Discussiones Mathematicae
General Algebra and Applications 23 (2003) 163–212

ADJOINTNESS BETWEEN THEORIES AND STRICT THEORIES

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Dedicated to Prof. Dr. habil. Klaus Denecke
on the occasion of his 60th birthday

Abstract

The categorical concept of a theory for algebras of a given type was foundet by Lawvere in 1963 (see [8]). Hoehnke extended this concept to partial heterogenous algebras in 1976 (see [5]). A partial theory is a *dhts*-category such that the object class forms a free algebra of type (2,0,0) freely generated by a nonempty set J in the variety determined by the identities $ox \approx o$ and $xo \approx o$, where o and i are the elements selected by the 0-ary operation symbols.

If the object class of a *dhts*-category forms even a monoid with unit element I and zero element O, then one has a strict partial theory.

In this paper is shown that every J-sorted partial theory corresponds in a natural manner to a J-sorted strict partial theory via a strongly d-monoidal functor. Moreover, there is a pair of adjoint functors between the category of all J-sorted theories and the category of all corresponding J-sorted strict theories.

This investigation needs an axiomatic characterization of the fundamental properties of the category $\underline{Par}$ of all partial function between arbitrary sets and this characterization leads to the concept of *dhts*- and $dhth\nabla s$-categories, respectively (see [5], [11], [13]).

Keywords: symmetric monoidal category, *dhts*-category, partial theory, adjoint functor.

2000 Mathematics Subject Classification: 18D10, 18D20, 18D99, 18A25, 08A55, 08C05, 08A02.

1. Introduction

Heterogeneous algebras (many-sorted algebras) are, as well-known, algebraic systems consisting of a family of carrier sets and a family of functions such that their definition domain are cartesian products of certain carrier sets and their values are elements of a distinguished carrier set. The concept of such algebraic systems was independently introduced and investigated by P.J. Higgins ([4]) and G. Birkhoff & J.D. Lipson ([1]).

The development of a functorial semantic of algebraic theories for heterogeneous partial algebras requires a good knowledge about diagonal-halfterminal-halfdiagonal-inversional-symmetric monoidal categories ($dhth\nabla s$-categories).

The morphism class of a category K will be denoted by K too, the object class of K by $|K|$, and the set of all morphisms in K out of an object A into an object B by $K[A,B]$.

The concept of a symmetric monoidal category in the sense of ([3]) is of fundamental importance.

Definition 1.1 ([3]). A sequence

$$K^\bullet = (K, \otimes, I, a, r, l, s)$$

is called *symmetric monoidal category*, if K is a category, $\otimes : K \times K \to K$ is a bifunctor, I is a distinguished object of K, $a = (a_{A,B,C} \in K[A \otimes (B \otimes C), (A \otimes B) \otimes C] \mid A, B, C \in |K|)$, $r = (r_A \in K[A \otimes I, A] \mid A \in |K|)$, $l = (l_A \in K[I \otimes A, A] \mid A \in |K|)$, $s = (s_{A,B} \in K[A \otimes B, B \otimes A] \mid A, B \in |K|)$ are families of isomorphisms in K (associativity, right-identity, left-identity, symmetry) such that

(F1) $\forall \rho, \rho' \in K \ (\operatorname{dom}(\rho \otimes \rho') = \operatorname{dom}\rho \otimes \operatorname{dom}\rho')$,

(F2) $\forall \rho, \rho' \in K \ (\operatorname{cod}(\rho \otimes \rho') = \operatorname{cod}\rho \otimes \operatorname{cod}\rho')$,

(F3) $\forall A, B \in |K| \ (1_{A \otimes B} = 1_A \otimes 1_B)$,

(F4) $\forall A, B, C, A', B', C' \in |K| \ \forall \rho \in K[A, B], \sigma \in K[B, C],$
$\rho' \in K[A', B'], \sigma' \in K[B', C'] \ ((\rho \otimes \rho')(\sigma \otimes \sigma') = \rho\sigma \otimes \rho'\sigma')$,

(M1) $\forall A, B, C, D \in |K|$
$(a_{A,B,C \otimes D} a_{A \otimes B, C, D} = (1_A \otimes a_{B,C,D}) a_{A, B \otimes C, D} (a_{A,B,C} \otimes 1_D))$,

(M2) $\forall A, B \in |K| \ (a_{A,I,B}(r_A \otimes 1_B) = 1_A \otimes l_B)$, (M3) $\forall A,B,C\in|K|(a_{A,B,C}s_{A\otimes B,C}a_{C,A,B}=(1_A\otimes s_{B,C})a_{A,C,B}(s_{A,C}\otimes 1_B))$,

(M4) $\forall A,B\in|K|\ (s_{A,B}s_{B,A}=1_{A\otimes B})$,

(M5) $\forall A\in|K|\ (s_{A,I}l_A=r_A)$,

(M6) $\forall A,B,C,A',B',C'\in|K|\ \forall\rho\in K[A,A'],\sigma\in K[B,B'],\tau\in K[C,C']$
$$(a_{A,B,C}((\rho\otimes\sigma)\otimes\tau)=(\rho\otimes(\sigma\otimes\tau))a_{A',B',C'}),$$

(M7) $\forall A,A'\in|K|\ \forall\rho\in K[A,A']\ (r_A\rho=(\rho\otimes 1_I)r_{A'})$,

(M8) $\forall A,B\in|K|\ \forall\rho\in K[A,A'],\sigma\in K[B,B']\ (s_{A,B}(\sigma\otimes\rho)=$
$$=(\rho\otimes\sigma)s_{A',B'}).$$

A symmetric monoidal category is called *symmetric strictly monoidal*, if all associativity, right-identity, and all left-identity isomorphisms, are unit morphisms, i.e. identity morphisms in K (in the other terminology), only.

The defining conditions determine a lot of properties as follows.

Corollary 1.2. *Let $K^\bullet$ be a symmetric monoidal category. Then*

(M9) $\forall A,B\in|K|\ (a_{I,A,B}(l_A\otimes 1_B)=l_{A\otimes B})$,

(M10) $\forall A,B\in|K|\ (a_{A,B,I}r_{A\otimes B}=1_A\otimes r_B)$,

(M11) $r_I=l_I$,

(M12) $s_{I,I}=1_{I\otimes I}$,

(M13) $\forall A\in|K|\ (s_{I,A}r_A=l_A)$,

(M14) $\forall A,A'\in|K|\ \forall\rho\in K[A,A']\ (l_A\rho=(1_I\otimes\rho)l_{A'})$,

(ASR) $\forall A,B\in|K|\ (a^{-1}_{A,B,I}(1_A\otimes s_{B,I})a_{A,I,B}=r_{A\otimes B}(r^{-1}_A\otimes 1_B))$,

(ASL) $\forall A,B\in|K|\ (a_{I,A,B}(s_{I,A}\otimes 1_B)a^{-1}_{A,I,B}=l_{A\otimes B}(1_A\otimes l^{-1}_B))$.

Defining

(B1) $b_{A,B,C,D}:=a_{A\otimes B,C,D}(a^{-1}_{A,B,C}(1_A\otimes s_{B,C})a_{A,C,B}\otimes 1_D)a^{-1}_{A\otimes C,B,D}$
for arbitrary $A,B,C,D\in|K|$,

one obtains furthermore

(B2) $\forall A, B, C, D \in |K|\ (b_{A,B,C,D} =$
$$= a^{-1}_{A,B,C\otimes D}(1_A \otimes a_{B,C,D}(s_{B,C} \otimes 1_D)a^{-1}_{C,B,D})a_{A,C,B\otimes D}),$$

(M15) $\forall A, B, C, D, A', B', C', D' \in |K|\ \forall \rho \in K[A, A'], \sigma \in K[B, B'],$
$$\lambda \in K[C, C'], \mu \in K[D, D']$$
$$(b_{A,B,C,D}((\rho \otimes \sigma) \otimes (\lambda \otimes \mu) = ((\rho \otimes \lambda) \otimes (\sigma \otimes \mu)b_{A',B',C',D'}),$$

(M16) $\forall A, B, C, D \in |K|\ (b_{A,B,C,D}b_{A,C,B,D} = 1_{A\otimes B} \otimes 1_{C\otimes D}),$

(M17) $\forall A, B, C, D \in |K|\ (b_{A,B,C,D}(s_{A,C} \otimes s_{B,D}) = s_{A\otimes B,C\otimes D}b_{C,D,A,B},$

