\neg a = a$,

(MV5) $a \oplus \neg 0 = \neg 0$,

(MV6) $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$.

Usually we denote $\neg 0$ by 1 and we read (MV5) as $a \oplus 1 = 1$.

MV-algebras were introduced by C.C. Chang as an algebraic counterpart of Łukasiewicz multiple valued logic. In what follows, we will check which of the axioms are satisfied by the assigned algebra of a bounded lattice with sectionally antitone involutions.

Let $\mathcal{A} = (A; \cdot, 0)$ be an algebra of type $(2,0)$ and $\mathcal{M} = (M; \oplus, \neg, 0)$ be an algebra of type $(2,1,0)$. Define the so called *assigned algebras* $\mathcal{M}(\mathcal{A})$ and $\mathcal{A}(\mathcal{M})$ as follows:

given $\mathcal{A}$ define

$$a \oplus b = (a \cdot 0) \cdot b \quad \text{and} \quad \neg a = a \cdot 0 \tag{G}$$

then $\mathcal{M}(\mathcal{A}) = (A; \oplus, \neg, 0)$.

Further, for given $\mathcal{M}$, define

$$a \cdot b = \neg a \oplus b \tag{H}$$

and put $\mathcal{A}(\mathcal{M}) = (M; \cdot, 0)$. We will denote $1 = \neg 0$ in $\mathcal{M}$ and $1 = 0 \cdot 0$ in $\mathcal{A}$.

Lemma 2. *Let* $\mathcal{A} = (A; \cdot, 0)$ *be an algebra satisfying the identities* (1), (2) *and* (3). *Then the assigned algebra* $\mathcal{M}(\mathcal{A}) = (A; \oplus, \neg, 0)$ *satisfies the axioms* (MV3), (MV4), (MV5) *and* (MV6).

Proof. Take $1 = 0 \cdot 0$. Then we have

(MV3): $a \oplus 0 = (a \cdot 0) \cdot 0 = a$;

(MV4): $\neg\neg a = (a \cdot 0) \cdot 0 = a$;

(MV5): $a \oplus \neg 0 = a \oplus 1 = (a \cdot 0) \cdot 1 = 1 = \neg 0$;

(MV6): $\neg(\neg a \oplus b) \oplus b = \neg(((a \cdot 0) \cdot 0) \cdot b) \oplus b = \neg(a \cdot b) \oplus b = (((a \cdot b) \cdot 0) \cdot 0) \cdot b = (a \cdot b) \cdot b$.

Analogously, we can show $\neg(\neg b \oplus a) \oplus a = (b \cdot a) \cdot a$ and, due to (2), we obtain (MV6). ∎

However, we can show that the assigned algebra $\mathcal{M}(\mathcal{A})$ of $\mathcal{A}$ satisfying (1), (2), (3) holds a bit more:

Lemma 3. *Let* $\mathcal{A} = (A; \cdot, 0)$ *be an algebra satisfying* (1), (2), (3). *Then its assigned algebra* $\mathcal{M}(\mathcal{A})$ *satisfies the conditions:*

(i) $0 \oplus a = a$;

(ii) $\neg a \oplus a = 1 = a \oplus \neg a$;

(iii) $1 \oplus a = 1$;

(iv) $a \oplus (\neg(a \oplus b) \oplus b) = 1$;

(v) $\neg a \oplus b = 1$ *if and only if* $b \oplus \neg a = 1$;

(vi) $\neg a \oplus b = 1$ *if and only if* $\neg(a \oplus c) \oplus (b \oplus c) = 1$ *for each* $c \in A$.

Proof. It is an elementary computation to verify (i), (ii) and (iii).

We prove (iv): by the proof of Lemma 2, $\neg(\neg x \oplus y) \oplus y = (x \cdot y) \cdot y$ which is the supremum $x \vee y$ with respect to the induced order $\leq$ on $\mathcal{A}$, see Theorem 1. Hence by (B) of Theorem 1 we conclude $\neg x \oplus (\neg(\neg x \oplus y) \oplus y) = x \cdot (x \vee y) = 1$.

Taking $x = \neg a, y = b$, we obtain (iv).

Prove the remaining properties.

(v): $\neg a \oplus b = 1$ iff $a \cdot b = 1$ iff $a \leq b$ iff $a \cdot 0 = a^0 \geq b^0 = b \cdot 0$ iff $(b \cdot 0) \cdot (a \cdot 0) = 1$ iff $b \oplus \neg a = 1$.

(vi): Suppose $\neg a \oplus b = 1$. Then $a \cdot b = 1$, i.e. $a \leq b$ which yields $a \cdot 0 \geq b \cdot 0$ and hence $(a \cdot 0) \cdot c \leq (b \cdot 0) \cdot c$. Thus $((a \cdot 0) \cdot c) \cdot ((b \cdot 0) \cdot c) = 1$ which gives $\neg(a \oplus c) \oplus (b \oplus c) = 1$ for each $c \in A$.

