# INFINITE INDEPENDENT SYSTEMS OF IDENTITIES <br> OF ALTERNATIVE COMMUTATIVE ALGEBRA OVER A FIELD OF CHARACTERISTIC THREE 

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#### Abstract

Let $\mathfrak{A}_{3}$ denote the variety of alternative commutative (Jordan) algebras defined by the identity $x^{3}=0$, and let $\mathfrak{S}_{2}$ be the subvariety of the variety $\mathfrak{A}_{3}$ of solvable algebras of solviability index 2 . We present an infinite independent system of identities in the variety $\mathfrak{A}_{3} \cap \mathfrak{S}_{2} \mathfrak{S}_{2}$. Therefore we infer that $\mathfrak{A}_{3} \cap \mathfrak{S}_{2} \mathfrak{S}_{2}$ contains a continuum of infinite based subvarieties and that there exist algebras with an unsolvable words problem in $\mathfrak{A}_{3} \cap \mathfrak{S}_{2} \mathfrak{S}_{2}$.

It is worth mentioning that these results were announced in 1999 in works of the international conference "Loops'99" (Prague).


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In [8] A.M. Slin'ko has formulated the question (Problem 1.129): if any variety of solvable alternative (Jordan) algebras would be finitely based. U.U. Umirbaev has got an affirmative answer to this question for alternative algebras over a field of characteristic $\neq 2,3$ (see [14]), and Yu.A. Medvedev [7] has given a negative answer for characteristic 2 . The main topic of this work is the construction of an example of an alternative commutative (Jordan) algebra also in the case of characteristic three*, which, together with

[^0]the former results, completes the settlement of Slin'ko's problem for solvable alternative algebras.

Let $(u, v, w)=u v \cdot w-u \cdot v w$ mean the associator in a considered algebra, let $\left(u_{1}, \ldots, u_{2 i-1}, u_{2 i}, u_{2 i+1}\right)=\left(\left(u_{1}, \ldots, u_{2 i-1}\right), u_{2 i}, u_{2 i+1}\right)$ and let $F$ be an infinite field of characteristic 3 . Let $\mathfrak{A}_{3}$ be the variety of alternative commutative (or Jordan) $F$-algebras, determined by the identities

$$
\begin{equation*}
x^{3}=0, \tag{1}
\end{equation*}
$$

$$
\begin{align*}
& \left(\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right),\left(x_{6}, x_{7}, x_{8}, x_{9}, x_{10}\right),\right.  \tag{2}\\
& \left.\left(x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\right),\left(x_{16}, x_{17}, x_{18}\right), x_{19}\right)=0 .
\end{align*}
$$

We denote by $\mathfrak{S}_{2}$ the variety of alternative commutative (Jordan) $F$-algebras being solvable of index 2 . We also write

$$
\begin{aligned}
\mu_{k}= & \left(\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right),\left(y_{1}, y_{2}, y_{3}\right), \ldots,\left(y_{12 k-2}, y_{12 k-1}, y_{12 k}\right),\right. \\
& \left(\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right),\left(y_{12 k+1}, y_{12 k+2}, y_{12 k+3}\right), \ldots\right. \\
& \left.\left.\ldots\left(y_{24 k-2}, y_{24 k-1}, y_{24 k}\right)\right),\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right)\right) .
\end{aligned}
$$

In this work it is proved that the system of identities $\left\{\mu_{k}=0 \mid k=1,2, \ldots\right\}$ is independent in the variety $\mathfrak{A}_{3} \cap \mathfrak{S}_{2} \mathfrak{S}_{2}$, i.e. no identity of this system follows from other identities of the system. From (1), it follows that the variety $\mathfrak{A}_{3} \cap \mathfrak{S}_{2} \mathfrak{S}_{2}$ is locally nilpotent [15]. Consequently, it is easy to show that any nilpotent variety of algebras, not necessary alternative or Jordan, has a finite basis of identities. We also note that in [6] it is shown that a lot of classic algebras being solvable of index 2, alternative and Jordan among them, have a finite basis of identities.

It follows from the main result of the work that the variety $\mathfrak{A}_{3} \cap \mathfrak{S}_{2} \mathfrak{S}_{2}$ contains a continuum of infinite based subvarieties and there are algebras with an unsolvable words problem in $\mathfrak{A}_{3} \cap \mathfrak{S}_{2} \mathfrak{S}_{2}$.

Now, we recall some notions and results from the theory of commutative Moufang loops (CML's), which can be found, e.g. in [1] (with some modifications). Any commutative Moufang loop ( $Q ; \cdot$ ) (CML $Q$, for short) is characterized by the identity $x^{2} \cdot y z=x y \cdot x z$. The inner mapping group $I(Q)$ of a CML $Q$ is the group generated by all the inner mappings $L(x, y)=L(x y)^{-1} L(x) L(y)$, where $L(x) y=x y$, of the CML $Q$. A subloop
$H$ of a CML $Q$ is normal in $Q$, if it is invariant under the group $I(Q)$. The associator (of multiplicity 1) $\left[x_{1}, x_{2}, x_{3}\right]$ of elements $x_{1}, x_{2}, x_{3} \in Q$ is determined by the equality $x_{1} x_{2} \cdot x_{3}=\left(x_{1} \cdot x_{2} x_{3}\right)\left[x_{1}, x_{2}, x_{3}\right]$. The associators of multiplicity $i$ are determined by induction: $\left[x_{1}, \ldots, x_{2 i-1}, x_{2 i}, x_{2 i+1}\right]=$ $\left[\left[x_{1}, \ldots, x_{2 i-1}\right], x_{2 i}, x_{2 i+1}\right]$. We denote by $Q_{i}$ the CML $Q$ generated by all the associators of multiplicity $i$. A CML $Q$ is centrally nilpotent of class $n$ if its lower central series has the form $Q=Q_{0} \supset Q_{1} \supset \ldots \supset Q_{n-1} \supset Q_{n}=\{1\}$. Let $f(Q)$ be the Frattini subloop of $Q$. If $Q$ is a centrally nilpotent loop, then $Q_{1} \subseteq f(Q)$. Hence a set $\left\{a_{i} \mid a_{i} \in Q, i \in I\right\}$ generates $Q$ if and only if the set $\left\{a_{i} Q_{1} \mid i \in I\right\}$ generates the abelian group $Q / Q_{1}$.

We recall (see [1], Chapter VIII, and [12]):
Lemma 1 (Bruck-Slaby's Theorem). Any finitely generated CML is centrally nilpotent.

Every CML satisfies the following identities:

$$
\begin{equation*}
[x, y, z]=[y, z, x]=[y, x, z]^{-1} \tag{3}
\end{equation*}
$$

$$
\begin{align*}
& {[[x, y, z], u, v]=[[x, u, v], y, z][x,[y, u, v], z][x, y,[z, u, v]] ;}  \tag{4}\\
& {[x y, u, v]=[x, u, v][[x, u, v], x, y] \cdot[y, u, v][[y, u, v], y, x] ;}
\end{align*}
$$

and the relation

$$
\begin{equation*}
\left[Q_{i}, Q_{j}, Q_{k}\right] \subseteq Q_{i+j+k+1} \tag{6}
\end{equation*}
$$

Let $Q$ be an arbitrary CML and let $F Q$ be its loop algebra. We remind [2] that $F Q$ is a free $F$-module with the basis $\{g \mid g \in Q\}$ and the product of elements of this basis is determined as their product in CML $Q$. We denote by $\omega H$ the ideal of algebra $F Q$, generated by all the elements $1-h$ $(h \in H)$, for a normal subloop $H$ of the CML $Q$. If $H=Q$, then $\omega Q$ is called the augmentation ideal of algebra $F Q$. Let $J$ denote the ideal of algebra $F Q$, generated by all the expressions $(u, v, w)+(v, u, w), u, v, w \in Q$. The Moufang identities hold in CML (see [1]), however these identities do not always hold in $F Q$, i.e. the algebra $F Q$ is not always alternative. (An algebra is called alternative if the identities $(x, x, y)=0$ and $(y, x, x)=0$ hold in it). It is shown in [13] that if $Q$ is a relatively free CML, then the
quotient algebra $F Q / J$ is alternative and the CML $Q$ can be embedded in the multiplication groupoid of algebra $F Q / J$. Now let $Q$ be a finite generated CML. By Lemma $1, Q$ is centrally nilpotent. Then $F\left(Q / Q_{1}\right)$ is a non-trivial associative algebra. Moreover an alternative algebra $F Q / J$ is non-trivial. CML $Q$ contains a minimal set of generators. Then, as in [12], we introduce for elements in $Q$ the notion of normal reduced word. Repeating the proof of Theorem 1 from [13] almost word for word, we prove that any finite generated CML $Q$ can be embedded in the multiplication groupoid of $F Q / J$. We identify CML $Q$ with this isomorphic image. In [13] the algebra $F Q / J$ is called a "loop algebra" of the CML $Q$ and $\omega Q / J$ (always $J \subseteq \omega Q$ ) an "augmentation ideal" (now we use these phrases in quotation marks) and are denoted by the same symbols $F Q$ and $\omega Q$, respectively.

Lemma 2 ([13]). Let $Q$ be a relatively free (or finite generated) CML and let $\phi$ be the homomorphism of "loop algebra" FQ. Then, by the homomorphism $\phi$, the image of CML $Q$ is CML.

Lemma 3 ([13]). Let $H$ be a normal subloop of relatively free (or finite generated) CML $Q$ and let $F Q, \omega Q$ be its "loop algebra" and "augmentation ideal", respectively. Then
(i) $\omega Q=\left\{\sum_{q \in Q} \lambda_{q} q \mid \sum_{q \in Q} \lambda_{q}=0\right\}$;
(ii) $F Q / \omega H \cong F(Q / H)$ and $\omega Q / \omega H \cong \omega(Q / H)$;
(iii) the "augmentation ideal" is generated as $F$-module by the elements of the form 1-q $\quad(q \in Q)$.

Lemma 4. The relatively free (or finite generated) CML $Q$ satisfies the identity

$$
\begin{equation*}
x^{3}=1 \tag{7}
\end{equation*}
$$

if and only if the "augmentation ideal" $\omega Q$ of the "loop algebra" $F Q$ satisfies the identity (1).

Proof. Let the CML $Q$ satisfy the identity (7). By (iii) of Lemma 3, any element $h$ in $\omega Q$ has the form $h=\lambda_{1} q_{1}+\ldots+\lambda_{n} q_{n}$, where $\lambda_{i} \in F, q_{i}=$ $1-g_{i}, g_{i} \in Q$. Since $F$ is a field of characteristic 3 , the equality $q^{3}=0$ follows from the equality $g^{3}=1$. Suppose that $h_{n-1}^{3}=0$, where $h_{n-1}=\lambda_{1} q_{1}+\ldots+$
$\lambda_{n-1} q_{n-1}$. Then, by the alternativity of $\omega Q$, we have $h^{3}=\left(h_{n-1}+\lambda_{n} q_{n}\right)^{3}=$ $\left(h_{n-1}^{2}+2 \lambda_{n} h_{n-1} q_{n}+\lambda_{n}^{2} q_{n}^{2}\right)\left(h_{n-1}+\lambda_{n} q_{n}\right)=h_{n-1}^{3}+\lambda_{n} h_{n-1}^{2} q_{n}+2 \lambda_{n} h_{n-1} q_{n}$. $h_{n-1}+2 \lambda_{n}^{2} h_{n-1} q_{n} \cdot q_{n}+\lambda_{n}^{2} q_{n}^{2} \cdot h_{n-1}+\lambda_{n}^{3} q_{n}^{3}=3 \lambda_{n} h_{n-1}^{2} q+3 \lambda_{n}^{2} h_{n-1} q_{n}^{2}=$ 0 . Therefore, the identity (1) holds in algebra $\omega Q$. Conversely, let the "augmentation ideal" $\omega Q$ satisfy the identity (1). Since the field $F$ is of characteristic 3 , we have $g^{3}=(1-(1-g))^{3}=1-3(1-g)+3(1-g)^{2}-(1-g)^{3}=$ 1 for $g \in Q$, as $1-g \in \omega Q$. Consequently, the CML $Q$ satisfies the identity (7). This completes the proof of Lemma 4.