(M18) $\forall A, A', B, B', C, C' \in |K|$
$$(b_{(A,(B\otimes C)),A',(B'\otimes C')}(1_{A\otimes A'} \otimes b_{B,C,B',C'})a_{A\otimes A',B\otimes B',C\otimes C'}$$
$$= (a_{A,B,C} \otimes a_{A',B',C'})b_{(A\otimes B),C,(A'\otimes B'),C'}(b_{A,B,A',B'} \otimes 1_{C\otimes C'})),$$

or equivalently,
$$\forall A, A', B, B', C, C' \in |K|$$
$$(a_{A\otimes A',B\otimes B',C\otimes C'}(b_{A,A',B,B'} \otimes 1_{C\otimes C'})b_{(A\otimes B),(A'\otimes B'),C,C'}$$
$$= (1_{A\otimes A'} \otimes b_{B,B',C,C'})b_{A,A',(B\otimes C),(B'\otimes C')}(a_{A,B,C} \otimes a_{A',B',C'})),$$

(M19) $\forall A, B \in |K|\ (b_{A,I,I,B} = 1_{A\otimes I} \otimes 1_{I\otimes B}),$

(M20) $\forall A, B \in |K|\ (b_{A,I,B,I} = (r_A \otimes r_B)((1_{A\otimes B} \otimes r_I)r_{A\otimes B})^{-1}),$

(M21) $\forall A, B \in |K|\ (b_{I,A,I,B} = (l_A \otimes l_B)((l_I \otimes 1_{A\otimes B})l_{A\otimes B})^{-1}),$

(M22) $\forall A, B \in |K|\ (b_{I,A,B,I} = s_{I\otimes A,B\otimes I}(s_{B,I} \otimes s_{I,A})),$

(M23) $\forall A, B \in |K|\ (b_{A,B,I,I} = (1_{A\otimes B} \otimes r_I)r_{A\otimes B}(r^{-1}_A \otimes r^{-1}_B)),$

(M24) $\forall A, B \in |K|\ (b_{I,I,A,B} = (l_I \otimes 1_{A\otimes B})l_{A\otimes B}(l^{-1}_A \otimes l^{-1}_B)).$ ■

Remark 1.3. By definition, the object class of a symmetric monoidal category $K^\bullet$ forms an illegitimate algebra $(|K|, \otimes, I)$ of type $(2, 0)$, because the carrier is not a set.

Especially, of interest are objects consisting of finitely many factors I in arbitrary brackets, namely objects of the subalgebra $\langle I\rangle$ generated by the one element set $\{I\}$ as follows:

$$\langle I\rangle^{(0)} := \{I\}, \qquad \langle I\rangle^{(n+1)} := \langle I\rangle^{(n)} \cup \{X \otimes Y \mid X, Y \in \langle I\rangle^{(n)}\},$$

$$\langle I\rangle := \bigcup_{n\,\in\,\mathbb{N}} \langle I\rangle^{(n)}.$$

This is in fact an algebra of type $(2, 0)$. The set $\langle I\rangle$ determines in a natural manner a symmetric monoidal subcategory $\langle I\rangle^\bullet$ of $K^\bullet$.

Moreover, every nonempty set $J \subseteq |K|$, $I \notin J$, determines a subalgebra H of type $(2, 0)$ as follows:

$$H^{(0)} := J \cup \{I\}, \qquad H^{(n+1)} := H^{(n)} \cup \{X \otimes Y \mid X,\, Y \in H^{(n)}\},$$

$$H := \bigcup_{n\,\in\,\mathbb{N}} H^{(n)}.$$

The symmetric monoidal subcategory of $K^\bullet$ generated by H, respectively by J, will be denoted by $H^\bullet$. Obviously, $H^\bullet$ is a small category, since the carrier is a set.

If $K^\bullet$ is a symmetric strictly monoidal category, then $(|K|, \otimes, I)$ is an illegitimate monoid, $\langle I\rangle$ is a one element set and every set J generates a monoid S with unit I.

Definition 1.4 ([10]). Let $K^\bullet$ be a symmetric monoidal category. The monoidal subcategory $\mathbf{C}^\bullet_K$ of $K^\bullet$ generated by the morphism class

$$\{1_X|X\in|K|\}\cup\{a_{X,Y,Z}|X,Y,Z\in|K|\}\cup\{r_X|X\in|K|\}\cup\{l_X|X\in|K|\}$$

$$\cup\{a^{-1}_{X,Y,Z}|X,Y,Z\in|K|\}\cup\{r^{-1}_X|X\in|K|\}\cup\{l^{-1}_X|X\in|K|\}$$

is called *central subcategory* of $K^\bullet$, its morphisms are called *central morphisms* of $K^\bullet$.

Remark 1.5. The class $\mathbf{C}_K$ of all central morphisms of a symmetric monoidal category $K^\bullet$ is given by the construction

$$\mathbf{C}^{(0)}_K := \{1_X \mid X\in|K|\}\cup\{a_{X,Y,Z} \mid X,Y,Z\in|K|\}\cup\{r_X \mid X\in|K|\}\cup\{l_X \mid X\in|K|\}$$

$$\cup\{a^{-1}_{X,Y,Z} \mid X,Y,Z\in|K|\}\cup\{r^{-1}_X \mid X\in|K|\}\cup\{l^{-1}_X \mid X\in|K|\},$$

$$\mathbf{C}^{(n+1)}_K := \mathbf{C}^{(n)}_K \cup \{c_1c_2 \mid c_1\in K[X,Y] \wedge c_2\in K[Y,P] \wedge c_1, c_2 \in \mathbf{C}^{(n)}_K$$

$$\wedge X,Y,P\in|K|\}\cup\{c_1\otimes c_2 \mid c_1,c_2\in\mathbf{C}^{(n)}_K\},$$

$$\mathbf{C}_K = \bigcup_{n\in\mathbb{N}} \mathbf{C}^{(n)}_K$$

and forms a monoidal subcategory $\mathbf{C}^\bullet_K$ of $K^\bullet$.

$\mathbf{C}_K$ consists of unit morphisms only, if $K^\bullet$ is symmetric strictly monoidal. The class of all unit morphisms of K is denotetd by Un_K.

Coherence principle ([9], [6], [7]). *Let $K^\bullet$ be a symmetric monoidal category. Then every planar closed diagram of central morphisms is commutative.*

Corollary 1.6. *Let $K^\bullet$ be a symmetric monoidal category. Then, by the coherence principle, there is at most one central morphism $c_{X,Y} \in K$ between objects X and Y for every $X, Y \in |K|$. The central morphisms are isomorphisms only.*

Let X and Y be arbitrary objects of $\langle I \rangle^\bullet$. Then there is exactly one central morphism in the set $\langle I \rangle[X, Y]$.

The isomorphisms

$$i^{(n)} : I^n \to I \text{ and } i^{*(n)} : \bigotimes_{k=1}^{n}{}^{*} I \to I,$$

$$\text{where } I^n := \bigotimes_{k=1}^{n} I \text{ and } \bigotimes_{k=1}^{1}{}^{*} I := I, \; \bigotimes_{k=1}^{n+1}{}^{*} I := I \otimes \left(\bigotimes_{K=1}^{n}{}^{*} I \right),$$

between the different powers of I and the object I are expressable in the following form:

$$i^{(1)} = 1_I, \; i^{(n+1)} = (i^{(n)} \otimes 1_I) r_I, \; n \geq 1, \text{ especially } i^{(2)} = r_I,$$

$$i^{*(1)} = 1_I, \; i^{*(n+1)} = (1_I \otimes i^{*(n)}) l_I, \; n \geq 1, \text{ especially } i^{*(2)} = l_I.$$

Proof. It remains to show the existence of an central morphism between arbitrary X and Y of $\langle I \rangle$.

a) One proves by induction over the complexity of X: $\forall X \in \langle I \rangle \; \exists c \in \langle I \rangle[X, I] \; (c \in \mathbf{C}_K)$:

$$\forall X \in \langle I \rangle^{(0)} \; (X = I \; \wedge \; 1_I \in \mathbf{C}_K);$$

$$\forall n \in \mathbb{N} \; [\forall X \in \langle I \rangle^{(n)} \; \exists c \in \langle I \rangle[X, I] \; (c \in \mathbf{C}_K) \Rightarrow$$

$$\Rightarrow \forall X \in \langle I \rangle^{(n+1)} \; \exists c \in \langle I \rangle[X, I] \; (c \in \mathbf{C}_K)],$$

since

$$\forall X \in (\langle I\rangle^{(n+1)} \setminus \langle I\rangle^{(n)})\ \exists X_1, X_2 \in \langle I\rangle^{(n)}\ \exists c_i \in \langle I\rangle[X_i, I] \cap \mathbf{C}_K\ (i = 1, 2)$$

$$(X = X_1 \otimes X_2\ \wedge\ c_1 \otimes c_2 \in \mathbf{C}_K \Rightarrow (c_1 \otimes c_2)r_I \in \langle I\rangle[X, I] \cap \mathbf{C}_K).$$

b) One proves by induction over the complexity of Y:

$$\forall X \in \langle I\rangle\ \forall Y \in \langle I\rangle\ \exists c \in \langle I\rangle[X, Y]\ (c \in \mathbf{C}_K).$$