The converse is trivial by using of $c = 0$ and applying (MV3). ■

We can prove the converse assertion:

Lemma 4. *Let* $\mathcal{M} = (M; \oplus, \neg, 0)$ *be an algebra of type* $(2, 1, 0)$ *satisfying* (MV3)-(MV6) *and* (iii). *Then* $\mathcal{M}$ *holds also* (i) *and* (ii).

Proof. Taking $b = 0$ in (MV6), we conclude $a = \neg(1 \oplus a) \oplus a$. Applying (iii), we get (i).

Taking $b = 1, a = \neg a$ in (MV6) and applying (i), we obtain $1 = a \oplus \neg a$ and for $a = \neg a$ we have also $1 = \neg a \oplus a$. Altogether, we get (ii). ■

Theorem 2. *Let* $\mathcal{M} = (M; \oplus, \neg, 0)$ *be an algebra satisfying* (MV3)-(MV6) *and* (iii)-(vi). *Then its assigned algebra* $\mathcal{A}(\mathcal{M}) = (M; \cdot, 0)$ *satisfies the identities* (1), (2) *and* (3).

Proof. Ad (1): $x \cdot 1 = \neg x \oplus 1 = 1$ by (MV5);

$1 \cdot x = \neg 1 \oplus x = 0 \oplus x = x$ since $\mathcal{M}$ satisfies (i) by Lemma 4;

$0 \cdot x = \neg 0 \oplus x = 1 \oplus x = 1$ by (iii).

Ad (2): $(x \cdot y) \cdot y = \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x = (y \cdot x) \cdot x$ by (MV6).

Ad (3): Define a binary relation ρ on M by the rule $a\rho b$ if and only if $\neg a \oplus b = 1$ (which is equivalent to $a \cdot b = 1$ due to (H)).

By (v), we have

$$(\star) \qquad a\rho b \quad \text{if and only if} \quad (\neg b)\rho(\neg a).$$

By $(\star)$ and (vi), we have

$$(\star\star) \qquad a\rho b \; \text{ if and only if } \; (\neg b \oplus c)\rho(\neg a \oplus c) \; \text{ for every } \; c \in A.$$

Finally, (iv) yields immediately

$$(\star\star\star) \qquad a\rho(a \cdot b) \cdot b$$

Taking $(\star\star\star)$ and $(\star)$, we have $x\rho(x \cdot y) \cdot y$ and hence $\neg((x \cdot y) \cdot y)\rho\neg x$, applying $(\star\star)$ we derive

$$((x \cdot y) \cdot y) \cdot z = (\neg((x \cdot y) \cdot y) \oplus z)\rho(\neg x \oplus z) = x \cdot z$$

thus, by our prescribtion for ρ, we have

$$(((x \cdot y) \cdot y) \cdot z) \cdot (x \cdot z) = 1 \text{ proving (3).}$$

■

Due to the previous results, we can assign to every algebra $\mathcal{A} = (A, \cdot, 0)$ satisfying (1), (2), (3) an algebra $\mathcal{M}(\mathcal{A}) = (A; \oplus, \neg, 0)$ satisfying (MV3)-(MV6) and (iii)-(vi) and, conversely, to every $\mathcal{M} = (M; \oplus, \neg, 0)$ satisfying (MV3)-(MV6) and (iii)-(vi) an algebra $\mathcal{A}(\mathcal{M}) = (M; \cdot, 0)$ satisfying (1), (2), (3). It is evident that $\mathcal{A}(\mathcal{M}(\mathcal{A})) = \mathcal{A}$ and $\mathcal{M}(\mathcal{A}(\mathcal{M})) = \mathcal{M}$. Due to Lemma 1, Theorem 1 and Lemma 3, we can assign also to every bounded lattice with sectionally antitone involutions $\mathcal{L}$ an algebra $\mathcal{M}(\mathcal{L}) = \mathcal{M}(\mathcal{A}(\mathcal{L}))$ satisfying (MV3)-(MV6) and (iii)-(vi) and, conversely, to every $\mathcal{M} = (M; \oplus, \neg, 0)$ satisfying (MV3)-(MV6) and (iii)-(vi) can be assigned a bounded lattice with sectionally antitone involutions $\mathcal{L}(\mathcal{M}) = \mathcal{L}(\mathcal{A}(\mathcal{M}))$. In what follows we can check by means of $\mathcal{A}(\mathcal{M})$ or $\mathcal{L}(\mathcal{M})$ whether $\mathcal{M} = (M; \oplus, \neg, 0)$ satisfies also (MV1) and/or (MV2).