Let now $A$ be an alternative commutative $F$-algebra with identity 1 and $B$ a subalgebra of $A$, satisfying (1). Then $1-B=\{1-b \mid b \in B\}$ is CML and $(1-b)^{-1}=1+b+b^{2}$.

Lemma 5. Let $A$ be an alternative commutative algebra with identity 1 and $B$ its subalgebra, satisfying (1). Then we have
$[1-u, 1-v, 1-w]=1-\left(\left(1+u+u^{2}\right) \cdot\left(1+v+v^{2}\right)\left(1+w+w^{2}\right)\right)(u, v, w)$ for all $u, v, w \in B$.

Proof. We put $1-u=a, 1-v=b$ and $1-w=c$. Then we have $[1-u, 1-v, 1-w]=(a \cdot b c)^{-1}(a b \cdot c)=(a \cdot b c)^{-1}(a b \cdot c)-(a \cdot b c)^{-1}(a \cdot b c)+1=$ $1+(a \cdot b c)^{-1}(a, b, c)=1+\left(\left((1-w)^{-1} \cdot(1-v)^{-1}\right)(1-u)^{-1}\right)(1-u, 1-v, 1-w)=$ $1-\left((1-w)^{-1}(1-v)^{-1} \cdot(1-u)^{-1}\right)(u, v, w)=1-\left(\left(1+w+w^{2}\right)\left(1+v+v^{2}\right)\right.$. $\left.\left(1+u+u^{2}\right)\right)(u, v, w)$. This completes the proof of Lemma 5 .

We write $\sum x=1+x+x^{2},\{x, y, z\}=\left(\sum x \cdot \sum y \sum z\right)(x, y, z)$, and $\left\{x_{1}, \ldots, x_{2 i-1}, x_{2 i}, x_{2 i+1}\right\}=\left\{\left\{x_{1}, \ldots, x_{2 i-1}\right\}, x_{2 i}, x_{2 i+1}\right\}$. If a CML $Q$ satisfies the identity (7), then from Lemmas 4 and 5 it follows that for $u, v, w \in$ $\omega Q[1-u, 1-v, 1-w]=1-\{u, v, w\}$, and consequently by induction, we get

$$
\begin{equation*}
\left[1-u_{1}, 1-u_{2}, \ldots, 1-u_{2 i+1}\right]=1-\left\{u_{1}, u_{2}, \ldots, u_{2 i+1}\right\} . \tag{8}
\end{equation*}
$$

In an arbitrary algebra $A$, we define by induction:

$$
A^{1}=A, A^{n}=\sum_{i+j=n} A^{i} \cdot A^{j}, A^{(1)}=A^{2}, A^{(n)}=\left(A^{(n-1)}\right)^{2}
$$

We remind that algebra $A$ is called nilpotent (respectively solvable) if there is an $n$, such that $A^{n}=0$ (respectively $A^{(n)}=0$ ). The least $n$ is called the nilpotent (respectively solvable) index. Let $f\left(x_{1}, x_{2}, \ldots, x_{i}\right)$ be a polynomial of free algebras. We say that $f\left(x_{1}, x_{2}, \ldots, x_{i}\right)=0$ is a partial identity of the algebra $A$ with the generating set $B$ if $f\left(b_{1}, b_{2}, \ldots, b_{i}\right)=0$ for any $b_{1}, b_{2}, \ldots, b_{i}$ in $B$.

Lemma 6. Let $A$ be an alternative commutative $F$-algebra and $I$ be an ideal of $A$. Then $I^{(n)}, n=1,2, \ldots$, is also an ideal of $A$.

Proof. As $F$ is a field of characteristic 3, we have
$(u, v, w)+(v, u, w)=0$,
$u v \cdot w-u \cdot v w+v u \cdot w-v \cdot u w=0$,
$2 u v \cdot w-u \cdot v w-v \cdot u w=0$,
$-u v \cdot w-u \cdot v w-v \cdot u w=0$,
and $u v \cdot w=-u \cdot v w-v \cdot u w$
for all $u, v, w \in A$.
We will prove the statement by induction on $n$. Let $x \in A, u, v \in I^{(n)}$ and assume that $I^{(n)}$ is an ideal of $A$. Then $x \cdot u v=-x u \cdot v-u \cdot x v$. But $x u, x v \in I^{(n)}$. Therefore $x I^{(n+1)} \subseteq I^{(n+1)}$. Consequently, $I^{(n+1)}$ is an ideal of $A$. The statement is proved by analogy for $n=1$. This completes the proof of Lemma 6 .

Let $L$ be a free CML that satisfies the identity (7), with a set of free generators $Y=\left\{y_{1}, y_{2}, \ldots\right\}$, where the cardinal number $|Y| \geq 5$. Let $\omega L$ be the "augmentation ideal" of the "loop algebra" $F L$. Let us consider the homomorphism $\alpha: F L \rightarrow F L /(\omega L)^{(2)}$. Then $H=\left\{h \in L \mid 1-h \in(\omega L)^{(2)}\right\}$ is the kernel of the homomorphism $\bar{\alpha}$ of the CML $L$, induced on $L$ by the homomorphism $\alpha$. By Lemma 2, the quotient loop $L / H=\bar{L}$ is a CML.

Lemma 7. Let $L$ be a free CML and $\bar{\alpha}: L \rightarrow \bar{L}$ be the homomorphism of CML defined above. Then the inequality $\left[\bar{y}_{1}, \bar{y}_{2}, \bar{y}_{3}, \bar{y}_{4}, \bar{y}_{5}\right] \neq 1$, where $\bar{y}_{i} \in \bar{Y}=\left\{\bar{\alpha} y_{i} \mid y_{i} \in Y\right\}$, holds in the CML $\bar{L}$.

Proof. First we construct an alternative commutative solvable $F$-algebra of index 2 , in which identity (1) holds and the following partial identity does not hold:

$$
\begin{equation*}
\left\{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right\}=0 . \tag{9}
\end{equation*}
$$

Let $M$ be a free $F$-module with a set of generators $X$ and let $N$ be the "exterior" algebra of module $M$, satisfying the identity $3 u v w=0$. We add a new symbol $b \notin N$ to the generators $X$ and assume that $B$ is an $F$-algebra generated by the set $X \cup\{b\}$ which besides the relations of the "exterior" algebra $N$ also satisfies the relations $b u \cdot v=b \cdot u v, b u=-u b$, for all $u, v \in X$. Let $E$ denote the $F$-submodule of module $B$ with the basis
consisting of the monomials of odd degree from $B$, except the monomials of the form $b^{2 k+1}$. Let $u, v$ be monomials from $E$. There is an odd number of generators from $X \cup\{b\}$ in the composition of $u, v$, bacause $u v=-v u$. Moreover, as there are necessary generators from $X$ in the composition of $u$, we have $u u=0$. Consequently, it follows easily that for the polynomials $s, t$ from $E$ the equalities $s t=-t s$ and $s s=0$ hold. Let

$$
\bar{u}=\left(\begin{array}{l}
u_{1} \\
u_{2} \\
u_{3}
\end{array}\right), \quad \text { where } u_{i} \in E,
$$

denote the elements of the direct product $R=E \times E \times E$. We define the product ( $\cdot$ ) on the set $R$ :

$$
\bar{u} \cdot \bar{v}=\left(\begin{array}{l}
b u_{2} v_{3}+b v_{2} u_{3}  \tag{10}\\
b u_{3} v_{1}+b v_{3} u_{1} \\
b u_{1} v_{2}+b v_{1} u_{2}
\end{array}\right) .
$$

We also define the sum ( + ) as the componentwise addition. Then, obviously, $(R,+, \cdot)$ becomes a commutative $F$-algebra and it satisfies the identity (1). Let us show that the algebra $R$ is alternative. Let $\bar{u}, \bar{v}, \bar{w} \in R$. Using (10), we obtain $(\bar{u}, \bar{v}, \bar{w})=\overline{u v} \cdot \bar{w}-\bar{u} \cdot \overline{v w}=$

$$
\begin{aligned}
& =\left(\begin{array}{l}
b u_{2} v_{3}+b v_{2} u_{3} \\
b u_{3} v_{1}+b v_{3} u_{1} \\
b u_{1} v_{2}+b v_{1} u_{2}
\end{array}\right)\left(\begin{array}{l}
w_{1} \\
w_{2} \\
w_{3}
\end{array}\right)-\left(\begin{array}{l}
u_{1} \\
u_{2} \\
u_{3}
\end{array}\right)\left(\begin{array}{l}
b v_{2} w_{3}+b v_{2} w_{3} \\
b v_{3} w_{1}+b v_{3} w_{1} \\
b v_{1} w_{2}+b v_{1} w_{2}
\end{array}\right)= \\
& =\left(\begin{array}{l}
b\left(b u_{3} v_{1}+b v_{3} u_{1}\right) w_{3}+b w_{2}\left(b u_{1} v_{2}+b v_{1} u_{2}\right)-b u_{2}\left(b v_{1} w_{2}+\right. \\
b\left(b u_{1} v_{2}+b v_{1} u_{2}\right) w_{1}+b w_{3}\left(b u_{2} v_{3}+b v_{2} u_{3}\right)-b u_{3}\left(b v_{2} w_{3}+\right. \\
b\left(b u_{2} v_{3}+b v_{2} u_{3}\right) w_{2}+b w_{1}\left(b u_{3} v_{1}+b v_{3} u_{1}\right)-b u_{1}\left(b v_{3} w_{1}+\right.
\end{array}\right. \\
& \left.\begin{array}{l}
\left.+b w_{1} v_{2}\right)-b\left(b v_{3} w_{1}+b w_{3} v_{1}\right) u_{3} \\
\left.+b w_{2} v_{3}\right)-b\left(b v_{1} w_{2}+b w_{1} v_{2}\right) u_{1} \\
\left.+b w_{3} v_{1}\right)-b\left(b v_{2} w_{3}+b w_{2} v_{3}\right) u_{2}
\end{array}\right)=\left(\begin{array}{l}
b^{2} u_{3} w_{3} v_{1}+b^{2} v_{3} u_{3} w_{1}+ \\
b^{2} u_{1} w_{1} v_{2}+b^{2} v_{1} u_{1} w_{2}+ \\
b^{2} u_{2} w_{2} v_{3}+b^{2} v_{2} u_{2} w_{3}+
\end{array}\right. \\
& +b^{2} w_{3} v_{3} u_{1}+b^{2} u_{2} w_{2} v_{1}+b^{2} v_{2} u_{2} w_{1}+b^{2} w_{2} v_{2} u_{1} \\
& \left.\begin{array}{l}
+b^{2} w_{1} v_{1} u_{2}+b^{2} u_{3} w_{3} v_{2}+b^{2} v_{3} u_{3} w_{2}+b^{2} w_{3} v_{3} u_{2} \\
+b^{2} w_{2} v_{2} u_{3}+b^{2} u_{1} w_{1} v_{3}+b^{2} v_{1} u_{1} w_{3}+b^{2} w_{1} v_{1} u_{3}
\end{array}\right) .
\end{aligned}
$$

If $\bar{u}=\bar{v}$, then the first component of the associator gets the form

$$
\begin{aligned}
& b^{2} u_{3} w_{3} u_{1}+b^{2} u_{3} u_{3} w_{1}+b^{2} w_{3} u_{3} u_{1}+b^{2} u_{2} w_{2} u_{1}+b^{2} u_{2} u_{2} w_{1}+ \\
& +b^{2} w_{2} u_{2} u_{1}=b^{2} u_{3} w_{3} u_{1}+b^{2} w_{3} u_{3} u_{1}+b^{2} u_{2} w_{2} u_{1}+b^{2} w_{2} u_{2} u_{1}= \\
& =b^{2} u_{3} w_{3} u_{1}-b^{2} u_{3} w_{3} u_{1}+b^{2} u_{2} w_{2} u_{1}-b^{2} u_{2} w_{2} u_{1}=0 .
\end{aligned}
$$

It is shown by analogy that all the other components are equal to zero. Therefore $(\bar{u}, \bar{u}, \bar{w})=0$, i.e. the algebra $R$ is alternative.