The truth of the assertion for an arbitrary $X \in \langle I\rangle$ and for $Y \in \langle I\rangle^{(0)}$ was shown in a).

$$\forall X \in \langle I\rangle\ \forall n \in \mathbb{N}\ [\forall Y \in \langle I\rangle^{(n)}\ \exists c \in \langle I\rangle[X, Y]\ (c \in \mathbf{C}_K) \Rightarrow$$

$$\Rightarrow \forall Y \in \langle I\rangle^{(n+1)}\ \exists c \in \langle I\rangle[X, Y]\ (c \in \mathbf{C}_K)],$$

since

$$\forall Y \in (\langle I\rangle^{(n+1)} \setminus \langle I\rangle^{(n)})\ \exists Y_1, Y_2 \in \langle I\rangle^{(n)}\ \exists c_1 \in \langle I\rangle[X, Y_1] \cap \mathbf{C}_K$$

$$\exists c_2 \in \langle I\rangle[I, Y_2] \cap \mathbf{C}_K$$

$$(Y = Y_1 \otimes Y_2\ \wedge\ c_1 \otimes c_2 \in \mathbf{C}_K \Rightarrow r_X^{-1}(c_1 \otimes c_2) \in \langle I\rangle[X, Y] \cap \mathbf{C}_K). \qquad \blacksquare$$

Definitions 1.7. Let $K^\bullet$ be a symmetric monoidal category in the sense of [3].

A sequence $(K^\bullet; d)$ is called *diagonal-symmetric monoidal category* (shortly *ds-category*) (in [2] considered in the strict case as a special Kronecker-category, in [13] as "diagonal-symmetrische Kategorie"), if $d = (d_A \in K[A, A \otimes A] \mid A \in |K|)$ is a family of morphisms of K such that

(D1) $\quad \forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ (\varphi d_{[A'} = d_A(\varphi \otimes \varphi))$,

(D2) $\quad \forall A \in |K|\ (d_A(d_A \otimes 1_A) = d_A(1_A \otimes d_A)a_{A,A,A})$,

(D3) $\quad \forall A \in |K|\ (d_A s_{A,A} = d_A)$,

(D4) $\quad \forall A, B \in |K|\ ((d_A \otimes d_B)b_{A,A,B,B} = d_{A \otimes B})$

are fulfilled.

$(K^\bullet, d, t)$ is called *diagonal-terminal-symmetric monoidal category* (*dts-category*) ([2]), if $(K^\bullet, d)$ is a *ds*-category with a family $t = (t_A \mid A \in |K|)$ of terminal morphisms $t_A \in K[A, I]$ such that the conditions

(T1) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ (\varphi t_{A'} = t_A)$
and
(DTR) $\forall A \in |K|\ (d_A(1_A \otimes t_A)r_A = 1_A)$

are right.

$(K^\bullet; d, t, o)$ will be called *diagonal-halfterminal-symmetric monoidal category* or *Hoehnke category* (shortly *dhts*-category) ([5], [11], [13]) , if d and t are morphism families as above and $o : I \to O$ is a distinguished morphism in K related to a distinguished object $O \in |K|$, such that

(D1) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ (d_A(\varphi \otimes \varphi) = \varphi d_{A'})$,

(DTR) $\forall A \in |K|\ (d_A(1_A \otimes t_A)r_A = 1_A)$,

(DTL) $\forall A \in |K|\ (d_A(t_A \otimes 1_A)l_A = 1_A)$,

(DTRL) $\forall A_1, A_2 \in |K| (d_{A_1 \otimes A_2}((1_{A_1} \otimes t_{A_2})r_{A_1} \otimes (t_{A_1} \otimes 1_{A_2})l_{A_2}) = 1_{A_1 \otimes A_2}))$,

(TT) $\forall A, B \in |K|\ (t_{A \otimes B} = (t_A \otimes t_B)t_{I \otimes I})$,

(O1) $\forall A \in |K|\ (A \otimes O = O \otimes A = O)$,

(o1) $\forall A \in |K|\ \forall \varphi \in K[A, O]\ (t_A o = \varphi)$,
and
(o2) $\forall A \in |K|\ \forall \psi \in K[O, A]\ ((1_A \otimes t_O)r_A = \psi)$

are fulfilled.
$(K^\bullet; d, t, \nabla, o)$ is called *diagonal-halfterminal-halfdiagonal-inversional-symmetric monoidal category* or *Hoehnke category with halfdiagonalinversions* (for short *dhth∇s-category*, in [13] named *dht∇-symmetric category)*, if $(K^\bullet; d, t, o)$ is a *dhts*-category endowed with a morphism family

$\nabla = (\nabla_A \in K[A \otimes A, A] \mid A \in |K|)$ fulfilling

(D_1^*) $\forall A \in |K|\ (d_A \nabla_A = 1_A)$,

(D_2^*) $\forall A \in |K|\ (\nabla_A d_A d_{A\otimes A} = d_{A\otimes A}(\nabla_A d_A \otimes 1_{A\otimes A}))$.

Any *ds*-, *dts*-, *dhts*-, and *dhth*∇*s*-category, respectively, is called *strict*, if the underlying symmetric monoidal category is strictly monoidal.

The zero morphisms $o_{A,B}$ absorb all other morphisms at composition and $\otimes$-operation in any *dhts*-category, i.e.

(o3) $\forall A, A', B, B' \in |K|\ \forall \rho \in K[A, A'],\ \sigma \in K[B, B']$

$$(\rho o_{A',B} = o_{A,B} \wedge\ o_{A,B}\sigma = o_{A,B'}),$$

(o4) $\forall A, B, C, D \in |K|\ \forall \xi \in K[C, D]$

$$(o_{A,B} \otimes \xi = o_{A\otimes C, B\otimes D}\ \wedge\ \xi \otimes o_{A,B} = o_{C\otimes A, B\otimes D}),$$

(o5) $\forall A \in |K|\ (o_{O,A} = (1_A \otimes t_O) r_A = (t_O \otimes 1_A) l_A)$.

Because of (o1) and (o2), the unit morphism 1_O is identical with the zero morphism $o_{O,O}$.

The category $\underline{Par}$ of all partial functions between arbitrary sets is an example for a *dhth*∇*s*-category.

In view of the properties of the category $\underline{Par}$ we will consider mainly *dhts*-categories fulfilling the conditions

(N_1) $\forall A,\ B \in |K|\ (A \otimes B = O \Rightarrow (A = O\ \vee\ B = O))$,

(N_2) $\forall A,\ B,\ C,\ D \in |K|\ \forall \varphi \in K[A, B]\ \forall \psi \in K[C, D]$

$$(\varphi \otimes \psi = o_{A\otimes C, B\otimes D} \Rightarrow (\varphi = o_{A,B}\ \vee\ \psi = o_{C,D})).$$

(N_3) $I \neq O$,

(N_4) $\forall A \in |K| \setminus \{\emptyset\}\ (1_A \neq o_{A,A})$.

Observe that $(K^\bullet; d)$ is a *ds*-category for each *dhts*-category $(K^\bullet; d, t, o)$ and ∇ is the only family in a $dhth\nabla s$-category with the properties (D_1^*) and (D_2^*), cf. [11].

Any *dhts*-category $\underline{K} = (K^\bullet; d, t, o)$ has the following properties:

• The class $T_K := \{\varphi \in K \mid \varphi t_{\mathrm{cod}\varphi} = t_{\mathrm{dom}\varphi}\}$ of so-called *total morphisms* of $\underline{K}$ forms a *dts*-subcategory $\underline{T}_K$ of $\underline{K}$ ([12]).

• $$(A \otimes B,\ (1_A \otimes t_B)r_A,\ (t_A \otimes 1_B)l_B)$$

is a categorical product in $\underline{T}_K$, but not in the whole category $\underline{K}$. The morphisms

$$p_1^{A,B} := (1_A \otimes t_B)r_A \text{ and } p_2^{A,B} := (t_A \otimes 1_B)l_B$$

are called the *canonical projections* concerning A and B ([5]).

• The class Iso_K of all *isomorphisms* of K forms a symmetric monoidal subcategory $Iso_K^\bullet$ and one has

$$Un_K \subseteq \mathbf{C}_K \subseteq Iso_K \subseteq Cor_K \subseteq T_K,$$

where Cor_K denotes the subcategory of all coretractions of K.

