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INFINITE INDEPENDENT SYSTEMS OF IDENTITIES OF ALTERNATIVE COMMUTATIVE ALGEBRA OVER A FIELD OF CHARACTERISTIC THREE

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Abstract

Let $\mathfrak{A}_3$ denote the variety of alternative commutative (Jordan) algebras defined by the identity $x^3 = 0$, and let $\mathfrak{S}_2$ be the subvariety of the variety $\mathfrak{A}_3$ of solvable algebras of solviability index 2. We present an infinite independent system of identities in the variety $\mathfrak{A}_3 \cap \mathfrak{S}_2\mathfrak{S}_2$. Therefore we infer that $\mathfrak{A}_3 \cap \mathfrak{S}_2\mathfrak{S}_2$ contains a continuum of infinite based subvarieties and that there exist algebras with an unsolvable words problem in $\mathfrak{A}_3 \cap \mathfrak{S}_2\mathfrak{S}_2$.

It is worth mentioning that these results were announced in 1999 in works of the international conference "Loops'99" (Prague).

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In [8] A.M. Slin'ko has formulated the question (Problem 1.129): *if any variety of solvable alternative (Jordan) algebras would be finitely based.* U.U. Umirbaev has got an affirmative answer to this question for alternative algebras over a field of characteristic $\neq 2, 3$ (see [14]), and Yu.A. Medvedev [7] has given a negative answer for characteristic 2. The main topic of this work is the construction of an example of an alternative commutative (Jordan) algebra also in the case of characteristic three*, which, together with

*Another example was constructed (independently) by A.V. Badeev, see **Added in proof** on the end of this paper.

the former results, completes the settlement of Slin'ko's problem for solvable alternative algebras.

Let $(u, v, w) = uv \cdot w - u \cdot vw$ mean the associator in a considered algebra, let $(u_1, \ldots, u_{2i-1}, u_{2i}, u_{2i+1}) = ((u_1, \ldots, u_{2i-1}), u_{2i}, u_{2i+1})$ and let F be an infinite field of characteristic 3. Let $\mathfrak{A}_3$ be the variety of alternative commutative (or Jordan) F-algebras, determined by the identities

$$x^3 = 0, \tag{1}$$

$$\begin{gathered} ((x_1, x_2, x_3, x_4, x_5), (x_6, x_7, x_8, x_9, x_{10}), \\ (x_{11}, x_{12}, x_{13}, x_{14}, x_{15}), (x_{16}, x_{17}, x_{18}), x_{19}) = 0. \end{gathered} \tag{2}$$

We denote by $\mathfrak{S}_2$ the variety of alternative commutative (Jordan) F-algebras being solvable of index 2. We also write

$$\begin{aligned} \mu_k = \ & ((x_1, x_2, x_3, x_4, x_5), (y_1, y_2, y_3), \ldots, (y_{12k-2}, y_{12k-1}, y_{12k}), \\ & ((x_1, x_2, x_3, x_4, x_5), (y_{12k+1}, y_{12k+2}, y_{12k+3}), \ldots \\ & \ldots (y_{24k-2}, y_{24k-1}, y_{24k})), (x_1, x_2, x_3, x_4, x_5)). \end{aligned}$$

In this work it is proved that the system of identities $\{\mu_k = 0 \mid k = 1, 2, \ldots\}$ is independent in the variety $\mathfrak{A}_3 \cap \mathfrak{S}_2\mathfrak{S}_2$, i.e. no identity of this system follows from other identities of the system. From (1), it follows that the variety $\mathfrak{A}_3 \cap \mathfrak{S}_2\mathfrak{S}_2$ is locally nilpotent [15]. Consequently, it is easy to show that any nilpotent variety of algebras, not necessary alternative or Jordan, has a finite basis of identities. We also note that in [6] it is shown that a lot of classic algebras being solvable of index 2, alternative and Jordan among them, have a finite basis of identities.

It follows from the main result of the work that the variety $\mathfrak{A}_3 \cap \mathfrak{S}_2\mathfrak{S}_2$ contains a continuum of infinite based subvarieties and there are algebras with an unsolvable words problem in $\mathfrak{A}_3 \cap \mathfrak{S}_2\mathfrak{S}_2$.

Now, we recall some notions and results from the theory of commutative Moufang loops (CML's), which can be found, e.g. in [1] (with some modifications). Any commutative Moufang loop $(Q; \cdot)$ (CML Q, for short) is characterized by the identity $x^2 \cdot yz = xy \cdot xz$. The *inner mapping group* $I(Q)$ of a CML Q is the group generated by all the inner mappings $L(x, y) = L(xy)^{-1}L(x)L(y)$, where $L(x)y = xy$, of the CML Q. A subloop

H of a CML Q is *normal* in Q, if it is invariant under the group $I(Q)$. The *associator* (of multiplicity 1) $[x_1, x_2, x_3]$ of elements $x_1, x_2, x_3 \in Q$ is determined by the equality $x_1x_2 \cdot x_3 = (x_1 \cdot x_2x_3)[x_1, x_2, x_3]$. The associators of multiplicity i are determined by induction: $[x_1, \dots, x_{2i-1}, x_{2i}, x_{2i+1}] = [[x_1, \dots, x_{2i-1}], x_{2i}, x_{2i+1}]$. We denote by Q_i the CML Q generated by all the associators of multiplicity i. A CML Q is *centrally nilpotent of class* n if its lower central series has the form $Q = Q_0 \supset Q_1 \supset \dots \supset Q_{n-1} \supset Q_n = \{1\}$. Let $f(Q)$ be the Frattini subloop of Q. If Q is a centrally nilpotent loop, then $Q_1 \subseteq f(Q)$. Hence a set $\{a_i \mid a_i \in Q,\ i \in I\}$ generates Q if and only if the set $\{a_iQ_1 \mid i \in I\}$ generates the abelian group Q/Q_1.

We recall (see [1], Chapter VIII, and [12]):

Lemma 1 (Bruck-Slaby's Theorem)**.** *Any finitely generated CML is centrally nilpotent.*

Every CML satisfies the following identities:

$$[x, y, z] = [y, z, x] = [y, x, z]^{-1}; \tag{3}$$

$$[[x, y, z], u, v] = [[x, u, v], y, z][x, [y, u, v], z][x, y, [z, u, v]]; \tag{4}$$

$$[xy, u, v] = [x, u, v][[x, u, v], x, y] \cdot [y, u, v][[y, u, v], y, x]; \tag{5}$$

and the relation

$$[Q_i, Q_j, Q_k] \subseteq Q_{i+j+k+1}. \tag{6}$$

■

Let Q be an arbitrary CML and let FQ be its *loop algebra*. We remind [2] that FQ is a free F-module with the basis $\{g \mid g \in Q\}$ and the product of elements of this basis is determined as their product in CML Q. We denote by ωH the ideal of algebra FQ, generated by all the elements 1-h ($h \in H$), for a normal subloop H of the CML Q. If $H = Q$, then ωQ is called the *augmentation ideal* of algebra FQ. Let J denote the ideal of algebra FQ, generated by all the expressions $(u, v, w) + (v, u, w)$, $u, v, w \in Q$. The Moufang identities hold in CML (see [1]), however these identities do not always hold in FQ, i.e. the algebra FQ is not always alternative. (An algebra is called *alternative* if the identities $(x, x, y) = 0$ and $(y, x, x) = 0$ hold in it). It is shown in [13] that if Q is a relatively free CML, then the quotient algebra FQ/J is alternative and the CML Q can be embedded in the multiplication groupoid of algebra FQ/J. Now let Q be a finite generated CML. By Lemma 1, Q is centrally nilpotent. Then $F(Q/Q_1)$ is a non-trivial associative algebra. Moreover an alternative algebra FQ/J is non-trivial. CML Q contains a minimal set of generators. Then, as in [12], we introduce for elements in Q the notion of normal reduced word. Repeating the proof of Theorem 1 from [13] almost word for word, we prove that any finite generated CML Q can be embedded in the multiplication groupoid of FQ/J. We identify CML Q with this isomorphic image. In [13] the algebra FQ/J is called a *"loop algebra"* of the CML Q and $\omega Q/J$ (always $J \subseteq \omega Q$) an *"augmentation ideal"* (now we use these phrases in quotation marks) and are denoted by the same symbols FQ and ωQ, respectively.

Lemma 2 ([13]). *Let Q be a relatively free (or finite generated) CML and let ϕ be the homomorphism of "loop algebra" FQ. Then, by the homomorphism ϕ, the image of CML Q is CML.* ■

Lemma 3 ([13]). *Let H be a normal subloop of relatively free (or finite generated) CML Q and let FQ, ωQ be its "loop algebra" and "augmentation ideal", respectively. Then*

(i) $\omega Q = \{\sum_{q\in Q} \lambda_q q | \sum_{q\in Q} \lambda_q = 0\}$;

(ii) $FQ/\omega H \cong F(Q/H)$ and $\omega Q/\omega H \cong \omega(Q/H)$;

(iii) *the "augmentation ideal" is generated as F-module by the elements of the form* 1-q $(q \in Q)$. ■

Lemma 4. *The relatively free (or finite generated) CML Q satisfies the identity*

$$x^3 = 1 \tag{7}$$

if and only if the "augmentation ideal" ωQ of the "loop algebra" FQ satisfies the identity (1).