Let $A$ denote the $F$-subalgebra of the algebra $R$, generated by the elements of the form $u=\left(u_{1}, u_{2}, 0\right)$, i.e. by the elements that have the third component equal to zero. Using (10) it is easy to see that the algebra $A$ is solvable of index 2. Then $\{u, v, w\}=\left(\left(1+u+u^{2}\right) \cdot\left(1+v+v^{2}\right)(1+w+\right.$ $\left.\left.w^{2}\right)\right)(u, v, w)=(1+u+v+w) \cdot(u, v, w)$. Let us write it in detail. We put $b^{2} u_{2} w_{2} v_{1}+b^{2} v_{2} u_{2} w_{1}+b^{2} w_{2} v_{2} u_{1}=c$ and $b^{2} u_{1} w_{1} v_{2}+b^{2} v_{1} u_{1} w_{2}+b^{2} w_{1} v_{1} u_{2}=d$. We have

$$
\begin{aligned}
\{u, v, w\}= & (1+u+v+w)(u, v, w)=\left(\begin{array}{c}
c \\
d \\
0
\end{array}\right)+\left(\begin{array}{c}
u_{1}+v_{1}+w_{1} \\
u_{2}+v_{2}+w_{2} \\
0
\end{array}\right)\left(\begin{array}{l}
c \\
d \\
0
\end{array}\right)= \\
& =\left(\begin{array}{c}
c \\
d \\
c\left(u_{2}+v_{2}+w_{2}\right)+\left(u_{1}+v_{1}+w_{1}\right) d
\end{array}\right)=\left(\begin{array}{l}
c \\
d \\
a
\end{array}\right),
\end{aligned}
$$

where $a=b^{3} w_{2} v_{2} u_{1} u_{2}+b^{3} u_{2} w_{2} v_{1} v_{2}+b^{3} v_{2} u_{2} w_{1} w_{2}+b^{3} u_{1} w_{1} v_{1} u_{2}+$ $b^{3} v_{1} u_{1} w_{1} v_{2}+b^{3} w_{1} v_{1} u_{1} w_{2}$. Consequently, we can choose generating elements $u, v, w, y, z$ in $A$ such that $\{\{u, v, w\}, y, z\} \neq 0$. Therefore, (9) is not a partial identity of the algebra $A$.

As we have shown before in Lemma 5 , the set $C=1-A$ is a CML. Since $F$ is a field of characteristic 3 , we have $(1-a)^{3}=1-3 a+3 a^{2}-a^{3}=1$ for $a \in A$, i.e. the CML $C$ satisfies the identity (7). It is obvious that $A=\omega C$. We suppose that the CML $C$ has the same number of generating elements as the free CML $L$ and let $C=L / H$. The loop homomorphism
$L \rightarrow C$ induces the algebra homomorphisms $\omega L \rightarrow \omega C, \omega L /(\omega L)^{(2)} \rightarrow$ $\omega C /(\omega C)^{(2)}=A / A^{(2)}$. Since the algebra $A$ is solvable of index 2 , then $A^{(2)}=(0)$ and we have the algebra homomorphism $\omega L /(\omega L)^{(2)} \rightarrow A$. As the partial identity (9) does not hold also in the algebra $A$, the same holds in algebra $\omega L /(\omega L)^{(2)}$. But $F L /(\omega L)^{(2)} \cong F(L / H)$. Then it follows from (8) that the inequality $\left[\bar{y}_{1}, \bar{y}_{2}, \bar{y}_{3}, \bar{y}_{4}, \bar{y}_{5}\right] \neq 1$ holds in the CML $\bar{L}=L / H$. This completes the proof of Lemma 7 .

Let $Q_{n}$ denote the CML, constructed in the proof of Theorem of [11]. It is the semi-direct product of $B_{n}$ and $G_{n}$, where $G_{n}$ is the free centrally nilpotent CML of class 2 with free generators $g_{1}, g_{2}, \ldots, g_{24 n}, B_{n}$ is the centrally nilpotent CML of class 2 , generated by the set $\left\{\alpha d \mid \alpha \in I\left(G_{n}\right)\right\}$, where $d \notin G_{n}, I\left(G_{n}\right)$ is the inner mapping group of CML $G_{n}$. It follows from the definition of the CML $B_{n}$ and $G_{n}$ that they satisfy the identity

$$
\begin{equation*}
\left[\left[x_{1}, x_{2}, x_{3}\right], x_{4}, x_{5}\right]=1 . \tag{11}
\end{equation*}
$$

Since the CML $Q_{n}$ is finite, it is centrally nilpotent by Lemma 1 . In the CML $Q_{n}$ the identities (7) and

$$
\begin{align*}
\lambda= & \lambda\left(x_{1}, x_{2}, \ldots, x_{19}\right)=\left[\left[x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right],\left[x_{6}, x_{7}, x_{8}, x_{9}, x_{10}\right],\right. \\
& {\left.\left[x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\right],\left[x_{16}, x_{17}, x_{18}\right], x_{19}\right]=1 } \tag{12}
\end{align*}
$$

hold, and for $k \neq n, k \neq 2 n$ also the following identities are satisfied:

$$
\left\{\begin{align*}
\tau_{k}= & \tau_{k}\left(y_{1}, y_{2}, y_{3}, y_{4}, y_{5} ; x_{1}, x_{2}, \ldots, x_{24 k}\right)=  \tag{13}\\
= & {\left[\left[y_{1}, y_{2}, y_{3}, y_{4}, y_{5}\right],\left[x_{1}, x_{2}, x_{3}\right], \ldots,\left[x_{12 k-2}, x_{12 k-1}, x_{12 k}\right]\right.} \\
& {\left[\left[y_{1}, y_{2}, y_{3}, y_{4}, y_{5}\right],\left[x_{12 k+1}, x_{12 k+2}, x_{12 k+3}\right], \ldots\right.} \\
& \left.\left.\ldots,\left[x_{24 k-2}, x_{24 k-1}, x_{24 k}\right]\right],\left[y_{1}, y_{2}, y_{3}, y_{4}, y_{5}\right]\right]=1,
\end{align*}\right.
$$

$$
\left\{\begin{align*}
\nu_{k}= & \nu_{k}\left(y_{2}, y_{3}, y_{5} ; x_{1}, x_{2}, \ldots, x_{24 k} ; u_{1}, u_{2}, \ldots, u_{12}\right)=  \tag{14}\\
= & {\left[\left[u_{1}, y_{2}, y_{3}, u_{2}, y_{5}\right],\left[x_{1}, x_{2}, x_{3}\right], \ldots,\left[x_{12 k-2}, x_{12 k-1}, x_{12 k}\right],\right.} \\
& {\left[\left[u_{3}, y_{2}, y_{3}, u_{4}, y_{5}\right],\left[x_{12 k+1}, x_{12 k+2}, x_{12 k+3}\right], \ldots\right.} \\
& \left.\left.\ldots,\left[x_{24 k-2}, x_{24 k-1}, x_{24 k}\right]\right],\left[u_{5}, y_{2}, y_{3}, u_{6}, y_{5}\right]\right] \times \\
& \times\left[\left[u_{7}, y_{2}, y_{3}, u_{8}, y_{5}\right],\left[x_{1}, x_{2}, x_{3}\right], \ldots,\left[x_{12 k-2}, x_{12 k-1}, x_{12 k}\right],\right. \\
& {\left[\left[u_{9}, y_{2}, y_{3}, u_{10}, y_{5}\right],\left[x_{12 k+1}, x_{12 k+2}, x_{12 k+3}\right], \ldots\right.} \\
& \left.\left.\ldots,\left[x_{24 k-2}, x_{24 k-1}, x_{24 k}\right]\right],\left[u_{11}, y_{2}, y_{3}, u_{12}, y_{5}\right]\right]=1,
\end{align*}\right.
$$

where either

1) $u_{1}=y_{s}, u_{3}=y_{t}, u_{7}=y_{t}, u_{11}=y_{s}, u_{2}=u_{4}=u_{6}=u_{8}=u_{10}=$ $u_{12}=y_{4}$, and $u_{5}=u_{9}=y_{1}$
or 2) $u_{2}=y_{s}, u_{4}=y_{t}, u_{8}=y_{t}, u_{12}=y_{s}, u_{1}=u_{3}=u_{5}=u_{7}=u_{9}=u_{11}=$ $y_{1}$, and $u_{6}=u_{10}=y_{4}$
or 3) $u_{1}=y_{s}, u_{5}=y_{t}, u_{7}=y_{t}, u_{9}=y_{s}, u_{2}=u_{4}=u_{6}=u_{8}=u_{10}=u_{12}=$ $y_{4}$, and $u_{3}=u_{11}=y_{1}$
or 4) $u_{2}=y_{s}, u_{6}=y_{t}, u_{8}=y_{t}, u_{10}=y_{s}, u_{1}=u_{3}=u_{5}=u_{7}=u_{9}=u_{11}=$ $y_{1}$, and $u_{4}=u_{12}=y_{4}$
or 5) $u_{3}=y_{s}, u_{5}=y_{t}, u_{11}=y_{t}, u_{9}=y_{s}, u_{2}=u_{4}=u_{6}=u_{8}=u_{10}=$ $u_{12}=y_{4}$, and $u_{1}=u_{7}=y_{1} ;$
or 6) $u_{4}=y_{s}, u_{6}=y_{t}, u_{12}=y_{t}, u_{10}=y_{s}, u_{1}=u_{3}=u_{5}=u_{7}=u_{9}=$ $u_{11}=y_{1}$, and $u_{2}=u_{8}=y_{4}$
and the following inequality holds:

$$
\left\{\begin{array}{l}
{\left[\left[y_{1}, y_{2}, y_{3}, y_{4}, y_{5}\right],\left[x_{1}, x_{2}, x_{3}\right], \ldots,\left[x_{12 n-2}, x_{12 n-1}, x_{12 n}\right],\right.}  \tag{15}\\
{\left[\left[y_{1}, y_{2}, y_{3}, y_{4}, y_{5}\right],\left[x_{12 n+1}, x_{12 n+2}, x_{12 n+3}\right], \ldots\right.} \\
\left.\left.\ldots,\left[x_{24 n-2}, x_{24 n-1}, x_{24 n}\right]\right],\left[y_{1}, y_{2}, y_{3}, y_{4}, y_{5}\right]\right] \neq 1 .
\end{array}\right.
$$

The identities (1), (12), (13) and the inequality (15) are proved in [11]. The identity (14) is proved as the proof of the identity $w_{k}=1$ of the Theorem of [11], using Lemma 9 of [11].