• The relation $\leq$ defined by

$$\varphi \leq \psi :\Leftrightarrow \exists A, A' \in |K|\ (\varphi, \psi \in K[A, A']\ \wedge\ \varphi = d_A(\varphi \otimes \psi)p_2^{A',A'})$$

is a partial order relation and it is compatible with composition and $\otimes$-operation of morphisms ([11]). Moreover, the following conditions are equivalent ([12]):

$$\varphi = d_A(\varphi \otimes \psi)p_2^{A',A'},$$

$$\varphi = d_A(\psi \otimes \varphi)p_1^{A',A'},$$

$$\varphi d_{A'} = d_A(\varphi \otimes \psi),$$

$$\varphi d_{A'} = d_A(\psi \otimes \varphi).$$

• Each morphism $\varphi \in K$ determines a so-called *subidentity* $\alpha(\varphi)$ as follows ([11]):

$$\alpha(\varphi) := d_{dom\varphi}(1_{dom\varphi} \otimes \varphi)p_1^{dom\varphi,cod\varphi} \leq 1_{dom\varphi}.$$

Moreover, each $dhth\nabla s$-category has the properties

(h∇_1) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ (\nabla_A \varphi d_{A'} = d_{A\otimes A}(\nabla_A \varphi \otimes (\varphi \otimes \varphi)\nabla_{A'}))$,

(hT$_1$) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ (\varphi t_{A'} d_I = d_A(\varphi t_{A'} \otimes t_A))$,

therefore $\nabla_A \varphi \leq (\varphi \otimes \varphi)\nabla_{A'}$ and $\varphi t_{A'} \leq t_A$ for all morphisms $\varphi \in K[A, A']$ and all objects A, $A' \in |K|$ ([15]).

Every morphism set $K[A, B]$ of a $dhth\nabla s$-category $\underline{K}$ forms a meet-semilattice with respect to $\varphi \wedge \psi = d_A(\varphi \otimes \psi)\nabla_B$. This semilattice has the minimum $o_{A,B}$, maximal elements are the total morphisms. Especially, the morphism sets $K[A, I]$ possess a maximum, namely t_A.

The basic morphisms related to the distinguished object I in any symmetric monoidal category, any *dhts*-category, or even any $dhth\nabla s$-category have some interesting properties as follows:

Lemma 1.8. *Let $K^\bullet$ be a symmetric monidal category. Then one has*

$$a_{I,I,I} = r_I^{-1} \otimes r_I.$$

Moreover, every dhts−category $\underline{K}$ has in addition the properties

$$d_I = r_I^{-1}, \quad r_I d_I = 1_{I\otimes I}, \quad t_I = 1_I \quad ([11]), \quad t_{I\otimes I} = r_I,$$

$$i \in Iso_K[I, I] \Rightarrow i = t_I,$$

$$\forall X \in |K|\ \forall x \in K[I, X]\ (x \in Iso_K \Rightarrow x^{-1} = t_X).$$

Finally, if $\underline{K}$ is a $dhth\nabla s$-category, then the additional property

$$\nabla_I = r_I$$

is true.

Proof. The identity $a_{A,I,B}(r_A \otimes 1_B) = 1_A \otimes l_B$ is one of the defining properties of monoidal-symmetric categories, hence $a_{I,I,I}(r_I \otimes 1_I) = 1_I \otimes r_I$ by $r_I = l_I$ and $a_{I,I,I} = (r_I^{-1} \otimes r_I)$, since all right-identity morphisms are isomorphisms.

In any *dhts*-category one has the defining identity $d_A(1_A \otimes t_A)r_A = 1_A$, hence $1_I = d_I(1_I \otimes t_I)r_I = d_I(1_I \otimes 1_I)r_I = d_I r_I$, since $t_I = 1_I$, consequently $d_I = r_I^{-1}$ and $r_I d_I = 1_{I\otimes I}$.

Each coretraction $\varphi \in K[A, B]$ of a *dhts*-category has the property $\varphi t_B = t_A$. Because d_I is even an isomorphism, one observes $d_I t_{I\otimes I} = t_I = 1_I$, therefore $t_{I\otimes I} = 1_{I\otimes I} t_{I\otimes I} = r_I d_I t_{I\otimes I} = r_I 1_I = r_I$.

One of the characterizing conditions of the diagonal inversions in a $dhth\nabla s$-category is $d_A \nabla_A = 1_A$. Therefore, $\nabla_I = 1_{I\otimes I}\nabla_I = r_I d_I \nabla_I = r_I$ as above. Now let $i \in K[I, I]$ be an isomorphism of a *dhts*-category $\underline{K}$. Then $i = i1_I = it_I = t_I$, because of $1_i = t_I$.

Let $x \in K[I, X]$ be an isomorphism in a *dhts*-category $\underline{K}$. Then one obtains in the same manner as above $1_I = t_I = xt_X$, hence the assertion. ■

Remark 1.9. Let $\underline{K}$ be a *dhts*-category. Then its object class $|K|$ forms an illegitimate algebra $(|K|, \otimes, I, O)$ of type $(2, 0, 0)$. Let J be a nonempty set such that $J \cap \{I, O\} = \emptyset$. Then J generates in $|K|$ a subalgebra H° of type $(2, 0, 0)$:

$$H^{\circ(0)} := J \cup \{I, O\}, \qquad H^{\circ(n+1)} := H^{\circ(n)} \cup \{X \otimes Y \mid X, Y \in H^{\circ(n)}\},$$

$$H^\circ := \bigcup_{n \in \mathbb{N}} H^{\circ(n)}.$$

The *dhts*-subcategory of $\underline{K}$ generated by H°, respectively by J, will be denoted by $\underline{H^\circ}$. Obviously, $\underline{H^\circ}$ is again a small category.

Let $\underline{K}$ be a strict *dhts*-category. Then the algebra $S^\circ := (H^\circ, \otimes, I, O)$ generated by a set J is a monoid with unit I and zero O.

2. Hoehnke theories

Let $\mathcal{G}$ denote the variety of all algebras of type type $(2, 0)$ (groupoids with a distinguished element I). Note that the distinguished element I does not play the role of a unit element in general. By the principles of General Algebra, every set J determines in $\mathcal{G}$ a free $\mathcal{G}$-algebra $\mathbf{F}_{\mathcal{G}}(J)$ freely generated by J. The algebra $\mathbf{F}_{\mathcal{G}}(J)$ contains a subalgebra $\underline{\langle I\rangle}$ consisting of all possible products of I as follows:

$$\langle I\rangle^{(0)} := \{I\},\ \langle I\rangle^{(n+1)} := \langle I\rangle^{(n)} \cup \{X \otimes Y \mid X, Y \in \langle I\rangle^{(n)}\},\ \langle I\rangle := \bigcup_{k\,\in\,\mathbb{N}} \langle I\rangle^{(k)}.$$

Every algebra $\underline{A} = (A; \otimes, I) \in \mathcal{G}$ can be transfered into an algebra $(A; \otimes, I, O)$ of type $(2, 0, 0)$ by addition of a distinguished element O with the property $\forall X \in A\ (X \otimes O = O = O \otimes X)$.

By $\mathcal{G}^{\circ}$ shall be denoted the variety of all algebras $(A; \otimes, I, O)$ of type $(2, 0, 0)$ (groupoids with distinguished element I and zero element O) such that $\forall X \in A\ (X \otimes O = O = O \otimes X)$. $\mathbf{F}_{\mathcal{G}^{\circ}}(J)$ denotes the free $\mathcal{G}^{\circ}$-algebra freely generated by a set J such that $J \cap \{I, O\} = \emptyset$. Clearly, $\mathbf{F}_{\mathcal{G}^{\circ}}(J)$ contains the trivial subalgebra $\underline{\langle I\rangle^{\circ}}$ with the carrier set $\langle I\rangle^{\circ} = \langle I\rangle \cup \{O\}$.

Let $\mathcal{M}$ be the variety of all monoids (algebras of type $(2, 0)$) and let $\mathcal{M}^{\circ}$ be the variety of all monoids with absorbing zero (algebras of type $(2, 0, 0)$ too).

The free $\mathcal{M}$-algebra ($\mathcal{M}^{\circ}$-algebra) freely generated by J will be denoted by $\mathbf{F}_{\mathcal{M}}(J)$ ($\mathbf{F}_{\mathcal{M}^{\circ}}(J)$). The trivial subalgebra $\underline{\langle I\rangle}$ ($\underline{\langle I\rangle^{\circ}}$) has the carrier set $\langle I\rangle = \{I\}$ ($\langle I\rangle^{\circ} = \{I, O\}$).

The identical embedding functions from J into the corresponding algebras will be denoted as follows:

$$\iota_H : J \hookrightarrow \mathbf{F}_{\mathcal{G}}(J),\ \ \iota_{H^{\circ}} : J \hookrightarrow \mathbf{F}_{\mathcal{G}^{\circ}}(J),$$
$$\iota_S : J \hookrightarrow \mathbf{F}_{\mathcal{M}}(J),\ \ \iota_{S^{\circ}} : J \hookrightarrow \mathbf{F}_{\mathcal{M}^{\circ}}(J).$$

Definition 2.1 ([5]). Let $\underline{\mathbf{T}}$ be a *dhts*-category, a *dhth*∇*s*-category, or a *dts*-category and let J be a nonempty set of objects of $\underline{\mathbf{T}}$ such that $I, O \notin J$.