Proof. Let the CML Q satisfy the identity (7). By (iii) of Lemma 3, any element h in ωQ has the form $h = \lambda_1 q_1 + \ldots + \lambda_n q_n$, where $\lambda_i \in F, q_i = 1 - g_i, g_i \in Q$. Since F is a field of characteristic 3, the equality $q^3 = 0$ follows from the equality $g^3 = 1$. Suppose that $h^3_{n-1} = 0$, where $h_{n-1} = \lambda_1 q_1 + \ldots +$

$\lambda_{n-1}q_{n-1}$. Then, by the alternativity of ωQ, we have $h^3 = (h_{n-1}+\lambda_n q_n)^3 = (h_{n-1}^2 + 2\lambda_n h_{n-1}q_n + \lambda_n^2 q_n^2)(h_{n-1} + \lambda_n q_n) = h_{n-1}^3 + \lambda_n h_{n-1}^2 q_n + 2\lambda_n h_{n-1}q_n \cdot h_{n-1} + 2\lambda_n^2 h_{n-1}q_n \cdot q_n + \lambda_n^2 q_n^2 \cdot h_{n-1} + \lambda_n^3 q_n^3 = 3\lambda_n h_{n-1}^2 q + 3\lambda_n^2 h_{n-1}q_n^2 = 0$. Therefore, the identity (1) holds in algebra ωQ. Conversely, let the "augmentation ideal" ωQ satisfy the identity (1). Since the field F is of characteristic 3, we have $g^3 = (1-(1-g))^3 = 1-3(1-g)+3(1-g)^2-(1-g)^3 = 1$ for $g \in Q$, as $1-g \in \omega Q$. Consequently, the CML Q satisfies the identity (7). This completes the proof of Lemma 4. ■

Let now A be an alternative commutative F-algebra with identity 1 and B a subalgebra of A, satisfying (1). Then $1 - B = \{1 - b \mid b \in B\}$ is CML and $(1-b)^{-1} = 1 + b + b^2$.

Lemma 5. *Let A be an alternative commutative algebra with identity* 1 *and B its subalgebra, satisfying* (1). *Then we have*
$[1-u, 1-v, 1-w] = 1 - ((1+u+u^2)\cdot(1+v+v^2)(1+w+w^2))(u,v,w)$
for all $u, v, w \in B$.

Proof. We put $1-u = a, 1-v = b$ and $1-w = c$. Then we have $[1-u, 1-v, 1-w] = (a\cdot bc)^{-1}(ab\cdot c) = (a\cdot bc)^{-1}(ab\cdot c) - (a\cdot bc)^{-1}(a\cdot bc) + 1 = 1 + (a\cdot bc)^{-1}(a,b,c) = 1 + (((1-w)^{-1}\cdot(1-v)^{-1})(1-u)^{-1})(1-u, 1-v, 1-w) = 1 - ((1-w)^{-1}(1-v)^{-1}\cdot(1-u)^{-1})(u,v,w) = 1 - ((1+w+w^2)(1+v+v^2)\cdot(1+u+u^2))(u,v,w)$. This completes the proof of Lemma 5. ■

We write $\sum x = 1 + x + x^2, \{x,y,z\} = (\sum x \cdot \sum y \sum z)(x,y,z)$, and $\{x_1, \dots, x_{2i-1}, x_{2i}, x_{2i+1}\} = \{\{x_1, \dots, x_{2i-1}\}, x_{2i}, x_{2i+1}\}$. If a CML Q satisfies the identity (7), then from Lemmas 4 and 5 it follows that for $u, v, w \in \omega Q$ $[1-u, 1-v, 1-w] = 1 - \{u, v, w\}$, and consequently by induction, we get

$$[1-u_1, 1-u_2, \dots, 1-u_{2i+1}] = 1 - \{u_1, u_2, \dots, u_{2i+1}\}. \tag{8}$$

In an arbitrary algebra A, we define by induction:
$A^1 = A$, $A^n = \sum_{i+j=n} A^i \cdot A^j$, $A^{(1)} = A^2$, $A^{(n)} = (A^{(n-1)})^2$.
We remind that algebra A is called *nilpotent* (respectively *solvable*) if there is an n, such that $A^n = 0$ (respectively $A^{(n)} = 0$). The least n is called the *nilpotent* (respectively *solvable*) *index*. Let $f(x_1, x_2, \dots, x_i)$ be a polynomial of free algebras. We say that $f(x_1, x_2, \dots, x_i) = 0$ is a *partial identity* of the algebra A with the generating set B if $f(b_1, b_2, \dots, b_i) = 0$ for any $b_1, b_2, \dots, b_i$ in B.

Lemma 6. *Let A be an alternative commutative F-algebra and I be an ideal of A. Then $I^{(n)}$, $n = 1, 2, \ldots$, is also an ideal of A.*

Proof. As F is a field of characteristic 3, we have

$(u, v, w) + (v, u, w) = 0,$

$uv \cdot w - u \cdot vw + vu \cdot w - v \cdot uw = 0,$

$2uv \cdot w - u \cdot vw - v \cdot uw = 0,$

$-uv \cdot w - u \cdot vw - v \cdot uw = 0,$

and $uv \cdot w = -u \cdot vw - v \cdot uw$

for all $u, v, w \in A$.

We will prove the statement by induction on n. Let $x \in A, u, v \in I^{(n)}$ and assume that $I^{(n)}$ is an ideal of A. Then $x \cdot uv = -xu \cdot v - u \cdot xv$. But $xu, xv \in I^{(n)}$. Therefore $xI^{(n+1)} \subseteq I^{(n+1)}$. Consequently, $I^{(n+1)}$ is an ideal of A. The statement is proved by analogy for $n = 1$. This completes the proof of Lemma 6. ∎

Let L be a free CML that satisfies the identity (7), with a set of free generators $Y = \{y_1, y_2, \ldots\}$, where the cardinal number $|Y| \geq 5$. Let ωL be the "augmentation ideal" of the "loop algebra" FL. Let us consider the homomorphism $\alpha : FL \to FL/(\omega L)^{(2)}$. Then $H = \{h \in L \mid 1 - h \in (\omega L)^{(2)}\}$ is the kernel of the homomorphism $\overline{\alpha}$ of the CML L, induced on L by the homomorphism α. By Lemma 2, the quotient loop $L/H = \overline{L}$ is a CML.

Lemma 7. *Let L be a free CML and $\overline{\alpha} : L \to \overline{L}$ be the homomorphism of CML defined above. Then the inequality $[\overline{y}_1, \overline{y}_2, \overline{y}_3, \overline{y}_4, \overline{y}_5] \neq 1$, where $\overline{y}_i \in \overline{Y} = \{\overline{\alpha} y_i \mid y_i \in Y\}$, holds in the CML $\overline{L}$.*

Proof. First we construct an alternative commutative solvable F-algebra of index 2, in which identity (1) holds and the following partial identity does not hold:

$$\{x_1, x_2, x_3, x_4, x_5\} = 0. \tag{9}$$

Let M be a free F-module with a set of generators X and let N be the "exterior" algebra of module M, satisfying the identity $3uvw = 0$. We add a new symbol $b \notin N$ to the generators X and assume that B is an F-algebra generated by the set $X \cup \{b\}$ which besides the relations of the "exterior" algebra N also satisfies the relations $bu \cdot v = b \cdot uv, bu = -ub$, for all $u, v \in X$. Let E denote the F-submodule of module B with the basis consisting of the monomials of odd degree from B, except the monomials of the form b^{2k+1}. Let u, v be monomials from E. There is an odd number of generators from $X \cup \{b\}$ in the composition of u, v, bacause $uv = -vu$. Moreover, as there are necessary generators from X in the composition of u, we have $uu = 0$. Consequently, it follows easily that for the polynomials s, t from E the equalities $st = -ts$ and $ss = 0$ hold. Let

$$\overline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \text{where } u_i \in E,$$

denote the elements of the direct product $R = E \times E \times E$. We define the product $(\cdot)$ on the set R:

$$\overline{u} \cdot \overline{v} = \begin{pmatrix} bu_2v_3 + bv_2u_3 \\ bu_3v_1 + bv_3u_1 \\ bu_1v_2 + bv_1u_2 \end{pmatrix}. \tag{10}$$

We also define the sum $(+)$ as the componentwise addition. Then, obviously, $(R, +, \cdot)$ becomes a commutative F-algebra and it satisfies the identity (1). Let us show that the algebra R is alternative. Let $\overline{u}, \overline{v}, \overline{w} \in R$. Using (10), we obtain $(\overline{u}, \overline{v}, \overline{w}) = \overline{uv} \cdot \overline{w} - \overline{u} \cdot \overline{vw} =$

$$= \begin{pmatrix} bu_2v_3 + bv_2u_3 \\ bu_3v_1 + bv_3u_1 \\ bu_1v_2 + bv_1u_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} - \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \begin{pmatrix} bv_2w_3 + bv_2w_3 \\ bv_3w_1 + bv_3w_1 \\ bv_1w_2 + bv_1w_2 \end{pmatrix} =$$

$$= \left(\begin{matrix} b(bu_3v_1 + bv_3u_1)w_3 + bw_2(bu_1v_2 + bv_1u_2) - bu_2(bv_1w_2 + \\ b(bu_1v_2 + bv_1u_2)w_1 + bw_3(bu_2v_3 + bv_2u_3) - bu_3(bv_2w_3 + \\ b(bu_2v_3 + bv_2u_3)w_2 + bw_1(bu_3v_1 + bv_3u_1) - bu_1(bv_3w_1 + \end{matrix} \right.$$

$$\left. \begin{matrix} + bw_1v_2) - b(bv_3w_1 + bw_3v_1)u_3 \\ + bw_2v_3) - b(bv_1w_2 + bw_1v_2)u_1 \\ + bw_3v_1) - b(bv_2w_3 + bw_2v_3)u_2 \end{matrix} \right) = \left(\begin{matrix} b^2u_3w_3v_1 + b^2v_3u_3w_1 + \\ b^2u_1w_1v_2 + b^2v_1u_1w_2 + \\ b^2u_2w_2v_3 + b^2v_2u_2w_3 + \end{matrix} \right.$$

$$\left. \begin{matrix} + b^2w_3v_3u_1 + b^2u_2w_2v_1 + b^2v_2u_2w_1 + b^2w_2v_2u_1 \\ + b^2w_1v_1u_2 + b^2u_3w_3v_2 + b^2v_3u_3w_2 + b^2w_3v_3u_2 \\ + b^2w_2v_2u_3 + b^2u_1w_1v_3 + b^2v_1u_1w_3 + b^2w_1v_1u_3 \end{matrix} \right).$$

If $\overline{u} = \overline{v}$, then the first component of the associator gets the form

$$b^2u_3w_3u_1 + b^2u_3u_3w_1 + b^2w_3u_3u_1 + b^2u_2w_2u_1 + b^2u_2u_2w_1 +$$

$$+ b^2w_2u_2u_1 = b^2u_3w_3u_1 + b^2w_3u_3u_1 + b^2u_2w_2u_1 + b^2w_2u_2u_1 =$$

$$= b^2u_3w_3u_1 - b^2u_3w_3u_1 + b^2u_2w_2u_1 - b^2u_2w_2u_1 = 0.$$

It is shown by analogy that all the other components are equal to zero. Therefore $(\overline{u}, \overline{u}, \overline{w}) = 0$, i.e. the algebra R is alternative.