By construction, the CML $Q_{n}$ is a semi-direct product of the CML's $B_{n}$ and $G_{n}$. Then, by (ii) of Lemma $3, F Q_{n} / \omega B_{n} \cong F G_{n}, \omega Q_{n} / \omega B_{n} \cong \omega G_{n}$, where $F B_{n}$ and $F G_{n}$ are subalgebras of "loop algebra" $F Q_{n}$, and $\omega G_{n}$ is the "augmentation ideal" of "loop algebra" $F G_{n}$. We will consider the homomorphism $\eta: F Q_{n} \rightarrow F Q_{n} /\left(\omega B_{n}\right)^{(2)}$. By Lemma 2, $\eta$ induces the homomorphism $\bar{\eta}$ of the CML $Q_{n}$.

We will show that $\bar{\eta}$ is the isomorphism of the CML $Q_{n}$. Indeed, let $\alpha$ and $\beta$ be the homomorphisms of CML $Q_{n}$ and $\bar{\eta} Q_{n}$, respectively, which, by Lemma 2, are induced by the homomorphisms of algebras $F Q_{n} \rightarrow F Q_{n} / \omega B_{n}$ and $F Q_{n} /\left(\omega B_{n}\right)^{(2)} \rightarrow\left(F Q_{n} /\left(\omega B_{n}\right)^{(2)}\right) /\left(\omega B_{n} /\left(\omega B_{n}\right)^{(2)}\right)$. By (i) of Lemma 3, $G_{n} \cap \omega B_{n}=\emptyset$. Then it follows from the relations $F G_{n} \cong F Q_{n} / \omega B_{n} \cong\left(F Q_{n} /\left(\omega B_{n}\right)^{(2)}\right) /\left(\omega B_{n} /\left(\omega B_{n}\right)^{(2)}\right)$ that $G_{n} \cong \alpha G_{n}$ and $\alpha G_{n} \cong \beta\left(\bar{\eta} G_{n}\right)$. Therefore, $\left|G_{n}\right|=\left|\beta\left(\bar{\eta} G_{n}\right)\right|$. We suppose that the homomorphism $\bar{\eta}$ of CML $G_{n}$ is not an isomorphism. By construction, the CML $G_{n}$ is finite. Then $\left|\bar{\eta} G_{n}\right|<\left|G_{n}\right|$ and $\left|\beta\left(\bar{\eta} G_{n}\right)\right|<\left|G_{n}\right|$. We have obtained a contradiction. Hence $\bar{\eta}$ is an isomorphism of CML $G_{n}$.

By construction, the CML $B_{n}$ is generated by the set $\left\{\varphi b \mid \varphi \in I\left(G_{n}\right)\right\}$, where $b \notin G_{n}$, and $I\left(G_{n}\right)$ is the inner mapping group of CML $G_{n}$. It is determined by the identities (7) and (11) and by the relations of the form $\left[\varphi_{1} b, \varphi_{2} b, \varphi_{3} b\right]=1$ or $\left[\varphi_{1} b, \varphi_{2} b, \varphi_{3} b\right]=t\left(\varphi_{1}, \varphi_{2}, \varphi_{3}\right) \neq 1$, where $\varphi_{1}, \varphi_{2}, \varphi_{3} \in$ $I\left(G_{n}\right)$. By Lemma 4 and (8), the system of identities (7), (11) is equivalent to the system consisting of the identity (1) and the partial identity (9) of the algebra $\omega B_{n}$. The meaning of the elements $t\left(\varphi_{1}, \varphi_{2}, \varphi_{3}\right)$ depends only on the inner mappings $\varphi_{1}, \varphi_{2}, \varphi_{3}$ of the CML $G_{n}$. Then, as $G_{n} \cong \bar{\eta} G_{n}$, the generators $\varphi b$ and the elements $t\left(\varphi_{1}, \varphi_{2}, \varphi_{3}\right)$ are not mapped on the unit of the CML $\bar{\eta} G_{n}$ under the homomorphism $\eta$. We suppose that $B_{n}=L / H$, where $L$ is a free CML, as in Lemma 7. By (ii) of Lemma 3,
$F B_{n} \cong F L / \omega H$. We will consider the homomorphism $F L \rightarrow F L /(\omega L)^{(2)}$, taking into account Lemma 6. By Lemma 2, a normal subloop $K$ of the CML $L$ corresponds to this homomorphism. We have $F L+\left(\omega H+(\omega L)^{(2)}\right)=$ $F L+\omega H+\left(\omega H+(\omega L)^{(2)}\right)=F L+\omega H+\left(\omega B_{n}\right)^{(2)}=F B_{n}+\left(\omega B_{n}\right)^{(2)}$. It means that the ideal $\omega H+(\omega L)^{(2)}$ is the kernel of the homomorphism $F L \rightarrow F B_{n} /\left(\omega B_{n}\right)^{(2)}$. The normal subloop $H K$ will be the kernel of the homomorphism $L \rightarrow \bar{\eta} B_{n}$, which by Lemma 2 , is induced by the homomorphism $F L \rightarrow F B_{n} /\left(\omega B_{n}\right)^{(2)}$. We suppose that the determining relation $\left[\varphi_{1} b, \varphi_{2} b, \varphi_{3} b\right] \neq 1$ of CML $B_{n}$ corresponds to the associator $\left[y_{1}, y_{2}, y_{3}\right]$ of the CML $L$ under the homomorphism $L \rightarrow B_{n}$. Then $\left[y_{1}, y_{2}, y_{3}\right] \notin H$ and, by Lemma $7,\left[y_{1}, y_{2}, y_{3}\right] \notin K$. Therefore, $\left[y_{1}, y_{2}, y_{3}\right] \notin H K$. It means that the determining relation $\left[\varphi_{1}, \varphi_{2} b, \varphi_{3} b\right]$ of the CML $B_{n}$ is not mapped on the unit of the CML $\bar{\eta} B_{n}$ under the homomorphism $B_{n} \rightarrow \bar{\eta} B_{n}$. Therefore the CMLs $B_{n}$ and $\bar{\eta} B_{n}$, which have the same determining relations, are isomorphic. Consequently, $\bar{\eta} Q_{n} / \bar{\eta} B_{n} \cong \bar{\eta} G_{n}$ and $\bar{\eta}$ is the isomorphism of the CML $Q_{n}$. Moreover, $\eta\left(F Q_{n}\right) / \eta\left(\omega B_{n}\right) \cong \eta\left(F G_{n}\right)$. As the homomorphism $\eta$ keeps the sum of the coefficients of the polynomials, by (i) of Lemma 3, $\eta\left(\omega Q_{n}\right) / \eta\left(\omega B_{n}\right) \cong \eta\left(\omega G_{n}\right)$.

Taking into account Lemma 6, we can consider the homomorphism $\xi: \eta\left(F Q_{n}\right) \rightarrow \eta\left(F G_{n}\right) /\left(\eta\left(F G_{n}\right)\right)^{(2)}$. Let $\bar{\xi}$ be the homomorphism of the CML $\bar{\eta} Q_{n}$ which, by Lemma 2, is induced by the homomorphism $\xi$. By the construction of the CML $G_{n}$ and then the CML $\bar{\eta} G_{n}$ is also a free centrally nilpotent of class 2. It follows from Lemma 7 that the CML $\bar{\xi} \bar{\eta} G_{n}$ also has such a property. Therefore, $\bar{\xi} \bar{\eta} G_{n} \cong \bar{\eta} G_{n}$.

Further, it follows from the relation $\eta\left(F Q_{n}\right) / \eta\left(\omega B_{n}\right) \cong \eta\left(F G_{n}\right)$ that zero of the algebra $\eta\left(F Q_{n}\right) / \eta\left(\omega B_{n}\right)$, namely $\eta\left(\omega B_{n}\right)$, is mapped on zero of the second algebra by the composition of homomorphisms: $\eta\left(F Q_{n}\right) \rightarrow \eta\left(F Q_{n}\right) / \eta\left(\omega B_{n}\right) \rightarrow \eta\left(F G_{n}\right) /\left(\eta\left(F G_{n}\right)\right)^{(2)}$. It means that the homomorphism $\xi$ does not impose any restrictions on the ideal $\eta\left(\omega B_{n}\right)$. By Lemma 2, the homomorphism $\eta\left(F Q_{n}\right) \rightarrow \eta\left(F Q_{n}\right) / \eta\left(\omega B_{n}\right)$ induces the homomorphism of the CML $\bar{\eta} Q_{n}$, whose kernel is the normal subloop $\left\{g \in \bar{\eta} Q_{n} \mid 1-g \in \eta\left(\omega B_{n}\right)\right\}=\bar{\eta} B_{n}$. Therefore, we infer that $\xi$ is an isomorphism of CML $\bar{\eta} B_{n}$. Consequently, we have the isomorphisms $\bar{\xi}\left(\bar{\eta} Q_{n}\right) \cong \bar{\eta} Q_{n} \cong Q_{n}$. Further we will identify the CML $\bar{\xi} \bar{\eta} Q_{n}$ with the $\mathrm{CML} Q_{n}$. We put $\xi \eta\left(\omega Q_{n}\right)=\omega \bar{Q}_{n}, \xi \eta\left(\omega B_{n}\right)=\omega \bar{B}_{n}$, and $\xi \eta\left(\omega G_{n}\right)=\omega \bar{G}_{n}$. As for the homomorphism $\eta$ it is proved that $\omega \bar{Q}_{n} / \omega \bar{B}_{n} \cong \omega \bar{G}_{n}$. It is obvious that $\omega \bar{B}_{n} \in \mathfrak{S}_{2}$ and $\omega \bar{G}_{n} \in \mathfrak{S}_{2}$. Then the algebra $\omega \bar{Q}_{n}$ belongs to the product of the varieties $\mathfrak{S}_{2} \mathfrak{S}_{2}$.

Further, $g_{i}, r_{i}$, $s_{i}$ will denote the elements of $\operatorname{CML} \bar{Q}_{n}$. Let $a_{i}=1-g_{i}, b_{i}=$ $1-r_{i}, c_{i}=1-s_{i}$. We also write

$$
\left\{\begin{array}{l}
\theta=\theta\left(x_{1}, x_{2}, \ldots, x_{19}\right)=\left\{\left\{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right\},\left\{x_{6}, x_{7}, x_{8}, x_{9}, x_{10}\right\},\right.  \tag{16}\\
\left.\left\{x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\right\},\left\{x_{16}, x_{17}, x_{18}\right\}, x_{19}\right\} ; \\
\xi_{k}=\xi_{k}\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5} ; y_{1}, y_{2}, \ldots, y_{24 k}\right)= \\
\left\{\left\{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right\},\left\{y_{1}, y_{2}, y_{3}\right\}, \ldots,\left\{y_{12 k-2}, y_{12 k-1}, y_{12 k}\right\},\right. \\
\left\{\left\{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right\},\left\{y_{12 k+1}, y_{12 k+2}, y_{12 k+3}\right\}, \ldots\right. \\
\left.\left.\ldots,\left\{y_{24 k-2}, y_{24 k-1}, y_{24 k}\right\}\right\},\left\{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right\}\right\} ; \\
\eta_{k}=\eta_{k}\left(x_{2}, x_{3}, x_{5} ; y_{1}, y_{2}, \ldots, y_{24 k} ; z_{1}, z_{2}, \ldots, z_{12}\right)= \\
\left\{\left\{z_{1}, x_{2}, x_{3}, z_{2}, x_{5}\right\},\left\{y_{1}, y_{2}, y_{3}\right\}, \ldots,\left\{y_{12 k-2}, y_{12 k-1}, y_{12 k}\right\},\right. \\
\left\{\left\{z_{3}, x_{2}, x_{3}, z_{4}, x_{5}\right\},\left\{y_{12 k+1}, y_{12 k+2}, y_{12 k+3}\right\}, \ldots\right. \\
\left.\left.\ldots,\left\{y_{24 k-2}, y_{24 k-1}, y_{24 k}\right\}\right\},\left\{z_{5}, x_{2}, x_{3}, z_{6}, x_{5}\right\}\right\}- \\
-\left\{\left\{z_{7}, x_{2}, x_{3}, z_{8}, x_{5}\right\},\left\{y_{1}, y_{2}, y_{3}\right\}, \ldots,\left\{y_{12 k-2}, y_{12 k-1}, y_{12 k}\right\},\right. \\
\left\{z_{9}, x_{2}, x_{3}, z_{10}, x_{5}\right\},\left\{\left\{z_{11}, x_{2}, x_{3}, z_{12}, x_{5}\right\},\right. \\
\left.\left\{y_{12 k+1}, y_{12 k+2}, y_{12 k+3}\right\}, \ldots,\left\{y_{24 k-2}, y_{24 k-1}, y_{24 k}\right\}\right\},
\end{array}\right.
$$