Then $\underline{\mathbf{T}}$ will be called
J-sorted dhts-theory or *J-sorted Hoehnke theory*,
J-sorted dhth∇*s-theorie* or
J-sorted Hoehnke theory with halfdiagonalinversions,
J-sorted dts-theory, respectively,

if $(|\mathbf{T}|; \otimes, I, O)$ is a free $\mathcal{G}^{\circ}$-algebra freely generated by J ($(|\mathbf{T}|; \otimes, I)$ is a free $\mathcal{G}$-algebra freely generated by J, $I \notin J$).

The class of all J-sorted *dhts*-theories (J-sorted *dhth*∇*s*-theories, J-sorted *dts*-theories) will be denoted by $|Th^{\circ}_{dht}(J)|$ ($|Th^{\circ}_{dhth\nabla}(J)|$, $|Th_{dt}(J)|$).

Besides the theory concept above we consider the following, more artifical, but simpler one, which arises in strict monoidal categories by replacing of the groupoid $\mathbf{F}_{\mathcal{G}^\circ}(J)$ ($\mathbf{F}_{\mathcal{G}}(J)$) by the monoid $\mathbf{F}_{\mathcal{M}^\circ}(J)$ ($\mathbf{F}_{\mathcal{M}}(J)$). So, one defines

Definition 2.2. Let $\underline{\mathbf{T}}$ be a *dhts*-category, a *dhth*∇*s*-category, or a *dts*-category such that the underlying symmetric monoidal category $\mathbf{T}^\bullet$ is strictly monoidal, i.e. all the morphisms a, r, and l are unit-morphims only ($A \otimes (B \otimes C) = (A \otimes B) \otimes C,\ A \otimes I = A = I \otimes A,\ a_{A,B,C} = 1_{A\otimes B\otimes C}$, $r_A = 1_A = l_A$ for all $A, B, C \in |\mathbf{T}|$).

Then $\underline{\mathbf{T}}$ will be called

J-sorted strict dhts-theory or *strict J-sorted Hoehnke theory*,

J-sorted strict dhth∇*s-theory* or

strict J-sorted Hoehnke theory with halfdiagonalinversions,

J-sorted strict dts-theory, respectively,

if there exists a nonempty set J in $|\mathbf{T}|$ such that $I, O \notin J$ and $(|\mathbf{T}|; \otimes, I, O)$ is a free $\mathcal{M}^\circ$-algebra $((|\mathbf{T}|; \otimes, I)$ is a free $\mathcal{M}$-algebra) freely generated by J. The class of all J-sorted strict *dhts*-theories (J-sorted strict *dhth*∇*s*-theories, J-sorted strict *dts*-theories) will be denoted by

$$|sTh^\circ_{dht}(J)| \quad (|sTh^\circ_{dhth\nabla}(J)|,\ |sTh_{dt}(J)|).$$

The categories of the classes $|Th^\circ_{dht}(J)|$, $|Th^\circ_{dhth\nabla}(J)|$, $|sTh^\circ_{dht}(J)|$, and $|sTh^\circ_{dhth\nabla}(J)|$ shortly will called *partial theories* (*Hoehnke theories*) and categories of $|Th_{dt}(J)|$ and $|sTh_{dt}(J)|$ are named *total theories.*

For a given set J one has on the one hand the free algebra $\mathbf{F}_{\mathcal{G}^\circ}(J)$ and on the other hand the free algebra $\mathbf{F}_{\mathcal{M}^\circ}(J)$ and both are algebras of the variety $\mathcal{G}^\circ$ of type $(2, 0, 0)$. Therefore, there arises the question about a connection between the two algebras.

Lemma 2.3. *Let* $\mathbf{F}_{\mathcal{G}^\circ}(J) =: (H^\circ; \otimes, I, O)$, $\mathbf{F}_{\mathcal{M}^\circ}(J) =: (S^\circ; \otimes, I, O)$, $\mathbf{F}_{\mathcal{G}}(J) =: (H; \otimes, I)$, *and* $\mathbf{F}_{\mathcal{M}}(J) =: (S; \otimes, I)$ *be the algebras defined as above. Then there is exactly one homomorphism* $W^* : \mathbf{F}_{\mathcal{G}^\circ}(J) \to \mathbf{F}_{\mathcal{M}^\circ}(J)$ ($W^* : \mathbf{F}_{\mathcal{G}}(J) \to \mathbf{F}_{\mathcal{M}}(J)$) *such that* $\iota_{H^\circ} W^* = \iota_{S^\circ}$ ($\iota_H W^* = \iota_S$).

The mapping W^* *works as follows:*

$$I \mapsto I =: IW^*,\ O \mapsto O =: OW^*,\ J \ni A \mapsto A =: AW^*,$$

$$\forall X, Y \in H\ ((X \otimes Y)W^* = XW^* \otimes YW^*).$$

Proof. Let $\underline{\mathbf{T}} \in |Th^{\circ}_{dht}(J)|$. The algebra $\mathbf{F}_{\mathcal{M}^{\circ}}(J) = (S^{\circ}; \otimes, I, O)$, generated by J, belongs to $\mathcal{G}^{\circ}$. Since $(H^{\circ} := |\mathbf{T}|; \otimes, I, O)$ is a a free $\mathcal{G}^{\circ}$-algebra freely generated by J, there is exactly one homomorphism W^{*} such that $\iota_{H^{\circ}} W^{*} = \iota_{S^{\circ}}$ and this homomorphism is surjective. The assertion about the working of the mapping becomes clear since $\iota_{S^{\circ}}$ is the identical embedding of J into S°.

The statement concerning groupoids and monoids without zero will be proved in the same manner. ∎

Corollary 2.4. *The mapping $W^{*} : H^{\circ} \to S^{\circ}$ has the following properties:*

$$\forall X \in \langle I \rangle \ (XW^{*} = I),$$

$$\forall Y \in H^{\circ} \ \forall X \in \langle I \rangle \ ((Y \otimes X)W^{*} = (X \otimes Y)W^{*} = YW^{*}),$$

$$\forall X, Y, Z \in H^{\circ} \ ((X \otimes (Y \otimes Z))W^{*} = ((X \otimes Y) \otimes Z)W^{*}),$$

$$\forall X \in H^{\circ} \setminus \langle I \rangle^{\circ} \ \exists!! A_1, A_2, ..., A_n \ (XW^{*} = A_1 \otimes A_2 \otimes \cdots \otimes A_n).$$

Proof. The first assertion one proves by induction over the complexity of the elements of $\langle I \rangle$.

By Lemma 2.3, $IW^{*} = I$. Assume that for any $n \in \mathbb{N}$ the condition

$$\forall Y \in \langle I \rangle^{(n)} \ (YW^{*} = I)$$

is valid. Then

$$\forall X \in \langle I \rangle^{(n+1)} \setminus \langle I \rangle^{(n)} \ \exists X_1, X_2 \in \langle I \rangle^{(n)}$$

$$(XW^{*} = (X_1 \otimes X_2)W^{*} = X_1 W^{*} \otimes X_2 W^{*} = I \otimes I = I),$$

hence $\forall n \in \mathbb{N} \ \forall X \in \langle I \rangle^{(n)} (XW^{*} = I)$.

Because of $(X \otimes Y)W^{*} = XW^{*} \otimes YW^{*}$, $XW^{*} = I$ for every $X \in \langle I \rangle$ and I is the unit element in the monoid, the second claim becomes true.

Let X, Y, and Z be elements of $|T| = H^{\circ}$. Then XW^{*}, YW^{*}, and ZW^{*} are elements of the monoid $\underline{S^{\circ}}$ and

$$(X \otimes (Y \otimes Z))W^{*} = XW^{*} \otimes YW^{*} \otimes ZW^{*} = ((X \otimes Y) \otimes Z)W^{*}.$$

Because of

$$H = \bigcup_{k \in \mathbb{N}} H^{(n)}, \quad H^{(0)} := J \cup \{I\},$$

$$H^{(n+1)} := H^{(n)} \cup \{X \otimes Y \mid X, Y \in H^{(n)}\},\ n \in \mathbb{N},$$

one shows the existence of such a representation by induction over the complexity of X.

$$X \in H^{(0)} \setminus \langle I \rangle \Rightarrow X = A \in J \Rightarrow XW^* = AW^* = A.$$

Assuming that for any $n \in \mathbb{N}$ each $X \in H^{(n)} \setminus \langle I \rangle$ fulfills the assertion one investigates an arbitrary $Y \in H^{(n+1)} \setminus H^{(n)} \setminus \langle I \rangle$. Then there are $X_1, X_2 \in H^{(n)} \setminus \langle I \rangle$ such that $YW^* = (X_1 \otimes X_2)W^* = X_1W^* \otimes X_2W^*$, hence there are $A_1, \cdots, A_j, B_1, \cdots B_k \in J$ such that $YW^* = (X_1 \otimes X_2)W^* = X_1W^* \otimes X_2W^* = A_1 \otimes \cdots \otimes A_j \otimes B_1 \otimes \cdots \otimes B_k$.