Let A denote the F-subalgebra of the algebra R, generated by the elements of the form $u = (u_1, u_2, 0)$, i.e. by the elements that have the third component equal to zero. Using (10) it is easy to see that the algebra A is solvable of index 2. Then $\{u, v, w\} = ((1 + u + u^2) \cdot (1 + v + v^2)(1 + w + w^2))(u, v, w) = (1 + u + v + w) \cdot (u, v, w)$. Let us write it in detail. We put $b^2u_2w_2v_1 + b^2v_2u_2w_1 + b^2w_2v_2u_1 = c$ and $b^2u_1w_1v_2 + b^2v_1u_1w_2 + b^2w_1v_1u_2 = d$. We have

$$\{u, v, w\} = (1 + u + v + w)(u, v, w) = \begin{pmatrix} c \\ d \\ 0 \end{pmatrix} + \begin{pmatrix} u_1 + v_1 + w_1 \\ u_2 + v_2 + w_2 \\ 0 \end{pmatrix} \begin{pmatrix} c \\ d \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} c \\ d \\ c(u_2 + v_2 + w_2) + (u_1 + v_1 + w_1)d \end{pmatrix} = \begin{pmatrix} c \\ d \\ a \end{pmatrix},$$

where $a = b^3w_2v_2u_1u_2 + b^3u_2w_2v_1v_2 + b^3v_2u_2w_1w_2 + b^3u_1w_1v_1u_2 + b^3v_1u_1w_1v_2 + b^3w_1v_1u_1w_2$. Consequently, we can choose generating elements u, v, w, y, z in A such that $\{\{u, v, w\}, y, z\} \neq 0$. Therefore, (9) is not a partial identity of the algebra A.

As we have shown before in Lemma 5, the set $C = 1 - A$ is a CML. Since F is a field of characteristic 3, we have $(1 - a)^3 = 1 - 3a + 3a^2 - a^3 = 1$ for $a \in A$, i.e. the CML C satisfies the identity (7). It is obvious that $A = \omega C$. We suppose that the CML C has the same number of generating elements as the free CML L and let $C = L/H$. The loop homomorphism

$L \to C$ induces the algebra homomorphisms $\omega L \to \omega C$, $\omega L/(\omega L)^{(2)} \to \omega C/(\omega C)^{(2)} = A/A^{(2)}$. Since the algebra A is solvable of index 2, then $A^{(2)} = (0)$ and we have the algebra homomorphism $\omega L/(\omega L)^{(2)} \to A$. As the partial identity (9) does not hold also in the algebra A, the same holds in algebra $\omega L/(\omega L)^{(2)}$. But $FL/(\omega L)^{(2)} \cong F(L/H)$. Then it follows from (8) that the inequality $[\overline{y}_1, \overline{y}_2, \overline{y}_3, \overline{y}_4, \overline{y}_5] \neq 1$ holds in the CML $\overline{L} = L/H$. This completes the proof of Lemma 7. ∎

Let Q_n denote the CML, constructed in the proof of Theorem of [11]. It is the semi-direct product of B_n and G_n, where G_n is the free centrally nilpotent CML of class 2 with free generators $g_1, g_2, \ldots, g_{24n}$, B_n is the centrally nilpotent CML of class 2, generated by the set $\{\alpha d \mid \alpha \in I(G_n)\}$, where $d \notin G_n, I(G_n)$ is the inner mapping group of CML G_n. It follows from the definition of the CML B_n and G_n that they satisfy the identity

$$[[x_1, x_2, x_3], x_4, x_5] = 1. \tag{11}$$

Since the CML Q_n is finite, it is centrally nilpotent by Lemma 1. In the CML Q_n the identities (7) and

$$\begin{aligned} \lambda = \lambda(x_1, x_2, \ldots, x_{19}) = [[x_1, x_2, x_3, x_4, x_5], [x_6, x_7, x_8, x_9, x_{10}], \\ [x_{11}, x_{12}, x_{13}, x_{14}, x_{15}], [x_{16}, x_{17}, x_{18}], x_{19}] = 1 \end{aligned} \tag{12}$$

hold, and for $k \neq n, k \neq 2n$ also the following identities are satisfied:

$$\left\{ \begin{aligned} \tau_k = {} & \tau_k(y_1, y_2, y_3, y_4, y_5; x_1, x_2, \ldots, x_{24k}) = \\ = {} & [[y_1, y_2, y_3, y_4, y_5], [x_1, x_2, x_3], \ldots, [x_{12k-2}, x_{12k-1}, x_{12k}], \\ & [[y_1, y_2, y_3, y_4, y_5], [x_{12k+1}, x_{12k+2}, x_{12k+3}], \ldots \\ & \ldots, [x_{24k-2}, x_{24k-1}, x_{24k}]], [y_1, y_2, y_3, y_4, y_5]] = 1, \end{aligned} \right. \tag{13}$$

$$
(14)\quad \begin{cases}
\nu_k = \nu_k(y_2, y_3, y_5; x_1, x_2, \dots, x_{24k}; u_1, u_2, \dots, u_{12}) = \\
\quad = [[u_1, y_2, y_3, u_2, y_5], [x_1, x_2, x_3], \dots, [x_{12k-2}, x_{12k-1}, x_{12k}], \\
\qquad [[u_3, y_2, y_3, u_4, y_5], [x_{12k+1}, x_{12k+2}, x_{12k+3}], \dots \\
\qquad \dots, [x_{24k-2}, x_{24k-1}, x_{24k}]], [u_5, y_2, y_3, u_6, y_5]] \times \\
\qquad \times [[u_7, y_2, y_3, u_8, y_5], [x_1, x_2, x_3], \dots, [x_{12k-2}, x_{12k-1}, x_{12k}], \\
\qquad [[u_9, y_2, y_3, u_{10}, y_5], [x_{12k+1}, x_{12k+2}, x_{12k+3}], \dots \\
\qquad \dots, [x_{24k-2}, x_{24k-1}, x_{24k}]], [u_{11}, y_2, y_3, u_{12}, y_5]] = 1,
\end{cases}
$$

where either

1) $u_1 = y_s$, $u_3 = y_t$, $u_7 = y_t$, $u_{11} = y_s$, $u_2 = u_4 = u_6 = u_8 = u_{10} = u_{12} = y_4$, and $u_5 = u_9 = y_1$

or 2) $u_2 = y_s$, $u_4 = y_t$, $u_8 = y_t$, $u_{12} = y_s$, $u_1 = u_3 = u_5 = u_7 = u_9 = u_{11} = y_1$, and $u_6 = u_{10} = y_4$

or 3) $u_1 = y_s$, $u_5 = y_t$, $u_7 = y_t$, $u_9 = y_s$, $u_2 = u_4 = u_6 = u_8 = u_{10} = u_{12} = y_4$, and $u_3 = u_{11} = y_1$

or 4) $u_2 = y_s$, $u_6 = y_t$, $u_8 = y_t$, $u_{10} = y_s$, $u_1 = u_3 = u_5 = u_7 = u_9 = u_{11} = y_1$, and $u_4 = u_{12} = y_4$

or 5) $u_3 = y_s$, $u_5 = y_t$, $u_{11} = y_t$, $u_9 = y_s$, $u_2 = u_4 = u_6 = u_8 = u_{10} = u_{12} = y_4$, and $u_1 = u_7 = y_1$;

or 6) $u_4 = y_s$, $u_6 = y_t$, $u_{12} = y_t$, $u_{10} = y_s$, $u_1 = u_3 = u_5 = u_7 = u_9 = u_{11} = y_1$, and $u_2 = u_8 = y_4$

and the following inequality holds:

$$(15)\qquad \begin{cases} [[y_1, y_2, y_3, y_4, y_5], [x_1, x_2, x_3], \dots, [x_{12n-2}, x_{12n-1}, x_{12n}], \\ [[y_1, y_2, y_3, y_4, y_5], [x_{12n+1}, x_{12n+2}, x_{12n+3}], \dots \\ \dots, [x_{24n-2}, x_{24n-1}, x_{24n}]], [y_1, y_2, y_3, y_4, y_5]] \neq 1. \end{cases}$$

The identities (1), (12), (13) and the inequality (15) are proved in [11]. The identity (14) is proved as the proof of the identity $w_k = 1$ of the Theorem of [11], using Lemma 9 of [11].