Theorem 3. *Let* $\mathcal{M} = (M, \oplus, \neg, 0)$ *be an algebra of type* $(2, 1, 0)$ *satisfying* (MV3). *Then* $\mathcal{M}$ *satisfies* (MV2) *if and only if the assigned algebra* $\mathcal{A}(\mathcal{M}) = (M; \cdot, 0)$ *satisfies the identity*

$$x \cdot (y \cdot 0) = y \cdot (x \cdot 0). \tag{4}$$

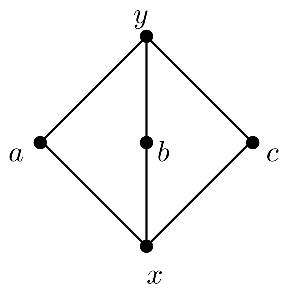
If $\mathcal{M}$ *satisfies* (MV2)-(MV6) *and* (iv), (vi), *then the induced lattice* $\mathcal{L}(\mathcal{M})$ *is distributive.*

Proof. Let $\mathcal{M}$ satisfies (MV3). If $\mathcal{A}(\mathcal{M})$ satisfies (4), then, by (MV3), $(x \cdot 0) \cdot 0 = x \oplus 0$ and $x \oplus y = (x \cdot 0) \cdot y = (x \cdot 0) \cdot ((y \cdot 0) \cdot 0) = (y \cdot 0) \cdot ((x \cdot 0) \cdot 0) = (y \cdot 0) \cdot x = y \oplus x$.

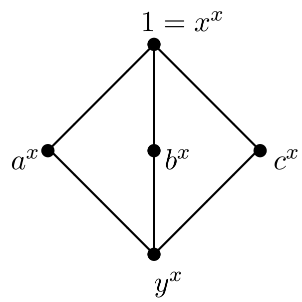
Conversely, let $\mathcal{A}$ satisfies (MV2) and (MV3). Then for each $a, b \in A$ we have $a = (a \cdot 0) \cdot 0$, $b = (b \cdot 0) \cdot 0$. Take $a \cdot 0 = x$, $b \cdot 0 = y$. Therefore, by (MV2), $a \cdot (b \cdot 0) = (x \cdot 0) \cdot ((y \cdot 0) \cdot 0) = (x \cdot 0) \cdot y = x \oplus y = y \oplus x = (y \cdot 0) \cdot x = (y \cdot 0) \cdot ((x \cdot 0) \cdot 0) = b \cdot (a \cdot 0)$ proving (4).

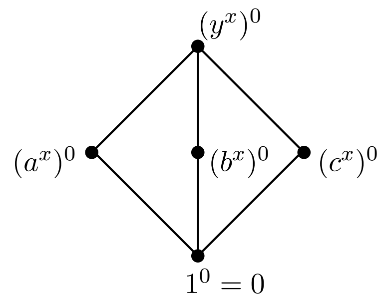
Suppose now that $\mathcal{M}$ satisfies (MV2)-(MV6) and (iv), (vi). Due to (MV2), it satisfies also (v). Due to (MV2) and (MV5), it satisfies also (iii). By Theorem 2, the assigned algebra $\mathcal{A}(\mathcal{M})$ satisfies (1), (2), (3) and, as shown previously, also (4). Hence $\mathcal{L}(\mathcal{M})$ is a lattice with sectionally antitone involutions by Theorem 1. Suppose now that $\mathcal{L}(\mathcal{M})$ is not distributive. Then it contains a sublattice isomorphic to M_3 or N_5.

(a) Let $\mathcal{L}(\mathcal{M})$ contain a sublattice



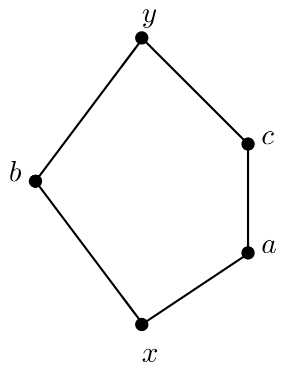
Since the antitone involutions are dual isomorphisms, it contains also the sublattices