The uniqueness of the factors of a $\otimes$-product which are elements of J is a consequence of the fact that $(S^\circ; \otimes, I, O)$ is a free $\mathcal{M}^\circ$-algebra freely generated by J. ■

Lemma 2.5. *Let be given H° and S° as above related to a fixed set J. Then there is a function $W : S^\circ \to H^\circ$ such that*

(W1) $\quad WW^* = 1_{S^\circ}$ *and*

(W2) $\quad \forall A, B \in S^\circ\ (A \otimes B = (AW \otimes BW)W^*)$.

The function $\Phi : H^\circ \to H^\circ$ defined by $\Phi := W^*W$ has the properties

(W3) $\quad \forall X \in \langle I \rangle\ (X\Phi = I)$,

(W4) $\quad \forall X \in H \setminus \langle I \rangle\ \exists!!\ A_1, ..., A_n \in J\ \left(X\Phi = \bigotimes_{j=1}^{n} A_j \right)$,

(W5) $\quad \forall X_1, X_2, Y_1, Y_2 \in H^\circ$

$$((X_1 \otimes X_2)\Phi = (Y_1 \otimes Y_2)\Phi \Leftrightarrow (X_1\Phi) \otimes (X_2\Phi) = (Y_1\Phi) \otimes (Y_2\Phi)).$$

Proof. Ad (W1): Defining

$$OW := O,\ IW := I,\ \forall A_1, ..., A_n \in J \left((A_1 \otimes \cdots \otimes A_n)W := \bigotimes_{j=1}^{n} A_j \right)$$

one gets immediately $WW^* = 1_{S^\circ}$.

Ad (W2): The assertion is trivial for $A = O$ or $B = O$. The same is true if $A = I$ or $B = I$. Now let $A, B \in S \setminus \{I\}$. Then, by definition,

$$\begin{aligned} A \otimes B =& A_1 \otimes \cdots \otimes A_n \otimes B_1 \otimes \cdots \otimes B_m = \left(\bigotimes_{k=1}^{n} A_k \right) W^* \otimes \left(\bigotimes_{j=1}^{m} B_j \right) W^* \\ &= (AW)W^* \otimes (BW)W^* = (AW \otimes BW)W^*. \end{aligned}$$

Ad (W3): The condition is valid for $X \in \{I, O\}$, since

$$I\Phi = IW^*W = IW = I \text{ and } O\Phi = OW^*W = OW = O.$$

Let X be an arbitrary element of $\langle I \rangle$. Then

$$X\Phi = (XW^*)W = IW = I.$$

Ad (W4): For all $X \in H \setminus \langle I \rangle$ one has

$$X\Phi = (XW^*)W = (A_1 \otimes \cdots \otimes A_n)W = \bigotimes_{j=1}^{n} A_j$$

and, by the properties of a free algebra,

$$\bigotimes_{j=1}^{n} A_j = \bigotimes_{k=1}^{m} A'_k \Rightarrow n = m \ \wedge\ A_j = A'_j \text{ for all } j \in \{1, ..., n\}.$$

Ad (W5):

$$\begin{aligned} (X_1 \otimes X_2)\Phi = (Y_1 \otimes Y_2)\Phi &\Leftrightarrow (X_1 \otimes X_2)W^*W = (Y_1 \otimes Y_2)W^*W \\ &\Leftrightarrow (X_1 \otimes X_2)W^* = (Y_1 \otimes Y_2)W^* \end{aligned}$$

$$\Leftrightarrow X_1W^* \otimes X_2W^* = Y_1W^* \otimes Y_2W^*$$

$$\Leftrightarrow X_1W^* = Y_1W^* \ \wedge \ X_2W^* = Y_2W^*$$

$$(\underline{S^\circ} \text{ is a free algebra})$$

$$\Leftrightarrow X_1W^*W = Y_1W^*W \ \wedge \ X_2W^*W = Y_2W^*W$$

$$\Leftrightarrow X_1\Phi = Y_1\Phi \ \wedge \ X_2\Phi = Y_2\Phi$$

$$\Leftrightarrow X_1\Phi \otimes X_2\Phi = Y_1\Phi \otimes Y_2\Phi \ (\underline{H^\circ} \text{ is a free algebra}).$$

■

Observe that the function $\Phi : H^\circ \to H^\circ$ maps O onto O, all elements of $\langle I\rangle \subseteq H$ onto I, and all elements $X \in H \setminus \langle I\rangle$ onto an $\otimes$-product of elements of J in canonical brackets consisting exactly of the factors of X which are different from I in the same order.

Lemma 2.6. *Let be* $\underline{H^\circ}$, $\underline{S^\circ}$, $\Phi : H^\circ \to H^\circ$ *as above. Then*

$$\forall X, Y, Z \in H^\circ \ ((X \otimes (Y \otimes Z))\Phi = ((X \otimes Y) \otimes Z)\Phi),$$

$$\forall n \in \mathbb{N} \setminus \{0\} \ \forall A_1, ..., A_n \in J \ \left(\left(\bigotimes_{j=1}^{n} A_j\right)\Phi = \bigotimes_{j=1}^{n} A_j\right),$$

$$\forall X \in \langle I\rangle \ \forall Y \in H^\circ \ ((Y \otimes X)\Phi = (X \otimes Y)\Phi = Y\Phi).$$

Proof.

$$(X \otimes (Y \otimes Z))\Phi = (X \otimes (Y \otimes Z))W^*W = (XW^* \otimes (YW^* \otimes ZW^*))W$$

$$= ((XW^* \otimes YW^*) \otimes ZW^*)W = ((X \otimes Y) \otimes Z)\Phi.$$

$$\left(\bigotimes_{j=1}^{n} A_j\right)\Phi = \left(\bigotimes_{j=1}^{n} A_j\right)W^*W = \left(\bigotimes_{j=1}^{n} A_j\right)W = \bigotimes_{j=1}^{n} A_j.$$

$(Y \otimes X)\Phi = (Y \otimes X)W^*W = (YW^* \otimes XW^*)W = (YW^* \otimes I)W = YW^*W = Y\Phi$,

$(X \otimes Y)\Phi = (X \otimes Y)W^*W = (XW^* \otimes YW^*)W = (I \otimes YW^*)W = YW^*W = Y\Phi$. ■

Corollary 2.7. *Let $\underline{\mathbf{T}}$ be any J-sorted Hoehnke theory and let $\Phi : H^\circ \to H^\circ$ be defined as above. Then there is exactly one central morphism $c_X := c_{X,X\Phi}$ in $\mathbf{C_T}$ for every $X \in |\mathbf{T}|$. The same statement is true, if $\underline{\mathbf{T}}$ is a J-sorted dts-theory and $\Phi : H \to H$.*

Moreover, $\forall X, Y \in |\mathbf{T}|\ (X\Phi = Y\Phi \Rightarrow \exists c_{X,Y} \in \mathbf{C_T}[X, Y])$.

Proof. The proof is organized by induction over the complexity of the objects $X \in |\mathbf{T}| = H^\circ$.

Because of $X\Phi = X$ for every $X \in J \cup \{I, O\} = H^{\circ(0)}$, $1_X \in \mathbf{C_T}[X, X\Phi]$, hence the start of induction is verified.

Let c_X exist in $\mathbf{C_T}$ for any $X \in H^{\circ(n)}$ and an arbitrary $n \in \mathbb{N}$. Let be $X \in H^{\circ(n+1)} \setminus H^{\circ(n)}$. Then there are $X_1, X_2 \in H^{\circ(n)}$ such that $X = X_1 \otimes X_2$ and $c_{X_1} \in \mathbf{C_T}[X_1, X_1\Phi]$, $c_{X_2} \in \mathbf{C_T}[X_2, X_2\Phi]$, hence $(c_{X_1} \otimes c_{X_2}) \in \mathbf{C_T}[X, X_1\Phi \otimes X_2\Phi]$.

Since $X_1\Phi = \underset{j=1}{\overset{n}{\otimes}} A_j$ and $X_2\Phi = \underset{j=n+1}{\overset{n+m}{\otimes}} A_j$ for suitable $A_j \in J$, $1 \le j \le n+m$, there is the canonical associativity isomorphism

$$a^{(n,m)}\langle X_1\Phi, X_2\Phi\rangle : X_1\Phi \otimes X_2\Phi \to (X_1 \otimes X_2)\Phi = X\Phi \text{ in } \mathbf{C_T},$$

therefore,

$$c_X := (c_{X_1} \otimes c_{X_2})a^{(n,m)}\langle X_1\Phi, X_2\Phi\rangle \in \mathbf{C_T}[X, X\Phi].$$

So, the existence of a central morphism c_X for every $X \in |\mathbf{T}| = H^\circ$ is proved. Moreover, $X\Phi = Y\Phi$ is sufficient for $c_{X,Y} := c_X(c_Y)^{-1} \in \mathbf{C_T}[X, Y]$.