By construction, the CML Q_n is a semi-direct product of the CML's B_n and G_n. Then, by (ii) of Lemma 3, $FQ_n/\omega B_n \cong FG_n$, $\omega Q_n/\omega B_n \cong \omega G_n$, where FB_n and FG_n are subalgebras of "loop algebra" FQ_n, and ωG_n is the "augmentation ideal" of "loop algebra" FG_n. We will consider the homomorphism $\eta : FQ_n \to FQ_n/(\omega B_n)^{(2)}$. By Lemma 2, η induces the homomorphism $\overline{\eta}$ of the CML Q_n.

We will show that $\overline{\eta}$ is the isomorphism of the CML Q_n. Indeed, let α and β be the homomorphisms of CML Q_n and $\overline{\eta}Q_n$, respectively, which, by Lemma 2, are induced by the homomorphisms of algebras $FQ_n \to FQ_n/\omega B_n$ and $FQ_n/(\omega B_n)^{(2)} \to (FQ_n/(\omega B_n)^{(2)})/(\omega B_n/(\omega B_n)^{(2)})$. By (i) of Lemma 3, $G_n \cap \omega B_n = \emptyset$. Then it follows from the relations $FG_n \cong FQ_n/\omega B_n \cong (FQ_n/(\omega B_n)^{(2)})/(\omega B_n/(\omega B_n)^{(2)})$ that $G_n \cong \alpha G_n$ and $\alpha G_n \cong \beta(\overline{\eta}G_n)$. Therefore, $|G_n| = |\beta(\overline{\eta}G_n)|$. We suppose that the homomorphism $\overline{\eta}$ of CML G_n is not an isomorphism. By construction, the CML G_n is finite. Then $|\overline{\eta}G_n| < |G_n|$ and $|\beta(\overline{\eta}G_n)| < |G_n|$. We have obtained a contradiction. Hence $\overline{\eta}$ is an isomorphism of CML G_n.

By construction, the CML B_n is generated by the set $\{\varphi b \mid \varphi \in I(G_n)\}$, where $b \notin G_n$, and $I(G_n)$ is the inner mapping group of CML G_n. It is determined by the identities (7) and (11) and by the relations of the form $[\varphi_1 b, \varphi_2 b, \varphi_3 b] = 1$ or $[\varphi_1 b, \varphi_2 b, \varphi_3 b] = t(\varphi_1, \varphi_2, \varphi_3) \neq 1$, where $\varphi_1, \varphi_2, \varphi_3 \in I(G_n)$. By Lemma 4 and (8), the system of identities (7), (11) is equivalent to the system consisting of the identity (1) and the partial identity (9) of the algebra ωB_n. The meaning of the elements $t(\varphi_1, \varphi_2, \varphi_3)$ depends only on the inner mappings $\varphi_1, \varphi_2, \varphi_3$ of the CML G_n. Then, as $G_n \cong \overline{\eta}G_n$, the generators φb and the elements $t(\varphi_1, \varphi_2, \varphi_3)$ are not mapped on the unit of the CML $\overline{\eta}G_n$ under the homomorphism η. We suppose that $B_n = L/H$, where L is a free CML, as in Lemma 7. By (ii) of Lemma 3,

$FB_n \cong FL/\omega H$. We will consider the homomorphism $FL \to FL/(\omega L)^{(2)}$, taking into account Lemma 6. By Lemma 2, a normal subloop K of the CML L corresponds to this homomorphism. We have $FL + (\omega H + (\omega L)^{(2)}) = FL + \omega H + (\omega H + (\omega L)^{(2)}) = FL + \omega H + (\omega B_n)^{(2)} = FB_n + (\omega B_n)^{(2)}$. It means that the ideal $\omega H + (\omega L)^{(2)}$ is the kernel of the homomorphism $FL \to FB_n/(\omega B_n)^{(2)}$. The normal subloop HK will be the kernel of the homomorphism $L \to \overline{\eta}B_n$, which by Lemma 2, is induced by the homomorphism $FL \to FB_n/(\omega B_n)^{(2)}$. We suppose that the determining relation $[\varphi_1 b, \varphi_2 b, \varphi_3 b] \neq 1$ of CML B_n corresponds to the associator $[y_1, y_2, y_3]$ of the CML L under the homomorphism $L \to B_n$. Then $[y_1, y_2, y_3] \notin H$ and, by Lemma 7, $[y_1, y_2, y_3] \notin K$. Therefore, $[y_1, y_2, y_3] \notin HK$. It means that the determining relation $[\varphi_1, \varphi_2 b, \varphi_3 b]$ of the CML B_n is not mapped on the unit of the CML $\overline{\eta}B_n$ under the homomorphism $B_n \to \overline{\eta}B_n$. Therefore the CMLs B_n and $\overline{\eta}B_n$, which have the same determining relations, are isomorphic. Consequently, $\overline{\eta}Q_n/\overline{\eta}B_n \cong \overline{\eta}G_n$ and $\overline{\eta}$ is the isomorphism of the CML Q_n. Moreover, $\eta(FQ_n)/\eta(\omega B_n) \cong \eta(FG_n)$. As the homomorphism η keeps the sum of the coefficients of the polynomials, by (i) of Lemma 3, $\eta(\omega Q_n)/\eta(\omega B_n) \cong \eta(\omega G_n)$.

Taking into account Lemma 6, we can consider the homomorphism $\xi : \eta(FQ_n) \to \eta(FG_n)/(\eta(FG_n))^{(2)}$. Let $\overline{\xi}$ be the homomorphism of the CML $\overline{\eta}Q_n$ which, by Lemma 2, is induced by the homomorphism ξ. By the construction of the CML G_n and then the CML $\overline{\eta}G_n$ is also a free centrally nilpotent of class 2. It follows from Lemma 7 that the CML $\overline{\xi}\overline{\eta}G_n$ also has such a property. Therefore, $\overline{\xi}\overline{\eta}G_n \cong \overline{\eta}G_n$.

Further, it follows from the relation $\eta(FQ_n)/\eta(\omega B_n) \cong \eta(FG_n)$ that zero of the algebra $\eta(FQ_n)/\eta(\omega B_n)$, namely $\eta(\omega B_n)$, is mapped on zero of the second algebra by the composition of homomorphisms: $\eta(FQ_n) \to \eta(FQ_n)/\eta(\omega B_n) \to \eta(FG_n)/(\eta(FG_n))^{(2)}$. It means that the homomorphism ξ does not impose any restrictions on the ideal $\eta(\omega B_n)$. By Lemma 2, the homomorphism $\eta(FQ_n) \to \eta(FQ_n)/\eta(\omega B_n)$ induces the homomorphism of the CML $\overline{\eta}Q_n$, whose kernel is the normal subloop $\{g \in \overline{\eta}Q_n \mid 1 - g \in \eta(\omega B_n)\} = \overline{\eta}B_n$. Therefore, we infer that ξ is an isomorphism of CML $\overline{\eta}B_n$. Consequently, we have the isomorphisms $\overline{\xi}(\overline{\eta}Q_n) \cong \overline{\eta}Q_n \cong Q_n$. Further we will identify the CML $\overline{\xi}\overline{\eta}Q_n$ with the CML Q_n. We put $\xi\eta(\omega Q_n) = \omega\overline{Q}_n$, $\xi\eta(\omega B_n) = \omega\overline{B}_n$, and $\xi\eta(\omega G_n) = \omega\overline{G}_n$. As for the homomorphism η it is proved that $\omega\overline{Q}_n/\omega\overline{B}_n \cong \omega\overline{G}_n$. It is obvious that $\omega\overline{B}_n \in \mathfrak{S}_2$ and $\omega\overline{G}_n \in \mathfrak{S}_2$. Then the algebra $\omega\overline{Q}_n$ belongs to the product of the varieties $\mathfrak{S}_2\mathfrak{S}_2$.

Further, g_i, r_i, s_i will denote the elements of CML $\overline{Q}_n$. Let $a_i = 1 - g_i, b_i = 1 - r_i, c_i = 1 - s_i$. We also write

$$(16)\quad \begin{cases} \theta = \theta(x_1, x_2, \dots, x_{19}) = \{\{x_1, x_2, x_3, x_4, x_5\}, \{x_6, x_7, x_8, x_9, x_{10}\}, \\ \{x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\}, \{x_{16}, x_{17}, x_{18}\}, x_{19}\}; \\ \xi_k = \xi_k(x_1, x_2, x_3, x_4, x_5; y_1, y_2, \dots, y_{24k}) = \\ \{\{x_1, x_2, x_3, x_4, x_5\}, \{y_1, y_2, y_3\}, \dots, \{y_{12k-2}, y_{12k-1}, y_{12k}\}, \\ \{\{x_1, x_2, x_3, x_4, x_5\}, \{y_{12k+1}, y_{12k+2}, y_{12k+3}\}, \dots \\ \dots, \{y_{24k-2}, y_{24k-1}, y_{24k}\}\}, \{x_1, x_2, x_3, x_4, x_5\}\}; \\ \eta_k = \eta_k(x_2, x_3, x_5; y_1, y_2, \dots, y_{24k}; z_1, z_2, \dots, z_{12}) = \\ \{\{z_1, x_2, x_3, z_2, x_5\}, \{y_1, y_2, y_3\}, \dots, \{y_{12k-2}, y_{12k-1}, y_{12k}\}, \\ \{\{z_3, x_2, x_3, z_4, x_5\}, \{y_{12k+1}, y_{12k+2}, y_{12k+3}\}, \dots \\ \dots, \{y_{24k-2}, y_{24k-1}, y_{24k}\}\}, \{z_5, x_2, x_3, z_6, x_5\}\} - \\ -\{\{z_7, x_2, x_3, z_8, x_5\}, \{y_1, y_2, y_3\}, \dots, \{y_{12k-2}, y_{12k-1}, y_{12k}\}, \\ \{z_9, x_2, x_3, z_{10}, x_5\}, \{\{z_{11}, x_2, x_3, z_{12}, x_5\}, \\ \{y_{12k+1}, y_{12k+2}, y_{12k+3}\}, \dots, \{y_{24k-2}, y_{24k-1}, y_{24k}\}\}, \end{cases}$$