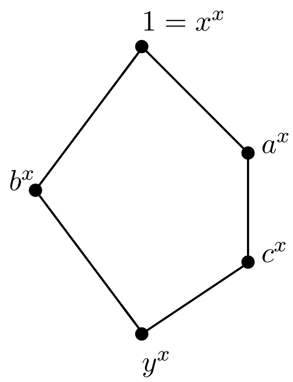


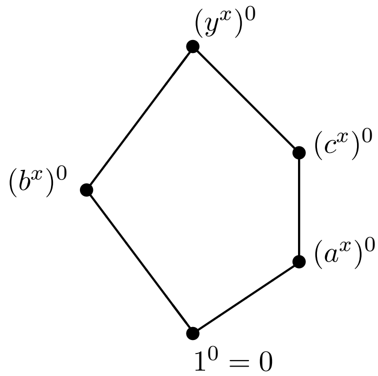
Then $(a^x)^0 \cdot (c^x \cdot 0) = (a^x)^0 \cdot (c^x)^0 = ((a^x)^0 \vee (c^x)^0)^{(c^x)^0} = ((y^x)^0)^{(c^x)^0}$ and $c^x \cdot ((a^x)^0 \cdot 0) = c^x \cdot ((a^x \cdot 0) \cdot 0) = c^x \cdot a^x = (c^x \vee a^x)^{a^x} = 1^{a^x} = a^x$. By (4), $((y^x)^0)^{(c^x)^0} = a^x$. Analogously, we can prove $((y^x)^0)^{(c^x)^0} = b^x$, thus $a^x = b^x$, i.e. $a = a^{xx} = b^{xx} = b$, a contradiction.

(b) Let $\mathcal{L}(\mathcal{M})$ contain a sublattice



Then it contains also the sublattices





Then $(a^x)^0 \cdot (b^x \cdot 0) = (a^x)^0 \cdot (b^x)^0 = ((a^x)^0 \vee (b^x)^0)^{(b^x)^0} = ((y^x)^0)^{(b^x)^0}$ and $b^x \cdot ((a^x)^0 \cdot 0) = b^x \cdot ((a^x \cdot 0) \cdot 0) = b^x \cdot a^x = (b^x \vee a^x)^{a^x} = 1^{a^x} = a^x$. Due to (4), we have $((y^x)^0)^{(b^x)^0} = a^x$. Analogously, it can be shown $((y^x)^0)^{(b^x)^0} = c^x$, thus $a = c$, a contradiction again. ■

We are going to check (MV1) by means of the properties of $\mathcal{A}(\mathcal{M})$.

Theorem 4. *Let* $\mathcal{M} = (M; \oplus, \neg, 0)$ *be an algebra of type* $(2,1,0)$ *satisfying* (MV3). *Then* $\mathcal{M}$ *satisfies* (MV1) *and* (MV2) *if and only if the assigned algebra* $\mathcal{A}(\mathcal{M}) = (M; \cdot, 0)$ *satisfies the* ***exchange identity***

$$x \cdot (y \cdot z) = y \cdot (x \cdot z). \tag{5}$$

Proof. Let $\mathcal{M}$ satisfies (MV3). Of course, the exchange identity yields (4) and, by Theorem 3, it gives (MV2). We prove (MV1). By (MV3), for each $x, y \in M$ we have $(x \cdot 0) \cdot 0 = x, (y \cdot 0) \cdot 0 = y$, thus

$(x \oplus y) \oplus z = z \oplus (x \oplus y) = (z \cdot 0) \cdot ((x \cdot 0) \cdot y) = (z \cdot 0) \cdot ((x \cdot 0) \cdot ((y \cdot 0) \cdot 0)) = (x \cdot 0) \cdot ((z \cdot 0) \cdot ((y \cdot 0) \cdot 0)) = (x \cdot 0) \cdot ((y \cdot 0) \cdot ((z \cdot 0) \cdot 0)) = (x \cdot 0) \cdot ((y \cdot 0) \cdot z) = x \oplus (y \oplus z)$.

Conversely, let $\mathcal{M}$ satisfies (MV1), (MV2), (MV3). Take $a = x \cdot 0$, $b = y \cdot 0$ and compute
$a \cdot (b \cdot c) = (x \cdot 0) \cdot ((y \cdot 0) \cdot c) = x \oplus (y \oplus c) = (x \oplus y) \oplus c = (y \oplus x) \oplus c = y \oplus (x \oplus c) = (y \cdot 0) \cdot ((x \cdot 0) \cdot c) = b \cdot (a \cdot c)$ proving (5). ∎

The following result is the final answer to our question (see also [4]):

Corollary. *Let* $\mathcal{M} = (M; \oplus, \neg, 0)$ *be an algebra satisfying* (MV3)-(MV6). *Then* $\mathcal{M}$ *is an MV-algebra if and only if its assigned algebra* $\mathcal{A}(\mathcal{M})$ *satisfies the exchange identity.*

REFERENCES

[1] J.C. Abbott, *Semi-boolean algebra*, Mat. Vestnik **4** (1967), 177–198.

[2] R.L.O. Cignoli, I.M.L. D'Ottaviano and D. Mundici, *Algebraic Foundations of Many-valued Reasoning*, Kluwer Acad. Publ., Dordrecht/Boston/London 2000.

[3] I. Chajda and R. Halaš, *Abbott's groupoids*, Multiple Valued Logic, to appear.

[4] I. Chajda, R. Halaš and J. Kühr, *Distributive lattices with sectionally antitone involutions*, preprint 2003.

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