The uniqueness is again a consequence of the coherence principle.

The claim concerning the *dts*-case will be proved similarly. ■

The function Φ defined as above induces in a natural manner a functor from a J-sorted theory $\underline{\mathbf{T}}$ into itself with additional interesting properties. This properties concern the monoidal structur of $\underline{\mathbf{T}}$.

3. Structure preserving functors

Considering different symmetric monoidal categories $K^\bullet$ and $K'^\bullet$ one has to distinguish between the operations and the basic morphisms of $K^\bullet$ and those of $K'^\bullet$, respectively, for instance between $r_A^{(K)}$ and $r_X^{(K')}$. If there is not danger of confusion, the upper index will be omitted.

Definition 3.1 ([14]). A functor $F : K^\bullet \to K'^\bullet$ between symmetric monoidal categories $K^\bullet$ and $K'^\bullet$ is called *monoidal*, iff there exists in K' a family of morphisms

$$\widetilde{F} = (\widetilde{F}\langle X, Y\rangle : XF \otimes YF \to (X \otimes Y)F \mid X, Y \in |K|)$$

and a morphism

$$i_F : I' \to IF,$$

such that the following conditions are fulfilled:

$(F \sim)$ $\forall X, Y \in |K|\ (\widetilde{F}\langle X, Y\rangle \in Iso_{K'})$,

(FI) $i_F \in Iso_{K'}$,

(FA) $$\forall X, Y, Z \in |K| \left(\left(1_{XF}^{(K')} \otimes \widetilde{F}\langle Y, Z\rangle\right) \widetilde{F}\langle X, Y \otimes Z\rangle \left(a_{X,Y,Z}^{(K)} F\right) = a_{XF,YF,ZF}^{(K')} \left(\widetilde{F}\langle X, Y\rangle \otimes 1_{ZF}^{(K')}\right) \widetilde{F}\langle X \otimes Y, Z\rangle\right),$$

(FR) $$\forall X \in |K| \ \left(\widetilde{F}\langle X, I\rangle \left(r_X^{(K)} F\right) = \left(1_{XF}^{(K')} \otimes i_F^{-1}\right) r_{XF}^{(K')}\right),$$

(FS) $$\forall X, Y \in |K| \ \left(\widetilde{F}\langle X, Y\rangle \left(s_{X,Y}^{(K)} F\right) = s_{XF,YF}^{(K')} \widetilde{F}\langle Y, X\rangle\right),$$

(FM) $$\forall \varphi : X \to Y, \psi : U \to V \in K \ ((\varphi F \otimes \psi F)\widetilde{F}\langle Y, V\rangle = \\ = \widetilde{F}\langle X, U\rangle(\varphi \otimes \psi)F).$$

A monoidal functor $F : K^\bullet \to K'^\bullet$ is called *strictly monoidal*, iff all morphisms of the family $\widetilde{F}$ as well as the morphism i_F are unit morphisms only.

Corollary 3.2 ([14]). *Let $F : K^\bullet \to K'^\bullet$ be a monoidal functor between symmetric monoidal categories with reference to $\widetilde{F}, i_F$. Then*

(FL) $$\forall X \in |K| \ \left(\widetilde{F}\langle I, X\rangle \left(l_X^{(K)} F\right) = \left(i_F^{-1} \otimes 1_{XF}^{(K')}\right) l_{XF}^{(K')}\right).$$ ■

In applications to theories of algebraic structures, functors $F : \underline{K} \to \underline{K'}$ between *dhts*-categories, $dhth\nabla s$-categories, or *dts*-categories are of interest which preserve in addition to the functor properties the *dhts*-, $dhth\nabla s$-, and the *dts*-structure, respectively, with respect to a family $\widetilde{F} = (\widetilde{F}\langle X, Y\rangle \mid X, Y \in |K|)$ of isomorphisms $\widetilde{F}\langle X, Y\rangle : XF \otimes YF \to (X \otimes Y)F$ in $\underline{K'}$ and an isomorphism i_F between I' and IF, where I and I' are the distinguished objects in $\underline{K}$ and $\underline{K'}$, respectively, ([5], [12], [14]). All symmetric monoidal categories with additional structures mentioned above are *ds*-categories. Of importance is the fact that a monoidal functor between at least *ds*-categories, which respects the diagonal morphisms except for isomorphisms, respects the canonical partial order relation and the distinguished terminal morphisms and the distinguished diagonal inversions, respectively, except for isomorphisms.

Definition 3.3 ([14]). A monoidal functor $F : \underline{K} \to \underline{K'}$ between *ds*-categories $\underline{K}$ and $\underline{K'}$ is called *d-monoidal*, if in addition the condition

$$\text{(FD)} \quad \forall A \in |K| \ \left(d_A^{(K)} F = d_{AF}^{(K')} \widetilde{F}\langle A, A\rangle\right)$$

holds with reference to the corresponding isomorphisms $\widetilde{F}$ and i_F. A strictly monoidal functor F fulfilling the condition (FD) is called *strictly d-monoidal.*

Obviously, the identical functor $\mathbf{1}_K$ of $K^\bullet$ forms a strictly monoidal functor with respect to

$$\widetilde{\mathbf{1}_K} = (\widetilde{\mathbf{1}_K}\langle X, Y\rangle = 1_{XF\otimes YF} \mid X, Y \in |K|),\ i_{\mathbf{1}_K} = 1_I$$

and the constant functor $E : K^\bullet \to K'^\bullet\,(X \mapsto I',\ \varphi \mapsto 1'_{I'})$ with reference to

$$\widetilde{E} = (\widetilde{E}\langle X, Y\rangle = 1'_{I'} \mid X, Y \in |K|),\ i_E = 1'_{I'},$$

too, where $K^\bullet$ and $K'^\bullet$ are arbitrary symmetric monoidal categories.

Both functors are even d-monoidal functors, if $\underline{K} = (K^\bullet; d)$ and $\underline{K'} = (K'^\bullet; d')$ are *ds*-categories.

Moreover: Each d-monoidal functor $F : \underline{K} \to \underline{K'}$ between *dhts*-categories possesses the following properties with respect to the corresponding $\widetilde{F}$, i_F ([11], [14]):

(FI*) $t_{IF}^{(K')} = i_F^{-1}$,

(Fmon) $\forall \varphi,\, \psi \in K\ (\varphi \leq \psi \Rightarrow \varphi F \leq \psi F)$,

(FT) $\forall X \in |K|\ \left(t_X^{(K)} F\, t_{IF}^{(K')} = t_{XF}^{(K')}\right)$,

(FP) $\forall X, Y \in |K|\ \left(p^{(K)}{}_j^{X,Y} F = (\widetilde{F}\langle X, Y\rangle)^{-1} p^{(K')}{}_j^{XF,YF}\ ;\ \ j = 1, 2\right)$,

(FE) $\forall A \in |K|\ \left(e \leq 1_A^{(K)} \Rightarrow eF \leq 1_{AF}^{(K')}\right)$,

(FEα) $\forall X,\, Y \in |K|\ \forall \varphi \in K[X, Y]\ \left((\alpha^{(K)}(\varphi))F = \alpha^{(K')}(\varphi F)\right)$.

Let $\underline{K}$, $\underline{K'}$ be $dhth\nabla s$-categories and let $F : \underline{K} \to \underline{K'}$ be a d-monoidal functor. Then, in addition to the the properties above, the following holds ([14]):

(Finf) $\forall X, Y \in |K|\ \forall \varphi, \psi \in K[X, Y]\ \Big(\left(d_X^{(K)}(\varphi \otimes \psi)\nabla_Y^{(K)}\right)$

$$F = d_{XF}^{(K')}(\varphi F \otimes \psi F)\nabla_{YF}^{(K')}\Big),$$

(Finj) $\forall X, Y \in |K|\ \forall \varphi \in K[X, Y]\ \Big((\varphi \otimes \varphi)\nabla_Y^{(K)} = \nabla_X^{(K)}\varphi$

$$\Rightarrow (\varphi F \otimes \varphi F)\nabla_{YF}^{(K')} = \nabla_{XF}^{(K')}(\varphi F)\Big),$$

(F∇) $\forall X \in |K|\ \left(\nabla_{XF}^{(K')} = \widetilde{F}\langle X, X\rangle \nabla_X^{(K)} F\right)$,

(F∇_1) $\forall X, Y, U \in |K|\ \forall \varphi \in K[X, U]\ \forall \psi \in K[Y, U]\ \Big(\left((\varphi \otimes \psi)\nabla_U^{(K)}\right) F$

$$= \widetilde{F}\langle X, Y\rangle \left((\varphi \otimes \psi)F\right)\nabla_{UF}^{(K')}\Big),$$

(F∇_2) $\forall X, Y \in |K|\ \forall \varphi.\psi \in K[X, Y]\ \Big((\varphi \otimes \psi)\nabla_Y^{(K)} = \nabla_X^{(K)}\varphi$

$$\Rightarrow (\varphi F \otimes \psi F)\nabla_{YF}^{(K')} = \nabla_{XF}^{(K')}(\varphi F)\Big).$$

Obviously, property (Finj) is a special case of (F∇_2) and it expresses once more the monotony of the functor F, namely $\varphi \leq \psi \Rightarrow \varphi F \leq \psi F$.