where either

1) $z_{1}=x_{s}, z_{3}=x_{t}, z_{7}=x_{t}, z_{11}=x_{s}, z_{2}=z_{4}=z_{6}=z_{8}=z_{10}=z_{12}=$ $x_{4}$, and $z_{5}=z_{9}=x_{1}$
or 2) $z_{2}=x_{s}, z_{4}=x_{t}, z_{8}=x_{t}, z_{12}=x_{s}, z_{1}=z_{3}=z_{5}=z_{7}=z_{9}=z_{11}=$ $x_{1}$, and $z_{6}=z_{10}=x_{4}$
or 3) $z_{1}=x_{s}, z_{5}=x_{t}, z_{7}=x_{t}, z_{9}=x_{s}, z_{2}=z_{4}=z_{6}=z_{8}=z_{10}=z_{12}=$ $x_{4}$, and $z_{3}=z_{11}=x_{1}$
```
or 4) \(z_{2}=x_{s}, z_{6}=x_{t}, z_{8}=x_{t}, z_{10}=x_{s}, z_{1}=z_{3}=z_{5}=z_{7}=z_{9}=z_{11}=\)
        \(x_{1}\), and \(z_{4}=z_{12}=x_{4}\)
or 5) \(z_{3}=x_{s}, z_{5}=x_{t}, z_{11}=x_{t}, z_{9}=x_{s}, z_{2}=z_{4}=z_{6}=z_{8}=z_{10}=z_{12}=\)
        \(x_{4}\), and \(z_{1}=z_{7}=x_{1}\)
or 6) \(z_{4}=x_{s}, z_{6}=x_{t}, z_{12}=x_{7}, z_{10}=x_{s}, z_{1}=z_{3}=z_{5}=z_{7}=z_{9}=z_{11}=\)
        \(x_{1}\), and \(z_{2}=z_{8}=x_{4}\).
```

Let $k \neq n, 2 n$ and let $I$ denote the ideal of the algebra $\omega \bar{Q}_{n}$, generated by the expressions of the forms:

$$
\begin{aligned}
& \theta\left(\alpha_{1} a_{1}, \ldots, \alpha_{19} a_{19}\right) \\
& \xi_{k}\left(\alpha_{1} a_{1}, \alpha_{2} a_{2}, \alpha_{3} a_{3}, \alpha_{4} a_{4}, \alpha_{5} a_{5} ; \beta_{1} b_{1}, \beta_{2} b_{2}, \ldots, \beta_{24 k} b_{24 k}\right) \\
& \bar{\eta}_{k}=\eta_{k}\left(\alpha_{2} a_{2}, \alpha_{3} a_{3}, \alpha_{5} a_{5} ; \beta_{1} b_{1}, \beta_{2} b_{2}, \ldots, \beta_{24 k} b_{24 k} ; \gamma_{1} c_{1}, \gamma_{2} c_{2}, \ldots, \gamma_{12} c_{12}\right)
\end{aligned}
$$

where
either 1) $c_{1}=a_{s}, c_{3}=a_{t}, c_{7}=a_{t}, c_{11}=a_{s}, c_{2}=c_{4}=c_{6}=c_{8}=c_{10}=$ $c_{12}=a_{4}$, and $c_{5}=c_{9}=a_{1}$
or 2) $c_{2}=a_{s}, c_{4}=a_{t}, c_{8}=a_{t}, c_{12}=a_{s}, c_{1}=c_{3}=c_{5}=c_{7}=c_{9}=c_{11}=$ $a_{1}$, and $c_{6}=c_{10}=a_{4}$
or 3) $c_{1}=a_{s}, c_{5}=a_{t}, c_{7}=a_{t}, c_{9}=a_{s}, c_{2}=c_{4}=c_{6}=c_{8}=c_{10}=c_{12}=$ $a_{4}$, and $c_{3}=c_{11}=a_{1}$
or 4) $c_{2}=a_{s}, c_{6}=a_{t}, c_{8}=a_{t}, c_{10}=a_{s}, c_{1}=c_{3}=c_{5}=c_{7}=c_{9}=c_{11}=$ $a_{1}$, and $c_{4}=c_{12}=a_{4}$
or 5) $c_{3}=a_{s}, c_{5}=a_{t}, c_{11}=a_{t}, c_{9}=a_{s}, c_{2}=c_{4}=c_{6}=c_{8}=c_{10}=$ $c_{12}=a_{4}$, and $c_{1}=c_{7}=a_{1}$
or 6) $c_{4}=a_{s}, c_{6}=a_{t}, c_{12}=a_{t}, c_{10}=a_{s}, c_{1}=c_{3}=c_{5}=c_{7}=c_{9}=$ $c_{11}=a_{1}$, and $c_{2}=c_{8}=a_{4}$.

Let $u_{i}, v_{i}, w_{i}$ denote the images of the elements $a_{i}, b_{i}, c_{i}$, respectively, under the homomorphism $\omega \bar{Q}_{n} \rightarrow \omega \bar{Q}_{n} / I$. Then the following equalities hold in the algebra $\omega \bar{Q}_{n} / I$ :

$$
\left\{\begin{array}{l}
\theta\left(\alpha_{1} u_{1}, \alpha_{2} u_{2}, \ldots, \alpha_{19} u_{19}\right)=0,  \tag{17}\\
\xi_{k}\left(\alpha_{1} u_{1}, \alpha_{2} u_{2}, \alpha_{3} u_{3}, \alpha_{4} u_{4}, \alpha_{5} u_{5} ; \beta_{1} v_{1}, \beta_{2} v_{2}, \ldots, \beta_{24 k} v_{24 k}\right)=0, \\
\eta_{k}=\eta_{k}\left(\alpha_{2} u_{2}, \alpha_{3} u_{3}, \alpha_{5} u_{5} ; \beta_{1} v_{1}, \beta_{2} v_{2}, \ldots, b_{24 k} v_{24 k} ;\right. \\
\left.\gamma_{1} w_{1}, \gamma_{2} w_{2}, \ldots, \gamma_{12} w_{12}\right)=0,
\end{array}\right.
$$

where
either 1) $w_{1}=u_{s}, w_{3}=u_{t}, w_{7}=u_{t}, w_{11}=u_{s}, w_{2}=w_{4}=w_{6}=w_{8}=$ $w_{10}=w_{12}=u_{4}$, and $w_{5}=w_{9}=u_{1}$
or 2) $w_{2}=u_{s}, w_{4}=u_{t}, w_{8}=u_{t}, w_{12}=u_{s}, w_{1}=w_{3}=w_{5}=w_{7}=w_{9}=$ $w_{11}=u_{1}$, and $w_{6}=w_{10}=u_{4}$
or 3) $w_{1}=u_{s}, w_{5}=u_{t}, w_{7}=u_{t}, w_{9}=u_{s}, w_{2}=w_{4}=w_{6}=w_{8}=w_{10}=$ $w_{12}=u_{4}$, and $w_{3}=w_{11}=u_{1}$
or 4) $w_{2}=u_{s}, w_{6}=u_{t}, w_{8}=u_{t}, w_{10}=u_{s}, w_{1}=w_{3}=w_{5}=w_{7}=w_{9}=$ $w_{11}=u_{1}$, and $w_{4}=w_{12}=u_{4}$
or 5) $w_{3}=u_{s}, w_{5}=u_{t}, w_{11}=u_{t}, w_{9}=u_{s}, w_{2}=w_{4}=w_{6}=w_{8}=$ $w_{10}=w_{12}=u_{4}$, and $w_{1}=w_{7}=u_{1}$
or 6) $w_{4}=u_{s}, w_{6}=u_{t}, w_{12}=u_{t}, w_{10}=u_{s}, w_{1}=w_{3}=w_{5}=w_{7}=$ $w_{9}=w_{11}=u_{1}$, and $w_{2}=w_{8}=u_{4}$.

By Lemma 2, the image $\overline{\bar{Q}}_{n}$ of the CML $\bar{Q}_{n}$ under the homomorphism $F \bar{Q}_{n} \rightarrow F \bar{Q}_{n} / I$ is CML.

Lemma 8. The identity $\tau_{n}=1$ does not hold in the $C M L \overline{\bar{Q}}_{n}$.

Proof. It follows from Lemma 4 that the identity (1) holds in the algebra $\omega \bar{Q}_{n}$. Then, as shown before Lemma 5, the set $T=1-\omega \bar{Q}_{n}$ forms a CML. It is obvious that $\bar{Q}_{n} \subseteq T$. Using (iii) of Lemma 3 it is easy to show that $\omega T=\omega \bar{Q}_{n}$. We denote by $H$ the subloop of the CML $T$, generated by all the expressions of the form

$$
\begin{aligned}
\bar{\lambda}= & \lambda\left(1-u_{1}, 1-u_{2}, \ldots, 1-u_{19}\right) \\
\bar{\tau}_{k}= & \tau_{k}\left(1-u_{1}, 1-u_{2}, \ldots, 1-u_{5} ; 1-v_{1}, 1-v_{2}, \ldots, 1-v_{24 k}\right) \\
\bar{\nu}_{k}= & \nu_{k}\left(1-u_{2}, 1-u_{3}, 1-u_{5} ; 1-v_{1}, 1-v_{2}, \ldots, 1-v_{24 k}\right. \\
& \left.1-w_{1}, 1-w_{2}, \ldots, 1-w_{12}\right)
\end{aligned}
$$

where $u_{i}, v_{i}, w_{i} \in \omega \bar{Q}_{n}$, with $k \neq n, k \neq 2 n$. It follows from (12)-(15) that the identity $\tau_{n}=1$ is not a corollary to the system of the identities $\lambda=1, \tau_{k}=1, \nu_{k}=1$ (for $k \neq n$ and $k \neq 2 n$ ). Then it follows from (15) and the isomorphism of the CMLs $Q_{n}$ and $\bar{Q}_{n}$ that for certain $g_{1}, g_{2}, \ldots, g_{5}, r_{1}, r_{2}, \ldots, r_{24 k} \in \bar{Q}_{n}$

$$
\begin{equation*}
\tau_{k}\left(g_{1}, g_{2}, \ldots, g_{5} ; r_{1}, r_{2}, \ldots, r_{24 k}\right) \notin H \tag{18}
\end{equation*}
$$