where either

1) $z_1 = x_s$, $z_3 = x_t$, $z_7 = x_t$, $z_{11} = x_s$, $z_2 = z_4 = z_6 = z_8 = z_{10} = z_{12} = x_4$, and $z_5 = z_9 = x_1$

or 2) $z_2 = x_s$, $z_4 = x_t$, $z_8 = x_t$, $z_{12} = x_s$, $z_1 = z_3 = z_5 = z_7 = z_9 = z_{11} = x_1$, and $z_6 = z_{10} = x_4$

or 3) $z_1 = x_s$, $z_5 = x_t$, $z_7 = x_t$, $z_9 = x_s$, $z_2 = z_4 = z_6 = z_8 = z_{10} = z_{12} = x_4$, and $z_3 = z_{11} = x_1$

or 4) $z_2 = x_s$, $z_6 = x_t$, $z_8 = x_t$, $z_{10} = x_s$, $z_1 = z_3 = z_5 = z_7 = z_9 = z_{11} = x_1$, and $z_4 = z_{12} = x_4$

or 5) $z_3 = x_s$, $z_5 = x_t$, $z_{11} = x_t$, $z_9 = x_s$, $z_2 = z_4 = z_6 = z_8 = z_{10} = z_{12} = x_4$, and $z_1 = z_7 = x_1$

or 6) $z_4 = x_s$, $z_6 = x_t$, $z_{12} = x_7$, $z_{10} = x_s$, $z_1 = z_3 = z_5 = z_7 = z_9 = z_{11} = x_1$, and $z_2 = z_8 = x_4$.

Let $k \neq n, 2n$ and let I denote the ideal of the algebra $\omega\overline{Q}_n$, generated by the expressions of the forms:

$\theta(\alpha_1 a_1, \ldots, \alpha_{19} a_{19})$,

$\xi_k(\alpha_1 a_1, \alpha_2 a_2, \alpha_3 a_3, \alpha_4 a_4, \alpha_5 a_5; \beta_1 b_1, \beta_2 b_2, \ldots, \beta_{24k} b_{24k})$,

$\overline{\eta}_k = \eta_k(\alpha_2 a_2, \alpha_3 a_3, \alpha_5 a_5; \beta_1 b_1, \beta_2 b_2, \ldots, \beta_{24k} b_{24k}; \gamma_1 c_1, \gamma_2 c_2, \ldots, \gamma_{12} c_{12})$,

where

either 1) $c_1 = a_s$, $c_3 = a_t$, $c_7 = a_t$, $c_{11} = a_s$, $c_2 = c_4 = c_6 = c_8 = c_{10} = c_{12} = a_4$, and $c_5 = c_9 = a_1$

or 2) $c_2 = a_s$, $c_4 = a_t$, $c_8 = a_t$, $c_{12} = a_s$, $c_1 = c_3 = c_5 = c_7 = c_9 = c_{11} = a_1$, and $c_6 = c_{10} = a_4$

or 3) $c_1 = a_s$, $c_5 = a_t$, $c_7 = a_t$, $c_9 = a_s$, $c_2 = c_4 = c_6 = c_8 = c_{10} = c_{12} = a_4$, and $c_3 = c_{11} = a_1$

or 4) $c_2 = a_s$, $c_6 = a_t$, $c_8 = a_t$, $c_{10} = a_s$, $c_1 = c_3 = c_5 = c_7 = c_9 = c_{11} = a_1$, and $c_4 = c_{12} = a_4$

or 5) $c_3 = a_s$, $c_5 = a_t$, $c_{11} = a_t$, $c_9 = a_s$, $c_2 = c_4 = c_6 = c_8 = c_{10} = c_{12} = a_4$, and $c_1 = c_7 = a_1$

or 6) $c_4 = a_s$, $c_6 = a_t$, $c_{12} = a_t$, $c_{10} = a_s$, $c_1 = c_3 = c_5 = c_7 = c_9 = c_{11} = a_1$, and $c_2 = c_8 = a_4$.

Let u_i, v_i, w_i denote the images of the elements a_i, b_i, c_i, respectively, under the homomorphism $\omega\overline{Q}_n \to \omega\overline{Q}_n/I$. Then the following equalities hold in the algebra $\omega\overline{Q}_n/I$:

$$(17) \quad \begin{cases} \theta(\alpha_1 u_1, \alpha_2 u_2, \ldots, \alpha_{19} u_{19}) = 0, \\ \xi_k(\alpha_1 u_1, \alpha_2 u_2, \alpha_3 u_3, \alpha_4 u_4, \alpha_5 u_5;\ \beta_1 v_1, \beta_2 v_2, \ldots, \beta_{24k} v_{24k}) = 0, \\ \eta_k = \eta_k(\alpha_2 u_2, \alpha_3 u_3, \alpha_5 u_5;\ \beta_1 v_1, \beta_2 v_2, \ldots, b_{24k} v_{24k}; \\ \gamma_1 w_1, \gamma_2 w_2, \ldots, \gamma_{12} w_{12}) = 0, \end{cases}$$

where

either 1) $w_1 = u_s$, $w_3 = u_t$, $w_7 = u_t$, $w_{11} = u_s$, $w_2 = w_4 = w_6 = w_8 = w_{10} = w_{12} = u_4$, and $w_5 = w_9 = u_1$

or 2) $w_2 = u_s$, $w_4 = u_t$, $w_8 = u_t$, $w_{12} = u_s$, $w_1 = w_3 = w_5 = w_7 = w_9 = w_{11} = u_1$, and $w_6 = w_{10} = u_4$

or 3) $w_1 = u_s$, $w_5 = u_t$, $w_7 = u_t$, $w_9 = u_s$, $w_2 = w_4 = w_6 = w_8 = w_{10} = w_{12} = u_4$, and $w_3 = w_{11} = u_1$

or 4) $w_2 = u_s$, $w_6 = u_t$, $w_8 = u_t$, $w_{10} = u_s$, $w_1 = w_3 = w_5 = w_7 = w_9 = w_{11} = u_1$, and $w_4 = w_{12} = u_4$

or 5) $w_3 = u_s$, $w_5 = u_t$, $w_{11} = u_t$, $w_9 = u_s$, $w_2 = w_4 = w_6 = w_8 = w_{10} = w_{12} = u_4$, and $w_1 = w_7 = u_1$

or 6) $w_4 = u_s$, $w_6 = u_t$, $w_{12} = u_t$, $w_{10} = u_s$, $w_1 = w_3 = w_5 = w_7 = w_9 = w_{11} = u_1$, and $w_2 = w_8 = u_4$.

By Lemma 2, the image $\overline{\overline{Q}}_n$ of the CML $\overline{Q}_n$ under the homomorphism $F\overline{Q}_n \to F\overline{Q}_n/I$ is CML.

Lemma 8. *The identity $\tau_n = 1$ does not hold in the CML $\overline{\overline{Q}}_n$.*

Proof. It follows from Lemma 4 that the identity (1) holds in the algebra $\omega\overline{Q}_n$. Then, as shown before Lemma 5, the set $T = 1 - \omega\overline{Q}_n$ forms a CML. It is obvious that $\overline{Q}_n \subseteq T$. Using (iii) of Lemma 3 it is easy to show that $\omega T = \omega\overline{Q}_n$. We denote by H the subloop of the CML T, generated by all the expressions of the form

$$\overline{\lambda} = \lambda(1 - u_1, 1 - u_2, \dots, 1 - u_{19}),$$

$$\overline{\tau}_k = \tau_k(1 - u_1, 1 - u_2, \dots, 1 - u_5; 1 - v_1, 1 - v_2, \dots, 1 - v_{24k}),$$

$$\overline{\nu}_k = \nu_k(1 - u_2, 1 - u_3, 1 - u_5; 1 - v_1, 1 - v_2, \dots, 1 - v_{24k};$$
$$1 - w_1, 1 - w_2, \dots, 1 - w_{12}),$$

where $u_i, v_i, w_i \in \omega\overline{Q}_n$, with $k \neq n, k \neq 2n$. It follows from (12)-(15) that the identity $\tau_n = 1$ is not a corollary to the system of the identities $\lambda = 1, \tau_k = 1, \nu_k = 1$ (for $k \neq n$ and $k \neq 2n$). Then it follows from (15) and the isomorphism of the CMLs Q_n and $\overline{Q}_n$ that for certain $g_1, g_2, \dots, g_5, r_1, r_2, \dots, r_{24k} \in \overline{Q}_n$

$$\tau_k(g_1, g_2, \ldots, g_5; r_1, r_2, \ldots, r_{24k}) \notin H. \qquad (18)$$

As $\overline{\lambda}, \overline{\tau}_k, \overline{\nu}_k$ are defined by the associators of the CML T, it is easy to show that the subloop H is invariant with respect to the inner mapping group of the CML H. Therefore, it is normal in T. We denote by kerφ the kernel of the homomorphism $F\overline{Q}_n \to F(T/H)$, where $\varphi(\sum_{t\in T}\alpha_t t) = \sum_{t\in T}\alpha_t(tH)$. Let the image of the element $\overline{\nu}_k \in T$ under the homomorphism $T \to T/H$ has the form $\alpha_k\beta_k$ whenever $u_i = \alpha_i(1-g_i), v_i = \beta_i(1-r_i), w_i = \gamma_i(1-s_i)$ $(\alpha_i, \beta_i, \gamma_i \in F)$ in $\overline{\tau}_k$ and $\overline{\nu}_k$. Then $\alpha_k\beta_k = 1, \alpha_k = \beta_k^{-1}$. Here α_k, β_k are associators of the CML T/H. With the help of (3) we present β_k^{-1} in the form ν_k in which the parenthesis distribution $[,]$ in ν_k coincides with the parenthesis distribution $\{,\}$ in the second member of the expression η_k (see the notations in (16)). The parenthesis distribution in α_k and in the first member of η_k coincide. Now we use the equality (8) for α_k and γ_k. We assume that $\alpha_k = 1 - \overline{\alpha}_k, \gamma_k = 1 - \overline{\gamma}_k$. As the identity $\alpha_k = \gamma_k$ holds in the CML T/H, it follows from the relation $F\overline{Q}_n/\text{ker}\varphi \cong F(T/H)$ that the identity $\overline{\alpha}_k = \overline{\gamma}_k$ holds in the algebra $F\overline{Q}_n/\text{ker}\varphi$. Consequently, $\overline{\alpha}_k - \overline{\gamma}_k \in \text{ker}\varphi$. But $\overline{\alpha}_k - \overline{\gamma}_k = \overline{\eta}_k$. Therefore $\overline{\eta}_k \in \text{ker}\varphi$. By analogy, we obtain $\overline{\theta}, \overline{\xi}_k \in \text{ker}\varphi$ from the relations $\overline{\lambda}, \overline{\tau}_k \in H$. Then it follows from the definition of the ideal I that $I \subseteq \text{ker}\varphi$. Finally, it follows from (18) that the identity $\tau_k = 1$ does not hold in the CML, being the image of the CML $\overline{Q}_n$ under the homomorphism $F\overline{Q}_n \to F\overline{Q}_n/\,\text{ker}\varphi$. Then it follows from the homomorphism $\overline{F\overline{Q}_n/I} \to F\overline{Q}_n/\text{ker}\varphi$ that the identity $\tau_k = 1$ does not hold in the CML $\overline{\overline{Q}_n}$ as well, being the image of the CML $\overline{Q}_n$ under the homomorphism $F\overline{Q}_n \to F\overline{Q}_n/I$. This completes the proof of Lemma 8. ■

Let $f = f(x_1, x_2, \ldots, x_t)$ be one of the polynomials θ, ξ_k, η_k appeared in (16). By the definition of $\{,,\}$, we pass to the operations $(+), (\cdot)$ in f and we introduce in the natural way the notions of *degree* on every variable x_i, *degree* and *homogeneity* of polynomials for the obtained polynomials. Let us write f in the form $f = f_0 + f_1 + \ldots + f_{r_1}$, where f_i is the sum of all the monomials of the polynomial f that have the degree i on x_1. Let $u_1, u_2, \ldots, u_t$ be the elements of the algebra $\omega\overline{C}_n/I$, determined above. For simplicity we write $f(\boldsymbol{u})$ instead of $f(u_1, u_2, \ldots, u_t)$. If $\alpha \in F$, then $f(\alpha u_1, u_2, \ldots, u_t) = f_0(\boldsymbol{u}) + \alpha f_1(\boldsymbol{u}) + \alpha^2 f_2(\boldsymbol{u}) + \ldots + \alpha^{r_1} f_{r_1}(\boldsymbol{u})$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{r_1}$ be arbitrary elements from F. Then, by (17), we get a system consisting of the r_1 equations

$$f_0(\boldsymbol{u}) + \alpha_i f_1(\boldsymbol{u}) + \ldots + \alpha_i^{r_1} f_{r_1}(\boldsymbol{u}) = 0$$

with variables $f_0(\boldsymbol{u}), f_1(\boldsymbol{u}), \ldots, f_{r_1}(\boldsymbol{u})$. By [4] (see p. 376), $d_1 f_j(\boldsymbol{u}) = 0$, where d_1 is the determinant of this system. We assumed that the field F is infinite. Then we can choose $\alpha_1, \alpha_2, \ldots, \alpha_{r_1}$ such that $d_1 \neq 0$. Then we obtain $f_j(\boldsymbol{u}) = 0$. Doing the same procedure with the polynomials f_{j_i} and variable $x_2 \ldots, x_t$ successively, we finally get the following statement:

Lemma 9. *Let $f = f_1(x_1, x_2, \ldots, x_t) + \ldots + f_i(x_1, x_2, \ldots, x_t) + \ldots + f_r(x_1, x_2, \ldots, x_t)$ be the decomposition of the polynomial f into homogeneous components $f_i(x_1, x_2, \ldots, x_t)$ and let $u_1, u_2, \ldots, u_t$ be the elements of the algebra $\omega\overline{Q}_n/I$ determined above. Then $f_i(u_1, u_2, \ldots, u_t) = 0$.* ∎

In particular, examining the homogeneous components of the least degree in each of the cases $\overline{\theta}, \overline{\xi}_k, \overline{\eta}_k$ and taking into account the identity $(x, y, z) = -(x, z, y)$ of the alternative algebra, we infer that the equalities:

$$((u_1, u_2, u_3, u_4, u_5), (u_6, u_7, u_8, u_9, u_{10}),$$

$$(u_{11}, u_{12}, u_{13}, u_{14}, u_{15}), (u_{16}, u_{17}, u_{18}), u_{19}) = 0;$$

$$((u_1, u_2, u_3, u_4, u_5), (v_1, v_2, v_3), \ldots, (v_{12k-2}, v_{12k-1}, v_{12k}),$$

$$((u_1, u_2, u_3, u_4, u_5), (v_{12k+1}, v_{12k+2}, v_{12k+3}), \ldots,$$

$$\ldots, (v_{24k-2}, v_{24k-1}, v_{24k})), (u_1, u_2, u_3, u_4, u_5)) = 0;$$

$$((w_1, u_2, u_3, w_2, u_5), (v_1, v_2, v_3), \ldots, (v_{12k-2}, v_{12k-1}, v_{12k}),$$

$$((w_3, u_2, u_3, w_4, u_5), (v_{12k+1}, v_{12k+2}, v_{12k+3}), \ldots$$

$$\ldots, (v_{24k-2}, v_{24k-1}, v_{24k})), (w_5, u_2, u_3, w_6, u_5))+$$

$$+ ((w_7, u_2, u_3, w_8, u_5), (v_1, v_2, v_3), \ldots, (v_{12k-2}, v_{12k-1}, v_{12k}),$$

$$((w_9, u_2, u_3, w_{10}, u_5), (v_{12k+1}, v_{12k+2}, v_{12k+3}), \ldots$$

$$\ldots, (v_{24k-2}, v_{24k-1}, v_{24k})), (w_{11}, u_2, u_3, w_{12}, u_5)) = 0,$$

hold in the algebra $\omega\overline{Q}_n/I$ (for $k \neq n, k \neq 2n$), where $w_1, w_2, \ldots, w_{12}$ take values u_1, u_4 exactly as in the previous case.

The algebra $\omega\overline{Q}_n$ is the homomorphic image of the “augmentation ideal” ωQ_n. Then it follows from (iii) of Lemma 3 that $\omega\overline{Q}_n$ is generated as F-module by the elements of the form $a_i = 1 - g_i$, where $g_i \in Q_n$. We denote by u_i the image of the element a_i under the homomorphism $\omega\overline{Q}_n \to \omega\overline{Q}_n/I$. Then any element v from $\omega\overline{Q}_n/I$ has the decomposition $v = \alpha_1 u_1 + \ldots + \alpha_t u_t$. Now, by induction on length t from the last equalities it is easy to prove the statement.

Lemma 10. *The identities* (2) *and* $\mu_k = 0$ *hold in the algebra* $\omega\overline{Q}_n/I$ *for* $k \neq n, k \neq 2n$. ■

Let G be CML and $a_1, a_2, \ldots, a_{2i+1}, b_1, b_2, \ldots, b_{2j+1}, c_1, c_2, \ldots, c_{2m+1}$ be elements in G. We will inductively define the associator of multiplicity k with the β^k parenthesis distribution. The associators of multiplicity 0 are the elements of the CML G, and the associators of multiplicity 1 with the β^1 parenthesis distribution are the associators from G of the form $[a_1, a_2, a_3]$. If $\beta^i(a_1, a_2, \ldots, a_{2i+1}), \beta^j(b_1, b_2, \ldots, b_{2j+1}), \beta^m(c_1, c_2, \ldots, c_{2m+1})$ are, respectively, associators of multiplicity i, j, m with the $\beta^i, \beta^j, \beta^m$ parenthesis distribution, then $[\beta^i(a_1, a_2, \ldots, a_{2i+1}), \beta^j(b_1, b_2, \ldots, b_{2j+1}), \beta^m(c_1, c_2, \ldots, c_{2m+1})]$ is an associator of multiplicity $i + j + m + 1$ with the $\beta^{i+j+m+1}$ parenthesis distribution.

Lemma 11. *Let a CML G with the lower central series $G = G_0 \supseteq G_1 \supseteq \ldots$ be generated by the elements $a_1, a_2, \ldots$ and let $\beta^k(b_1, b_2, \ldots, b_{2k+1})$ be the associator of the CML G of multiplicity k with a certain β^k parenthesis distribution. Then:*

1) $\beta^k(b_1, b_2, \ldots, b_{2k+1}) \in G_k$;
2) *the quotient loop G_k/G_{k+1} is generated by those cosets that contain associators of the form*

$$[a_{i_1}, a_{i_2}, \ldots, a_{i_{2k+1}}], \tag{19}$$

where $a_{i_j} \in \{a_1, a_2, \ldots\}$.