As $\bar{\lambda}, \bar{\tau}_{k}, \bar{\nu}_{k}$ are defined by the associators of the CML $T$, it is easy to show that the subloop $H$ is invariant with respect to the inner mapping group of the CML $H$. Therefore, it is normal in $T$. We denote by $\operatorname{ker} \varphi$ the kernel of the homomorphism $F \bar{Q}_{n} \rightarrow F(T / H)$, where $\varphi\left(\sum_{t \in T} \alpha_{t} t\right)=\sum_{t \in T} \alpha_{t}(t H)$. Let the image of the element $\bar{\nu}_{k} \in T$ under the homomorphism $T \rightarrow$ $T / H$ has the form $\alpha_{k} \beta_{k}$ whenever $u_{i}=\alpha_{i}\left(1-g_{i}\right), v_{i}=\beta_{i}\left(1-r_{i}\right), w_{i}=$ $\gamma_{i}\left(1-s_{i}\right)\left(\alpha_{i}, \beta_{i}, \gamma_{i} \in F\right)$ in $\bar{\tau}_{k}$ and $\bar{\nu}_{k}$. Then $\alpha_{k} \beta_{k}=1, \alpha_{k}=\beta_{k}^{-1}$. Here $\alpha_{k}, \beta_{k}$ are associators of the CML $T / H$. With the help of (3) we present $\beta_{k}^{-1}$ in the form $\nu_{k}$ in which the parenthesis distribution [,] in $\nu_{k}$ coincides with the parenthesis distribution $\{$,$\} in the second member of the expres-$ sion $\eta_{k}$ (see the notations in (16)). The parenthesis distribution in $\alpha_{k}$ and in the first member of $\eta_{k}$ coincide. Now we use the equality (8) for $\alpha_{k}$ and $\gamma_{k}$. We assume that $\alpha_{k}=1-\bar{\alpha}_{k}, \gamma_{k}=1-\bar{\gamma}_{k}$. As the identity $\alpha_{k}=\gamma_{k}$ holds in the CML $T / H$, it follows from the relation $F \bar{Q}_{n} / \operatorname{ker} \varphi \cong F(T / H)$ that the identity $\bar{\alpha}_{k}=\bar{\gamma}_{k}$ holds in the algebra $F Q_{n} / \operatorname{ker} \varphi$. Consequently, $\bar{\alpha}_{k}-\bar{\gamma}_{k} \in \operatorname{ker} \varphi$. But $\bar{\alpha}_{k}-\bar{\gamma}_{k}=\bar{\eta}_{k}$. Therefore $\bar{\eta}_{k} \in \operatorname{ker} \varphi$. By analogy, we obtain $\bar{\theta}, \bar{\xi}_{k} \in \operatorname{ker} \varphi$ from the relations $\bar{\lambda}, \bar{\tau}_{k} \in H$. Then it follows from the definition of the ideal $I$ that $I \subseteq \operatorname{ker} \varphi$. Finally, it follows from (18) that the identity $\tau_{k}=1$ does not hold in the CML, being the image of the CML $\bar{Q}_{n}$ under the homomorphism $F \bar{Q}_{n} \rightarrow F \bar{Q}_{n} / \operatorname{ker} \varphi$. Then it follows from the homomorphism $F \bar{Q}_{n} / I \rightarrow F \bar{Q}_{n} / \operatorname{ker} \varphi$ that the identity $\tau_{k}=1$ does not hold in the CML $\overline{\bar{Q}}_{n}$ as well, being the image of the CML $\bar{Q}_{n}$ under the homomorphism $F \bar{Q}_{n} \rightarrow F \bar{Q}_{n} / I$. This completes the proof of Lemma 8 .

Let $f=f\left(x_{1}, x_{2}, \ldots, x_{t}\right)$ be one of the polynomials $\theta, \xi_{k}, \eta_{k}$ appeared in (16). By the definition of $\{,$,$\} , we pass to the operations (+),(\cdot)$ in $f$ and we introduce in the natural way the notions of degree on every variable $x_{i}$, degree and homogeneity of polynomials for the obtained polynomials. Let us write $f$ in the form $f=f_{0}+f_{1}+\ldots+f_{r_{1}}$, where $f_{i}$ is the sum of all the monomials of the polynomial $f$ that have the degree $i$ on $x_{1}$. Let $u_{1}, u_{2}, \ldots, u_{t}$ be the elements of the algebra $\omega \bar{C}_{n} / I$, determined above. For simplicity we write $f(\boldsymbol{u})$ instead of $f\left(u_{1}, u_{2}, \ldots, u_{t}\right)$. If $\alpha \in F$, then $f\left(\alpha u_{1}, u_{2}, \ldots, u_{t}\right)=$ $f_{0}(\boldsymbol{u})+\alpha f_{1}(\boldsymbol{u})+\alpha^{2} f_{2}(\boldsymbol{u})+\ldots+\alpha^{r_{1}} f_{r_{1}}(\boldsymbol{u})$. Let $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{r_{1}}$ be arbitrary elements from $F$. Then, by (17), we get a system consisting of the $r_{1}$ equations

$$
f_{0}(\boldsymbol{u})+\alpha_{i} f_{1}(\boldsymbol{u})+\ldots+\alpha_{i}^{r_{1}} f_{r_{1}}(\boldsymbol{u})=0
$$

with variables $f_{0}(\boldsymbol{u}), f_{1}(\boldsymbol{u}), \ldots, f_{r_{1}}(\boldsymbol{u})$. By [4] (see p. 376), $d_{1} f_{j}(\boldsymbol{u})=0$, where $d_{1}$ is the determinant of this system. We assumed that the field $F$ is infinite. Then we can choose $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{r_{1}}$ such that $d_{1} \neq 0$. Then we obtain $f_{j}(\boldsymbol{u})=0$. Doing the same procedure with the polynomials $f_{j_{i}}$ and variable $x_{2} \ldots, x_{t}$ successively, we finally get the following statement:

Lemma 9. Let $f=f_{1}\left(x_{1}, x_{2}, \ldots, x_{t}\right)+\ldots+f_{i}\left(x_{1}, x_{2}, \ldots, x_{t}\right)+\ldots+f_{r}\left(x_{1}\right.$, $x_{2}, \ldots, x_{t}$ ) be the decomposition of the polynomial $f$ into homogeneous components $f_{i}\left(x_{1}, x_{2}, \ldots, x_{t}\right)$ and let $u_{1}, u_{2}, \ldots, u_{t}$ be the elements of the algebra $\omega \bar{Q}_{n} / I$ determined above. Then $f_{i}\left(u_{1}, u_{2}, \ldots, u_{t}\right)=0$.

In particular, examining the homogeneous components of the least degree in each of the cases $\bar{\theta}, \bar{\xi}_{k}, \bar{\eta}_{k}$ and taking into account the identity $(x, y, z)=$ $-(x, z, y)$ of the alternative algebra, we infer that the equalities:

$$
\begin{aligned}
& \left(\left(u_{1}, u_{2}, u_{3}, u_{4}, u_{5}\right),\left(u_{6}, u_{7}, u_{8}, u_{9}, u_{10}\right),\right. \\
& \left.\left(u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\right),\left(u_{16}, u_{17}, u_{18}\right), u_{19}\right)=0 \\
& \left(\left(u_{1}, u_{2}, u_{3}, u_{4}, u_{5}\right),\left(v_{1}, v_{2}, v_{3}\right), \ldots,\left(v_{12 k-2}, v_{12 k-1}, v_{12 k}\right),\right. \\
& \left(\left(u_{1}, u_{2}, u_{3}, u_{4}, u_{5}\right),\left(v_{12 k+1}, v_{12 k+2}, v_{12 k+3}\right), \ldots,\right. \\
& \left.\left.\ldots,\left(v_{24 k-2}, v_{24 k-1}, v_{24 k}\right)\right),\left(u_{1}, u_{2}, u_{3}, u_{4}, u_{5}\right)\right)=0 ; \\
& \left(\left(w_{1}, u_{2}, u_{3}, w_{2}, u_{5}\right),\left(v_{1}, v_{2}, v_{3}\right), \ldots,\left(v_{12 k-2}, v_{12 k-1}, v_{12 k}\right),\right. \\
& \left(\left(w_{3}, u_{2}, u_{3}, w_{4}, u_{5}\right),\left(v_{12 k+1}, v_{12 k+2}, v_{12 k+3}\right), \ldots\right. \\
& \left.\left.\ldots,\left(v_{24 k-2}, v_{24 k-1}, v_{24 k}\right)\right),\left(w_{5}, u_{2}, u_{3}, w_{6}, u_{5}\right)\right)+ \\
& +\left(\left(w_{7}, u_{2}, u_{3}, w_{8}, u_{5}\right),\left(v_{1}, v_{2}, v_{3}\right), \ldots,\left(v_{12 k-2}, v_{12 k-1}, v_{12 k}\right),\right. \\
& \left(\left(w_{9}, u_{2}, u_{3}, w_{10}, u_{5}\right),\left(v_{12 k+1}, v_{12 k+2}, v_{12 k+3}\right), \ldots\right. \\
& \left.\left.\ldots,\left(v_{24 k-2}, v_{24 k-1}, v_{24 k}\right)\right),\left(w_{11}, u_{2}, u_{3}, w_{12}, u_{5}\right)\right)=0,
\end{aligned}
$$

hold in the algebra $\omega \bar{Q}_{n} / I$ (for $k \neq n, k \neq 2 n$ ), where $w_{1}, w_{2}, \ldots, w_{12}$ take values $u_{1}, u_{4}$ exactly as in the previous case.

The algebra $\omega \bar{Q}_{n}$ is the homomorphic image of the "augmentation ideal" $\omega Q_{n}$. Then it follows from (iii) of Lemma 3 that $\omega \bar{Q}_{n}$ is generated as $F$-module by the elements of the form $a_{i}=1-g_{i}$, where $g_{i} \in Q_{n}$. We denote by $u_{i}$ the image of the element $a_{i}$ under the homomorphism $\omega \bar{Q}_{n} \rightarrow \omega \bar{Q}_{n} / I$. Then any element $v$ from $\omega \bar{Q}_{n} / I$ has the decomposition $v=\alpha_{1} u_{1}+\ldots+\alpha_{t} u_{t}$. Now, by induction on length $t$ from the last equalities it is easy to prove the statement.

Lemma 10. The identities (2) and $\mu_{k}=0$ hold in the algebra $\omega \bar{Q}_{n} / I$ for $k \neq n, k \neq 2 n$.

Let $G$ be CML and $a_{1}, a_{2}, \ldots, a_{2 i+1}, b_{1}, b_{2}, \ldots, b_{2 j+1}, c_{1}, c_{2}, \ldots, c_{2 m+1}$ be elements in $G$. We will inductively define the associator of multiplicity $k$ with the $\beta^{k}$ parenthesis distribution. The associators of multiplicity 0 are the elements of the CML $G$, and the associators of multiplicity 1 with the $\beta^{1}$ parenthesis distribution are the associators from $G$ of the form $\left[a_{1}, a_{2}, a_{3}\right]$. If $\beta^{i}\left(a_{1}, a_{2}, \ldots, a_{2 i+1}\right), \beta^{j}\left(b_{1}, b_{2}, \ldots, b_{2 j+1}\right), \beta^{m}\left(c_{1}, c_{2}, \ldots, c_{2 m+1}\right)$ are, respectively, associators of multiplicity $i, j, m$ with the $\beta^{i}, \beta^{j}, \beta^{m}$ parenthesis distribution, then $\left[\beta^{i}\left(a_{1}, a_{2}, \ldots, a_{2 i+1}\right), \beta^{j}\left(b_{1}, b_{2}, \ldots, b_{2 j+1}\right), \beta^{m}\left(c_{1}, c_{2}, \ldots, c_{2 m+1}\right)\right]$ is an associator of multiplicity $i+j+m+1$ with the $\beta^{i+j+m+1}$ parenthesis distribution.