Proof. The first assertion follows easily from (6) by induction on k. The second assertion will be also proved by induction on k. Under $k = 0$ the elements of form (19) are generators of CML and, as a consequence, the

cosets that contain these elements generate the quotient loop G_0/G_1. Let us assume that the quotient loop G_k/G_{k+1} is generated by the cosets that contain elements of form (19). As $G_{k+1} = [G_k, G, G]$ is generated by the elements $[h, g, f]$, where $h \in G_k$ and $g, f \in G$, it is obvious that the quotient loop G_{k+1}/G_{k+2} is generated by the cosets that contain these elements. Moreover, by induction hypotheses, $h = \prod_{i=1}^n h_i^{\epsilon_i} \cdot h'$, where $\epsilon_i = \pm 1$, $h' \in G_{k+1}$, $h \in G_k$ and every h_i is an associator of form (19). It follows from (5), (6) and (3) that
$[h, g, f] = [\prod_{i=1}^n h_i^{\epsilon_i} \cdot h', g, f] = [\prod_{i=1}^n h_i^{\epsilon_i}, g, f][h', g, f] \pmod{G_{k+2}} = \prod_{i=1}^n [h_i^{\epsilon_i}, g, f][h', g, f] \pmod{G_{k+2}} = \prod_{i=1}^n [h_i^{\epsilon_i}, g, f] \pmod{G_{k+2}} = \prod_{i=1}^n [h_i, g, f]^{\epsilon_i} \pmod{G_{k+2}}$.

Further, suppose that $g = \prod_{j=1}^r a_j^{\bar{\epsilon}_j}$, $f = \prod_{m=1}^s a_m^{\tilde{\epsilon}_m}$. Therefore it follows again from (5), (6), and (3) that
$[h, g, f] = [\prod_{i=1}^n h_i, g, f]^{\epsilon_i} \pmod{G_{k+2}} = \prod_{i=1}^n [h_i, \prod_{j=1}^r a_j, f]^{\epsilon_i} \pmod{G_{k+2}}$ $= \prod_{i=1}^n (\prod_{j=1}^r [h_i, a_j, f]^{\bar{\epsilon}_j})^{\epsilon_i} \pmod{G_{k+2}} = \prod_{i=1}^n (\prod_{j=1}^r [h_i, a_j, \prod_{m=1}^s a_m^{\tilde{\epsilon}_m}]^{\bar{\epsilon}_j})^{\epsilon_i}$ $\pmod{G_{k+2}} = \prod_{i=1}^n (\prod_{j=1}^r (\prod_{m=1}^s [h_i, a_j, a_m^{\tilde{\epsilon}_m}]^{\bar{\epsilon}_j})^{\epsilon_i} \pmod{G_{k+2}} = \prod_{i=1}^n (\prod_{j=1}^r (\prod_{m=1}^s [h_i, a_j, a_m]^{\tilde{\epsilon}_m})^{\bar{\epsilon}_j})^{\epsilon_i} \pmod{G_{k+2}}$.

Thus $[h_i, a_j, a_m]$ have the form indicated in (19). This completes the proof of Lemma 11. ∎

We remind that a 3-*Lie algebra* $(L; (\,,\,,\,))$ is a linear space L over the associative and commutative ring with identity with a certain 3-linear operation $(\,,\,,\,)$ on Q which satisfies the identities (see [3]):

$$(x, x, y) = 0, (x, y, x) = 0, (y, x, x) = 0,$$

$$((x, y, z), u, v) = ((x, u, v), y, z) + (x, (y, u, v), z) + (x, y, (z(u, v)). \tag{20}$$

In an arbitrary alternative commutative algebra A the identity $((x, y, z), u, v) = ((x, u, v),\ y, z) + ((y, u, v), z, x) + ((z, u, v), x, y)$ holds, where $(x, y, z) = xy \cdot z - x \cdot yz$ (see [9]). Then, by the bi-associativity of alternative algebra (cf. [15]), the set A with respect to the ternary operation (x, y, z) becomes a 3-Lie algebra. Let us denote it by $\Lambda(A)$.

Let now G be an arbitrary centrally nilpotent CML that satisfies the identity (7) and let $G = G_0 \supset G_1 \supset \ldots G_s = \{1\}$ be its lower central series. As in the case of groups and Lie algebras [5], we tie the 3-Lie algebra $L(G)$ with CML G. By (6) we have $G_{i+1} \supset G_{3i+1} = [G_i, G_i, G_i]$; then

$C_i = G_i/G_{i+1}$ is an abelian group. Let $L(C)$ be a direct sum of groups $C_1, C_2, \ldots, C_{s-1}$. We define the addition $\oplus$ on $L(G)$ by the formula

$$g \oplus h = g_1h_1 + g_2h_2 + \ldots + g_{s-1}h_{s-1}, \tag{21}$$

where $g = g_1 + g_2 + \ldots + g_{s-1}, h = h_1 + h_2 + \ldots + h_{s-1}$. It is obvious that "zero" of the group $L(G)$ is the element $1 + 1 + \ldots$ and the element $g_1^{-1} + g_2^{-1} + \ldots$ is "opposite" to g.

We introduce on group $L(G)$ the ternary operation $(,,)$. Let $a \in G_i, b \in G_j, c \in G_k, u \in G_{i+1}, v \in G_{j+1}, w \in G_{k+1}$. Then it follows from (5) and (6) that

$$[au, bv, cw]G_{i+j+k+2} = [a, b, c]G_{i+j+k+2}.$$

Now it is clear that if $g_i = aG_{i+1}, g_j = bG_{j+1}, g_k = cG_{k+1}$, then $(g_i, g_j, g_k) = [a,b,c]G_{i+j+k+2}$ is a certain element of group $C_{i+j+k+1} = G_{i+j+k+1}/G_{i+j+k+2}$. We extend operation $(,,)$ on the whole group $L(G)$ by the formula

$$(g, h, r) = \sum_{i,j,k+1}^{s-1} (g_i, h_j, r_k), \tag{22}$$

where g, h, r are elements in $L(G)$, and $\sum$ means addition $\oplus$ in the group $L(G)$. Let us show that the operation $(,,)$ is distributive with respect to $\oplus$. Let $f, g, h, r \in L(G)$. Let us show that the expressions

$$(f, g, h \oplus r), \quad (f, g, h) \oplus (f, g, r) \tag{23}$$

are equal in $L(G)$. The first of the expressions (23) is, by definition, equal to

$$\sum_{i,j,k=1}^{s-1} (f_i, g_j, h_k r_k).$$

Let $f_i = aG_{i+1}, g_j = bG_{j+1}, h_k = cG_{k+1}, r_k = dG_{k+1}$. Then by (5) and (6)

$$(f_i, g_j, h_k r_k) = (aG_{i+1}, bG_{j+1}, (cd)G_{k+1}) = (a, b, cd)G_{i+j+k+2} =$$

$$(a, b, c)(a, b, d)G_{i+j+k+2} = (a, b, c)G_{i+j+k+2} \cdot (a, b, d)G_{i+j+k+2} =$$

$$(aG_{i+1}, bG_{j+1}, cG_{k+1}) \cdot (aG_{i+1}, bG_{j+1}, dG_{k+1}) = (f_i, g_j, h_k) \cdot (f_i, g_j, r_k).$$

Consequently, we obtain

$$\sum_{i,j,k=1}^{s-1}(f_i,g_j,h_kr_k)=\sum_{i,j,k=1}^{s-1}(f_i,g_j,h_k)\cdot(f_i,g_j,r_k)=(f,g,h)\cdot(f,h,r).$$

In such a way we have seen that both expressions (23) coincide. Other relations of distributivity can be proved by analogy. Finally, it follows from the di-associativity of CML (cf. [1]) and the identity (4) that $L(G)$ is a 3-Lie algebra. Consequently, we have proved

Proposition 1. *Let G be an arbitrary centrally nilpotent CML with the lower central series $G=G_0\supset G_1\supset\ldots\supset G_s=\{1\}$. Then the direct sum $L(G)$ of the modules G_i/G_{i+1}, $i=0,1,\ldots,s-1$, on the operations* (21) *and* (22) *will be a* 3*-Lie algebra.* ∎

Let us now suppose that a CML G, that satisfies the identity (7) is generated by the set $X=\{x_1,x_2,\ldots,x_t\}$. We put $y_i=1-x_i$. It follows from the definition of the "augmentation ideal" ωG of the "loop algebra" FQ that $y_i\in\omega G$. We denote by A the subalgebra of algebra ωG, generated by the elements $y_1,y_2,\ldots,y_t$. By Lemma 4, the algebra A satisfies the identity (1), so is nilpotent [14]. Then for every monomial $v\in A$ there exists a number m such that $v\in A^m\backslash A^{m+1}$. The number m will be called the *weight* of the monomial v. The polynomial that only consists of monomials of the weight m will be called *homogeneous of the weight m*. Let U be a word of CML G from the generating set X. We consider in U the generators y_i of the algebra A, using the relation $x_i=1-y_i$. Let us assume that U has the decomposition

$$U=1-(u_m+u_{m+1}+\ldots,u_r,)\tag{24}$$

in A, where u_i is a homogeneous polynomial from A of the weight i and u_m is a polynomial of the smallest weight. We determine the mapping $\delta:G\to A$ by the formula: $\delta(U)=0$ if $U=1$, and $\delta(U)=u_m$ for all the other cases.

Lemma 12. *Let U, V, W be words $(\neq 1)$ of CML G from the generating set X and let $\delta(U)=u_m$, $\delta(V)=v_k$, $\delta(W)=w_n$. Then for every integer l*

$$\delta(U^l)=lu_m.\tag{25}$$

If $m < k$, then

$$\delta(UV) = u_k. \tag{26}$$

If $m = k$ and $u_k + v_k \neq 0$, then

$$\delta(UV) = u_k + v_k. \tag{27}$$

If $m = k$ and $u_k + v_k = 0$, then $UV = 1$ or $\delta(UV)$ belongs to A^t, where $t > m$. If $(u_m, v_k, w_n) \neq 0$, then

$$\delta([U, V, W]) = (u_m, v_k, w_n). \tag{28}$$

If $(u_m, v_k, w_n) = 0$, then $[U, V, W] = 1$ or $\delta([U, V, W])$ belongs to A^t, where $t > m + k + n$.

Proof. We put $u_m + m_{m+1} + \ldots, u_r = u$. Then $U = 1 - u$. To prove (25) we use the decomposition $(1-u)^l = \sum_{t=0}^{l} (-1)^t \binom{l}{t} u^t$, where $\binom{l}{t} = \frac{l(l-1)\ldots(l-t+1)}{t!}$. As $u \in A$, all non-constant members of the smallest weight of the element $(1-u)^l$ are of the form $-lu$. Then (25) is proved.

The assertions (26), (27) follow from the multiplication rules, and the remaining assertions follow from Lemma 5. ∎

We put $D_k = \{g \in G \mid 1 - g \in (\omega G)^k\}$. It is easy to see that D_k is the kernel of homomorphism, induced on the CML G by the natural homomorphism $FG \to FG/(\omega G)^k$. By Lemma 2, G/D_k is a CML, so D_k is a normal subloop of the CML G (see [1]). It follows from Lemma 12.

Lemma 13. *If G_m is the m-th member of the lower central series of a CML G, then $G_m \subseteq D_{2m+1}$.*

Proof. We will use the induction on m. We have $G_o = G = D_1$. Let us suppose that $G_m \subseteq D_{2m+1}$ and let $a \in G_m, u, v \in G$. Then $[a, u, v] = 1$, or $\delta([a, u, v])$ has a weight not less than $2m + 3$, as $\delta(a)$ has a weight not less than $2m + 1$. In any case $[a, u, v] \in D_{2m+3}$, and therefore $G_{m+1} \subseteq D_{2m+3}$. This completes the proof of Lemma 13. ∎

The CML G is generated by the finite set X. Then by Lemma 1 it is centrally nilpotent. Assume that its lower central series has the form $G = G_0 \supset G_1 \supset \ldots \supset G_s = \{1\}$.

Proposition 2. *Let G and A be the algebras considered above. Then the mapping $x_iG_1 \to y_i$ induces the monomorphism of 3-Lie algebra $L(G)$ into 3-Lie algebra $A \subseteq \Lambda(\omega G)$. Obviously, the monomorphism is determined in the following way:*

Let $\beta^k(x_{i_1}, x_{i_2}, \ldots, x_{2k+1})$, where $x_{i_j} \in X$, be an associator of multiplicity k of the CML G with the β^k parenthesis distribution and let $\beta^k(x_{i_1}, x_{i_2}, \ldots, x_{i_{2k+1}}) \in G_{\mu(k)}/G_{\mu(k)+1}$. Then the mapping

$$\beta^k(x_{i_1}, x_{i_2}, \ldots, x_{i_{2k+1}})G_{\mu(k)+1} \to \beta^k(y_{i_1}, y_{i_2}, \ldots, y_{i2k+1})$$

is a monomorphism of quotient loop $G_{\mu(k)}/G_{\mu(k)+1}$ in the additive group $\Lambda_{\mu(k)}(A)$, where $\Lambda_{\mu(k)}(A)$ is a submodule of module $\Lambda(A)$, consisting of homogeneous polynomials of the weight $\mu(k)$, and the parenthesis distribution β^k means multiplicity in $\Lambda(A)$.

Proof. By the definition (22) of the multiplication operation in the algebra $L(G)$ and (6), and also by the relation between the operation of taking the associator into the group G_k/G_{k+1} and the multiplication in the algebra $\Lambda(\omega G)$, indicated in (28), the expression $\beta^k(x_{i_1}, x_{i_2}, \ldots, x_{i_{2k+1}})$ obviously turns into an element $\beta^k(y_{i_1}, y_{y_2}, \ldots, y_{i_{2k+1}})$ of the algebra $\Lambda(A)$.

Further, by Lemma 13, the arbitrary element U from G_k/G_{k+1}, under the mapping $x_i \to y_i$, turns into an element of the algebra A of the form

$$1 + u_{2k+1} + u_{2k+2} + \ldots + u_t,$$

where u_j has the weight j or equals zero, and $j > 2k + 1$. This lemma also shows that the equality

$$\delta(UG_{k+1}) = \delta(U) = u_{2k+1}$$

determines the mapping δ_{2k+1} of group $C_k = G_k/G_{k+1}$ in the set of homogeneous elements of the weight $2k + 1$ of the algebra A.

Moreover, by the definition (21) of the addition operation of the algebra $L(G)$ and (24)-(27), δ_{2k+1} is a linear mapping C_k in A^{2k+1}. By Lemma 11 the associators of the form $[x_1, x_2, \ldots, x_{2k+1}]$ generate the subloop G_k, therefore the mapping

$$\delta(V) = \delta_1(v_1) + \delta_3(v_3) + \ldots, +\delta_{2k+1}(v_{2k+1}) + \ldots$$

is a linear mapping of Z_3-module $L(G)$ into Z_3-module A, where Z_3 means the field of three elements. Consequently, the mapping $x_iG_1 \rightarrow y_i$ induces the homomorphism of 3-Lie algebra $L(G)$ in A.

By [10] the subloop G_1, generated by all the associators of the CML G, belongs to the Frattini subloop. Therefore the mapping $x_iG_1 \rightarrow y_i$ is one-to-one. If a, b, c are elements in G, then it follows from Lemma 5 that $[a, b, c] = 1 - (a^{-1} \cdot b^{-1}c^{-1})(a, b, c)$. Therefore, if $[a, b, c] \neq 1$ then $(a, b, c) \neq 0$. Consequently, it is easy to show by induction that if $\beta^k(x_{i_1}, x_{i_2}, \ldots, x_{i_{2k+1}}) \neq 1$, then $\beta^k(y_{i_1}, y_{i_2}, \ldots, x_{i_{2k+1}}) \neq 0$. Then it follows from (28) that the mapping $x_iG_1 \rightarrow y_i$ induces the monomorphism of 3-Lie algebra $L(G)$ into the 3-Lie algebra A. This completes the proof of Proposition 2. ■

It follows from Lemma 8 and Proposition 2 that

Lemma 14. *In the algebra $\omega\overline{Q}_n/I$ the identity $\mu_n = 0$ does not hold.* ■

It is obvious that the identities $xy = yx$ and $x^2 \cdot yx = x^2y \cdot x$ hold in the algebra $\omega\overline{Q}_n/I$, i.e. the algebra $\omega\overline{Q}_n/I$ is Jordan. Then from Lemmas 10 and 14 we immediately obtain

Theorem 1. *The infinite system of identities $\{\mu_k = 0\}(k = 1, 2, \ldots)$ is independent in the variety $\mathfrak{A}_3 \cap \mathfrak{S}_2\mathfrak{S}_2$ of alternative commutative (Jordan) algebras over the infinite field of characteristic* 3. ■

If a certain identity is deduced from the system of identities $\{\mu_k = 0\}, k = 1, 2, \ldots$, then in its deduction only a finite number of identities of this system can be used. Therefore, if the system of identities $\{\mu_k = 0\}$ were equivalent to a certain finite system of identities, then it would be equivalent to one of its finite subsystem. Consequently, from Theorem 1 we obtain

Corollary 1. *Any infinite subset of the system of identities $\{\mu_k = 0\}, k = 1, 2, \ldots$, is not equivalent to any finite system of identities.*

Corollary 2. *In the variety* $\mathfrak{A}_3 \cap \mathfrak{S}_2\mathfrak{S}_2$ *of alternative commutative (Jordan) algebras over the infinite field of characteristic* 3 *there exists an algebra, given by the enumerable set of relations, in which the word problem is unsolvable.*

Proof. Let S be some recursively enumerable and non-recursive set of numbers. Let us examine the algebra A of the variety $\mathfrak{A}_3 \cap \mathfrak{S}_2\mathfrak{S}_2$, defined by the identical relations $\{\mu_n = 0\}$ for $n \in S$. It is obvious that each relation of the algebra A is an identical relation. By Theorem 1 the arbitrary identity from $\{\mu_n = 0\}$ under a given n is true in A if and only if $n \in S$. Therefore, in A the problem of word equality is unsolvable. ∎

Corollary 3. *The variety* $\mathfrak{A}_3 \cap \mathfrak{S}_2\mathfrak{S}_2$ *of alternative commutative (Jordan) algebras over the infinite field of characteristic* 3 *contains a continuum of different infinite based subvarieties.*

This statement follows directly from Theorem 1 and Corollary 1. ∎

Added in proof (by the editors): A.V. Badeev's thesis: "On Spechtness of varieties of commutative alternative algebras over a field of characteristic 3 and commutative Moufang loops" (Moscow 1999) is closely related to topics of the paper. See also the paper:
A.V. Badeev, *On the Specht property of varieties of commutative alternative algebras over a field of characteristic* 3 *and of commutative Moufang loops*, Sibirsk. Mat. Zh. **41** (2000), 1252–1268.

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