Lemma 11. Let a CML $G$ with the lower central series $G=G_{0} \supseteq G_{1} \supseteq \ldots$ be generated by the elements $a_{1}, a_{2}, \ldots$ and let $\beta^{k}\left(b_{1}, b_{2}, \ldots, b_{2 k+1}\right)$ be the associator of the CML $G$ of multiplicity $k$ with a certain $\beta^{k}$ parenthesis distribution. Then:

1) $\beta^{k}\left(b_{1}, b_{2}, \ldots, b_{2 k+1}\right) \in G_{k}$;
2) the quotient loop $G_{k} / G_{k+1}$ is generated by those cosets that contain associators of the form

$$
\begin{equation*}
\left[a_{i_{1}}, a_{i_{2}}, \ldots, a_{i_{2 k+1}}\right] \tag{19}
\end{equation*}
$$

where $a_{i_{j}} \in\left\{a_{1}, a_{2}, \ldots\right\}$.
Proof. The first assertion follows easily from (6) by induction on $k$. The second assertion will be also proved by induction on $k$. Under $k=0$ the elements of form (19) are generators of CML and, as a consequence, the
cosets that contain these elements generate the quotient loop $G_{0} / G_{1}$. Let us assume that the quotient loop $G_{k} / G_{k+1}$ is generated by the cosets that contain elements of form (19). As $G_{k+1}=\left[G_{k}, G, G\right]$ is generated by the elements $[h, g, f]$, where $h \in G_{k}$ and $g, f \in G$, it is obvious that the quotient loop $G_{k+1} / G_{k+2}$ is generated by the cosets that contain these elements. Moreover, by induction hypotheses, $h=\prod_{i=1}^{n} h_{i}^{\epsilon_{i}} \cdot h^{\prime}$, where $\epsilon_{i}= \pm 1, h^{\prime} \in$ $G_{k+1}, h \in G_{k}$ and every $h_{i}$ is an associator of form (19). It follows from (5), (6) and (3) that
$[h, g, f]=\left[\prod_{i=1}^{n} h_{i}^{\epsilon_{i}} \cdot h^{\prime}, g, f\right]=\left[\prod_{i=1}^{n} h_{i}^{\epsilon_{i}}, g, f\right]\left[h^{\prime}, g, f\right]\left(\bmod G_{k+2}\right)=$ $\prod_{i=1}^{n}\left[h_{i}^{\epsilon_{i}}, g, f\right]\left[h^{\prime}, g, f\right]\left(\bmod G_{k+2}\right)=\prod_{i=1}^{n}\left[h_{i}^{\epsilon_{i}}, g, f\right]\left(\bmod G_{k+2}\right)=$ $\prod_{i=1}^{n}\left[h_{i}, g, f\right]^{\epsilon_{i}}\left(\bmod G_{k+2}\right)$.

Further, suppose that $g=\prod_{j=1}^{r} a_{j}^{\bar{\epsilon}_{j}}, f=\prod_{m=1}^{s} a_{m}^{\tilde{\epsilon}_{m}}$. Therefore it follows again from (5), (6), and (3) that $[h, g, f]=\left[\prod_{i=1}^{n} h_{i}, g, f\right]^{\epsilon_{i}}\left(\bmod G_{k+2}\right)=\prod_{i=1}^{n}\left[h_{i}, \prod_{j=1}^{r} a_{j}, f\right]^{\epsilon_{i}}\left(\bmod G_{k+2}\right)$ $=\prod_{i=1}^{n}\left(\prod_{j=1}^{r}\left[h_{i}, a_{j}, f\right]^{\bar{\epsilon}_{j}}\right)^{\epsilon_{i}}\left(\bmod G_{k+2}\right)=\prod_{i=1}^{n}\left(\prod_{j=1}^{r}\left[h_{i}, a_{j}, \prod_{m=1}^{s} a_{m}^{\tilde{\epsilon}_{m}}\right]^{\epsilon_{j}}\right)^{\epsilon_{i}}$ $\left(\bmod G_{k+2}\right)=\prod_{i=1}^{n}\left(\prod_{j=1}^{r}\left(\prod_{m=1}^{s}\left[h_{i}, a_{j}, a_{m}^{\epsilon_{m}}\right]^{\bar{\epsilon}_{j}}\right)^{\epsilon_{i}} \quad\left(\bmod G_{k+2}\right)=\right.$ $\prod_{i=1}^{n}\left(\prod_{j=1}^{r}\left(\prod_{m=1}^{s}\left[h_{i}, a_{j}, a_{m}\right]^{\tilde{\epsilon}_{m}}\right)^{\bar{\epsilon}_{j}}\right)^{\epsilon_{i}}\left(\bmod G_{k+2}\right)$.

Thus $\left[h_{i}, a_{j}, a_{m}\right]$ have the form indicated in (19). This completes the proof of Lemma 11.

We remind that a 3-Lie algebra $(L ;(,)$,$) is a linear space L$ over the associative and commutative ring with identity with a certain 3 -linear operation $(,$,$) on Q$ which satisfies the identities (see [3]):

$$
(x, x, y)=0,(x, y, x)=0,(y, x, x)=0
$$

$$
\begin{equation*}
((x, y, z), u, v)=((x, u, v), y, z)+(x,(y, u, v), z)+(x, y,(z(u, v)) . \tag{20}
\end{equation*}
$$

In an arbitrary alternative commutative algebra $A$ the identity $((x, y, z), u, v)$ $=((x, u, v), y, z)+((y, u, v), z, x)+((z, u, v), x, y)$ holds, where $(x, y, z)=$ $x y \cdot z-x \cdot y z$ (see [9]). Then, by the bi-associativity of alternative algebra (cf. [15]), the set $A$ with respect to the ternary operation ( $x, y, z$ ) becomes a 3 -Lie algebra. Let us denote it by $\Lambda(A)$.

Let now $G$ be an arbitrary centrally nilpotent CML that satisfies the identity (7) and let $G=G_{0} \supset G_{1} \supset \ldots G_{s}=\{1\}$ be its lower central series. As in the case of groups and Lie algebras [5], we tie the 3-Lie algebra $L(G)$ with CML $G$. By (6) we have $G_{i+1} \supset G_{3 i+1}=\left[G_{i}, G_{i}, G_{i}\right]$; then
$C_{i}=G_{i} / G_{i+1}$ is an abelian group. Let $L(C)$ be a direct sum of groups $C_{1}, C_{2}, \ldots, C_{s-1}$. We define the addition $\oplus$ on $L(G)$ by the formula

$$
\begin{equation*}
g \oplus h=g_{1} h_{1}+g_{2} h_{2}+\ldots+g_{s-1} h_{s-1} \tag{21}
\end{equation*}
$$

where $g=g_{1}+g_{2}+\ldots+g_{s-1}, h=h_{1}+h_{2}+\ldots+h_{s-1}$. It is obvious that "zero" of the group $L(G)$ is the element $1+1+\ldots$ and the element $g_{1}^{-1}+g_{2}^{-1}+\ldots$ is "opposite" to $g$.

We introduce on group $L(G)$ the ternary operation (, , ). Let $a \in G_{i}, b \in$ $G_{j}, c \in G_{k}, u \in G_{i+1}, v \in G_{j+1}, w \in G_{k+1}$. Then it follows from (5) and (6) that

$$
[a u, b v, c w] G_{i+j+k+2}=[a, b, c] G_{i+j+k+2}
$$

Now it is clear that if $g_{i}=a G_{i+1}, g_{j}=b G_{j+1}, g_{k}=c G_{k+1}$, then $\left(g_{i}, g_{j}, g_{k}\right)=$ $[a, b, c] G_{i+j+k+2}$ is a certain element of group $C_{i+j+k+1}=G_{i+j+k+1} / G_{i+j+k+2}$. We extend operation $(,$,$) on the whole group L(G)$ by the formula

$$
\begin{equation*}
(g, h, r)=\sum_{i, j, k+1}^{s-1}\left(g_{i}, h_{j}, r_{k}\right), \tag{22}
\end{equation*}
$$

where $g, h, r$ are elements in $L(G)$, and $\sum$ means addition $\oplus$ in the group $L(G)$. Let us show that the operation $(,$,$) is distributive with respect to \oplus$. Let $f, g, h, r \in L(G)$. Let us show that the expressions

$$
\begin{equation*}
(f, g, h \oplus r), \quad(f, g, h) \oplus(f, g, r) \tag{23}
\end{equation*}
$$

are equal in $L(G)$. The first of the expressions (23) is, by definition, equal to

$$
\sum_{i, j, k=1}^{s-1}\left(f_{i}, g_{j}, h_{k} r_{k}\right)
$$

Let $f_{i}=a G_{i+1}, g_{j}=b G_{j+1}, h_{k}=c G_{k+1}, r_{k}=d G_{k+1}$. Then by (5) and (6)

$$
\begin{aligned}
& \left(f_{i}, g_{j}, h_{k} r_{k}\right)=\left(a G_{i+1}, b G_{j+1},(c d) G_{k+1}\right)=(a, b, c d) G_{i+j+k+2}= \\
& (a, b, c)(a, b, d) G_{i+j+k+2}=(a, b, c) G_{i+j+k+2} \cdot(a, b, d) G_{i+j+k+2}= \\
& \left(a G_{i+1}, b G_{j+1}, c G_{k+1}\right) \cdot\left(a G_{i+1}, b G_{j+1}, d G_{k+1}\right)=\left(f_{i}, g_{j}, h_{k}\right) \cdot\left(f_{i}, g_{j}, r_{k}\right)
\end{aligned}
$$

Consequently, we obtain

$$
\sum_{i, j, k=1}^{s-1}\left(f_{i}, g_{j}, h_{k} r_{k}\right)=\sum_{i, j, k=1}^{s-1}\left(f_{i}, g_{j}, h_{k}\right) \cdot\left(f_{i}, g_{j}, r_{k}\right)=(f, g, h) \cdot(f, h, r)
$$

In such a way we have seen that both expressions (23) coincide. Other relations of distributivity can be proved by analogy. Finally, it follows from the di-associativity of CML (cf. [1]) and the identity (4) that $L(G)$ is a 3 -Lie algebra. Consequently, we have proved

Proposition 1. Let $G$ be an arbitrary centrally nilpotent CML with the lower central series $G=G_{0} \supset G_{1} \supset \ldots \supset G_{s}=\{1\}$. Then the direct sum $L(G)$ of the modules $G_{i} / G_{i+1}, i=0,1, \ldots, s-1$, on the operations (21) and (22) will be a 3 -Lie algebra.

Let us now suppose that a CML $G$, that satisfies the identity (7) is generated by the set $X=\left\{x_{1}, x_{2}, \ldots, x_{t}\right\}$. We put $y_{i}=1-x_{i}$. It follows from the definition of the "augmentation ideal" $\omega G$ of the "loop algebra" $F Q$ that $y_{i} \in \omega G$. We denote by $A$ the subalgebra of algebra $\omega G$, generated by the elements $y_{1}, y_{2}, \ldots, y_{t}$. By Lemma 4, the algebra $A$ satisfies the identity (1), so is nilpotent [14]. Then for every monomial $v \in A$ there exists a number $m$ such that $v \in A^{m} \backslash A^{m+1}$. The number $m$ will be called the weight of the monomial $v$. The polynomial that only consists of monomials of the weight $m$ will be called homogeneous of the weight $m$. Let $U$ be a word of CML $G$ from the generating set $X$. We consider in $U$ the generators $y_{i}$ of the algebra $A$, using the relation $x_{i}=1-y_{i}$. Let us assume that $U$ has the decomposition

$$
\begin{equation*}
U=1-\left(u_{m}+u_{m+1}+\ldots, u_{r},\right) \tag{24}
\end{equation*}
$$

in $A$, where $u_{i}$ is a homogeneous polynomial from $A$ of the weight $i$ and $u_{m}$ is a polynomial of the smallest weight. We determine the mapping $\delta: G \rightarrow A$ by the formula: $\delta(U)=0$ if $U=1$, and $\delta(U)=u_{m}$ for all the other cases.

Lemma 12. Let $U, V, W$ be words $(\neq 1)$ of $C M L G$ from the generating set $X$ and let $\delta(U)=u_{m}, \delta(V)=v_{k}, \delta(W)=w_{n}$. Then for every integer $l$

$$
\begin{equation*}
\delta\left(U^{l}\right)=l u_{m} \tag{25}
\end{equation*}
$$

If $m<k$, then

$$
\begin{equation*}
\delta(U V)=u_{k} \tag{26}
\end{equation*}
$$

If $m=k$ and $u_{k}+v_{k} \neq 0$, then

$$
\begin{equation*}
\delta(U V)=u_{k}+v_{k} . \tag{27}
\end{equation*}
$$

If $m=k$ and $u_{k}+v_{k}=0$, then $U V=1$ or $\delta(U V)$ belongs to $A^{t}$, where $t>m$. If $\left(u_{m}, v_{k}, w_{n}\right) \neq 0$, then

$$
\begin{equation*}
\delta([U, V, W])=\left(u_{m}, v_{k}, w_{n}\right) \tag{28}
\end{equation*}
$$

If $\left(u_{m}, v_{k}, w_{n}\right)=0$, then $[U, V, W]=1$ or $\delta([U, V, W])$ belongs to $A^{t}$, where $t>m+k+n$.

Proof. We put $u_{m}+m_{m+1}+\ldots, u_{r}=u$. Then $U=1-u$. To prove (25) we use the decomposition $(1-u)^{l}=\sum_{t=0}^{l}(-1)^{t}\binom{l}{t} u^{t}$, where $\binom{l}{t}=\frac{l(l-1) \ldots(l-t+1)}{t!}$. As $u \in A$, all non-constant members of the smallest weight of the element $(1-u)^{l}$ are of the form $-l u$. Then (25) is proved.

The assertions (26), (27) follow from the multiplication rules, and the remaining assertions follow from Lemma 5.

We put $D_{k}=\left\{g \in G \mid 1-g \in(\omega G)^{k}\right\}$. It is easy to see that $D_{k}$ is the kernel of homomorphism, induced on the CML $G$ by the natural homomorphism $F G \rightarrow F G /(\omega G)^{k}$. By Lemma $2, G / D_{k}$ is a CML, so $D_{k}$ is a normal subloop of the CML $G$ (see [1]). It follows from Lemma 12.

Lemma 13. If $G_{m}$ is the $m$-th member of the lower central series of a CML $G$, then $G_{m} \subseteq D_{2 m+1}$.

Proof. We will use the induction on $m$. We have $G_{o}=G=D_{1}$. Let us suppose that $G_{m} \subseteq D_{2 m+1}$ and let $a \in G_{m}, u, v \in G$. Then $[a, u, v]=1$, or $\delta([a, u, v])$ has a weight not less than $2 m+3$, as $\delta(a)$ has a weight not less than $2 m+1$. In any case $[a, u, v] \in D_{2 m+3}$, and therefore $G_{m+1} \subseteq D_{2 m+3}$. This completes the proof of Lemma 13.

The CML $G$ is generated by the finite set $X$. Then by Lemma 1 it is centrally nilpotent. Assume that its lower central series has the form $G=G_{0} \supset G_{1} \supset$ $\ldots \supset G_{s}=\{1\}$.

Proposition 2. Let $G$ and $A$ be the algebras considered above. Then the mapping $x_{i} G_{1} \rightarrow y_{i}$ induces the monomorphism of 3-Lie algebra $L(G)$ into 3-Lie algebra $A \subseteq \Lambda(\omega G)$. Obviously, the monomorphism is determined in the following way:

Let $\beta^{k}\left(x_{i_{1}}, x_{i_{2}}, \ldots, x_{2 k+1}\right)$, where $x_{i_{j}} \in X$, be an associator of multiplicity $k$ of the $C M L G$ with the $\beta^{k}$ parenthesis distribution and let $\beta^{k}\left(x_{i_{1}}, x_{i_{2}}\right.$, $\left.\ldots, x_{i_{2 k+1}}\right) \in G_{\mu(k)} / G_{\mu(k)+1}$. Then the mapping

$$
\beta^{k}\left(x_{i_{1}}, x_{i_{2}}, \ldots, x_{i_{2 k+1}}\right) G_{\mu(k)+1} \rightarrow \beta^{k}\left(y_{i_{1}}, y_{i_{2}}, \ldots, y_{i 2 k+1}\right)
$$

is a monomorphism of quotient loop $G_{\mu(k)} / G_{\mu(k)+1}$ in the additive group $\Lambda_{\mu(k)}(A)$, where $\Lambda_{\mu(k)}(A)$ is a submodule of module $\Lambda(A)$, consisting of homogeneous polynomials of the weight $\mu(k)$, and the parenthesis distribution $\beta^{k}$ means multiplicity in $\Lambda(A)$.

Proof. By the definition (22) of the multiplication operation in the algebra $L(G)$ and (6), and also by the relation between the operation of taking the associator into the group $G_{k} / G_{k+1}$ and the multiplication in the algebra $\Lambda(\omega G)$, indicated in (28), the expression $\beta^{k}\left(x_{i_{1}}, x_{i_{2}}, \ldots, x_{i_{2 k+1}}\right)$ obviously turns into an element $\beta^{k}\left(y_{i_{1}}, y_{y_{2}}, \ldots, y_{i_{2 k+1}}\right)$ of the algebra $\Lambda(A)$.

Further, by Lemma 13, the arbitrary element $U$ from $G_{k} / G_{k+1}$, under the mapping $x_{i} \rightarrow y_{i}$, turns into an element of the algebra $A$ of the form

$$
1+u_{2 k+1}+u_{2 k+2}+\ldots+u_{t}
$$

where $u_{j}$ has the weight $j$ or equals zero, and $j>2 k+1$. This lemma also shows that the equality

$$
\delta\left(U G_{k+1}\right)=\delta(U)=u_{2 k+1}
$$

determines the mapping $\delta_{2 k+1}$ of group $C_{k}=G_{k} / G_{k+1}$ in the set of homogeneous elements of the weight $2 k+1$ of the algebra $A$.

Moreover, by the definition (21) of the addition operation of the algebra $L(G)$ and (24)-(27), $\delta_{2 k+1}$ is a linear mapping $C_{k}$ in $A^{2 k+1}$. By Lemma 11 the associators of the form $\left[x_{1}, x_{2}, \ldots, x_{2 k+1}\right]$ generate the subloop $G_{k}$, therefore the mapping

$$
\delta(V)=\delta_{1}\left(v_{1}\right)+\delta_{3}\left(v_{3}\right)+\ldots,+\delta_{2 k+1}\left(v_{2 k+1}\right)+\ldots
$$

is a linear mapping of $Z_{3}$-module $L(G)$ into $Z_{3}$-module $A$, where $Z_{3}$ means the field of three elements. Consequently, the mapping $x_{i} G_{1} \rightarrow y_{i}$ induces the homomorphism of 3-Lie algebra $L(G)$ in $A$.

By [10] the subloop $G_{1}$, generated by all the associators of the CML $G$, belongs to the Frattini subloop. Therefore the mapping $x_{i} G_{1} \rightarrow y_{i}$ is one-toone. If $a, b, c$ are elements in $G$, then it follows from Lemma 5 that $[a, b, c]=$ $1-\left(a^{-1} \cdot b^{-1} c^{-1}\right)(a, b, c)$. Therefore, if $[a, b, c] \neq 1$ then $(a, b, c) \neq 0$. Consequently, it is easy to show by induction that if $\beta^{k}\left(x_{i_{1}}, x_{i_{2}}, \ldots, x_{i_{2 k+1}}\right) \neq 1$, then $\beta^{k}\left(y_{i_{1}}, y_{i_{2}}, \ldots, x_{i_{2 k+1}}\right) \neq 0$. Then it follows from (28) that the mapping $x_{i} G_{1} \rightarrow y_{i}$ induces the monomorphism of 3-Lie algebra $L(G)$ into the 3 -Lie algebra $A$. This completes the proof of Proposition 2.

It follows from Lemma 8 and Proposition 2 that
Lemma 14. In the algebra $\omega \bar{Q}_{n} / I$ the identity $\mu_{n}=0$ does not hold.
It is obvious that the identities $x y=y x$ and $x^{2} \cdot y x=x^{2} y \cdot x$ hold in the algebra $\omega \bar{Q}_{n} / I$, i.e. the algebra $\omega \bar{Q}_{n} / I$ is Jordan. Then from Lemmas 10 and 14 we immediately obtain

Theorem 1. The infinite system of identities $\left\{\mu_{k}=0\right\}(k=1,2, \ldots)$ is independent in the variety $\mathfrak{A}_{3} \cap \mathfrak{S}_{2} \mathfrak{S}_{2}$ of alternative commutative (Jordan) algebras over the infinite field of characteristic 3 .

If a certain identity is deduced from the system of identities $\left\{\mu_{k}=0\right\}, k=$ $1,2, \ldots$, then in its deduction only a finite number of identities of this system can be used. Therefore, if the system of identities $\left\{\mu_{k}=0\right\}$ were equivalent to a certain finite system of identities, then it would be equivalent to one of its finite subsystem. Consequently, from Theorem 1 we obtain

Corollary 1. Any infinite subset of the system of identities $\left\{\mu_{k}=0\right\}, k=$ $1,2, \ldots$, is not equivalent to any finite system of identities.

Corollary 2. In the variety $\mathfrak{A}_{3} \cap \mathfrak{S}_{2} \mathfrak{S}_{2}$ of alternative commutative (Jordan) algebras over the infinite field of characteristic 3 there exists an algebra, given by the enumerable set of relations, in which the word problem is unsolvable.

Proof. Let $S$ be some recursively enumerable and non-recursive set of numbers. Let us examine the algebra $A$ of the variety $\mathfrak{A}_{3} \cap \mathfrak{S}_{2} \mathfrak{S}_{2}$, defined by the identical relations $\left\{\mu_{n}=0\right\}$ for $n \in S$. It is obvious that each relation of the algebra $A$ is an identical relation. By Theorem 1 the arbitrary identity from $\left\{\mu_{n}=0\right\}$ under a given $n$ is true in $A$ if and only if $n \in S$. Therefore, in $A$ the problem of word equality is unsolvable.

Corollary 3. The variety $\mathfrak{A}_{3} \cap \mathfrak{S}_{2} \mathfrak{S}_{2}$ of alternative commutative (Jordan) algebras over the infinite field of characteristic 3 contains a continuum of different infinite based subvarieties.

This statement follows directly from Theorem 1 and Corollary 1.
Added in proof (by the editors): A.V. Badeev's thesis: "On Spechtness of varieties of commutative alternative algebras over a field of characteristic 3 and commutative Moufang loops" (Moscow 1999) is closely related to topics of the paper. See also the paper:
A.V. Badeev, On the Specht property of varieties of commutative alternative algebras over a field of characteristic 3 and of commutative Moufang loops, Sibirsk. Mat. Zh. 41 (2000), 1252-1268.

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[^0]:    *Another example was constructed (independently) by A.V. Badeev, see Added in proof on the end of this paper.