The so-called zero functor $Z : \underline{K} \to \underline{K'}$ is defined by $XZ = O^{(K')}$ for all objects $X \in |K|$ and $\varphi Z = 1_{O^{(K')}}^{(K')}$ for all morphisms $\varphi \in K$. Trivially, this functor is a d-monoidal one.

Proposition 3.4 ([14]). *Let $F : \underline{K} \to \underline{K'}$ be a d-monoidal functor between Hoehnke categories such that $F \neq Z$. Then one obtains:*

$$\forall X \in |K| \ \left(\widetilde{F}\langle X, O\rangle = \widetilde{F}\langle O, X\rangle = 1^{(K')}_{O^{(K')}}\right),$$

$$\forall X, Y \in |K| \ \left(o^{(K)}_{X,Y} F = o^{(K')}_{XF,YF}\right),$$

$$o^{(K)} F = t^{(K')}_{IF} o^{(K')} \quad \left(\Leftrightarrow o^{(K')} = i_F(o^{(K)} F)\right). \qquad \blacksquare$$

By the structure of any Hoehnke categories $\underline{K}$ and $\underline{K'}$, each functor $F : \underline{K} \to \underline{K'}$ determines with respect to every pair of objects $X, Y \in |K|$ the morphism

$$F^*\langle X, Y\rangle := d^{(K')}_{(X\otimes Y)F}\left(p^{(K)}{}^{X,Y}_{1} F \otimes p^{(K)}{}^{X,Y}_{2} F\right) \in K'[(X \otimes Y)F, XF \otimes YF]$$

in the category K'.

Proposition 3.5 ([5]). *In the case that $F : \underline{K} \to \underline{K'}$ is a d-monoidal functor with reference to $\widetilde{F}$ and i_F, the morphisms $\widetilde{F}\langle X, Y\rangle$ are uniquely determined by*

$$(\widetilde{F}\langle X, Y\rangle)^{-1} = d^{(K')}_{(X\otimes Y)F}\left(p^{(K)}{}^{X,Y}_{1} F \otimes p^{(K)}{}^{X,Y}_{2} F\right) = F^*\langle X, Y\rangle.$$

$\blacksquare$

Moreover:

Theorem 3.6 ([5], [14]). *Assume that $F : \underline{K} \to \underline{K'}$ is any functor from a dhts-category $\underline{K}$ into a dhts-category $\underline{K'}$ satisfying the following conditions:*

(F$*$) $\forall X, Y \in |K| (F^*\langle X, Y\rangle \in \ Iso_{K'})$,

(FI*) $t^{(K')}{}_{IF} \in Iso_{K'}$,

(FM*) $\forall \varphi, \psi \in K \ ((\varphi \otimes \psi)F\, F^*\langle X', Y'\rangle = F^*\langle X, Y\rangle(\varphi F \otimes \psi F))$.

Then $F : \underline{K} \to \underline{K'}$ is d-monoidal with reference to the morphisms

$$\widetilde{F}\langle X, Y\rangle := (F^*\langle X, Y\rangle)^{-1}, \quad i_F := t^{(K')}{}^{-1}_{IF}. \qquad \blacksquare$$

The statements in 3.5 and 3.6 allow us to speak about d-monoidal functors between Hoehnke categories as such.

Hoehnke has shown in [5] that the composition of *dht*-symmetric functors $F : \underline{K} \to \underline{K'}$ and $G : \underline{K'} \to \underline{K''}$ between Hoehnke categories $\underline{K}$, $\underline{K'}$, $\underline{K''}$, respectively, yields a *dht*-symmetric functor $FG : \underline{K} \to \underline{K''}$. The same is true for *d*-monoidal functors between Hoehnke categories. More precisely:

Proposition 3.7. *Let $F : \underline{K} \to \underline{K'}$ and $G : \underline{K'} \to \underline{K''}$ be d-monoidal functors between Hoehnke categories $\underline{K}$, $\underline{K'}$, $\underline{K''}$. Then the functor $FG : \underline{K} \to \underline{K''}$ is a d-monoidal functor too.*

Proof. Ad (F$*$): Since every functor maps isomorphisms to isomorphism and

$$(FG)^*\langle X,Y\rangle = d^{(K'')}_{(X\otimes Y)(FG)}\left(p_1^{(K)X,Y}(FG)\otimes p_2^{(K)X,Y}(FG)\right)$$

$$= d^{(K'')}_{((X\otimes Y)F)G}\left(\left(p_1^{(K)X,Y}F\right)G\otimes (p_2^{(K)X,Y}F)G\right)$$

$$= \left(d^{(K')}_{(X\otimes Y)F}\right)GG^*\langle (X\otimes Y)F,(X\otimes Y)F\rangle\left(\left(p_1^{(K)X,Y}F\right)G\otimes\left(p_2^{(K)X,Y}F\right)G\right)$$

$$= \left(d^{(K')}_{(X\otimes Y)F}\right)G\left(p_1^{(K)X,Y}F\otimes\left(p_2^{(K)X,Y}F\right)GG^*\langle XF,YF\rangle\right.$$

$$= \left(d^{(K')}_{(X\otimes Y)F}\left(p_1^{(K)X,Y}F\otimes p_2^{(K)X,Y}F\right)GG^*\langle XF,YF\rangle\right.$$

$$= (F^*\langle X,Y\rangle))GG^*\langle XF,YF\rangle$$

one obtains $(FG)^*\langle X,Y\rangle \in Iso_{K''}$.

Ad (FI*): $t^{(K'')}_{I(FG)} = t^{(K'')}_{(IF)G} = \left(t^{(K')}_{IF}\right)Gt^{(K'')}_{I^{(K')}G)} \in Iso_{K''}$

since $t^{(K'')}_{I^{(K')}G} \in Iso_{K''}$ and $t^{(K')}_{IF} \in Iso_{K'}$.

Ad (FM*): $(\varphi\otimes\psi)(FG)(FG)^*\langle U,V\rangle = ((\varphi\otimes\psi)F)G(F^*\langle U,V\rangle)GG^*\langle UF,VF\rangle$

$$= ((\varphi\otimes\psi)FF^*\langle U,V\rangle)GG^*\langle UF,VF\rangle$$

$$= (F^*\langle X,Y\rangle(\varphi F\otimes\psi F))GG^*\langle UF,VF\rangle$$

$$= (F^*\langle X,Y\rangle)G(\varphi F\otimes\psi F)GG^*\langle UF,VF\rangle$$

$$= (F^*\langle X,Y\rangle)GG^*\langle XF,YF\rangle((\varphi F)G\otimes(\psi F)G)$$

$$= (FG)^*\langle X,Y\rangle(\varphi(FG)\otimes\psi(FG)).$$

∎

Lemma 3.8. *Let $F : \underline{K} \to \underline{K'}$ be a functor from a Hoehnke category $\underline{K}$ into a Hoehnke category $\underline{K'}$ such that the conditions*

$$\text{(sFD)} \qquad \forall X \in |K| \left(d_X^{(K)} F = d_{XF}^{(K')} \right),$$

$$\text{(sFT)} \qquad \forall X \in |K| \left(t_X^{(K)} F = t_{XF}^{(K')} \right), \text{ and}$$

$$\text{(sFM)} \qquad \forall \varphi, \psi \in K \ ((\varphi \otimes \psi)F = (\varphi F \otimes \psi F))$$

are fulfilled.

Then F has the properties

$$\text{(sF}*) \qquad \forall X, Y \in |K| (F^* \langle X, Y\rangle \in \ Un_{K'}) \text{ and}$$

$$\text{(sFI}^*) \qquad t^{(K')}{}_{IF} \in Un_{K'},$$

i.e. $F : \underline{K} \to \underline{K'}$ is a strictly d-monoidal functor.

Proof. Assuming (sFT) one has $1_{IF}^{(K')} = 1_I^{(K)} F = t_I^{(K)} F = t_{IF}^{(K')}$, hence $IF = I^{(K')}$ and (sFI*) is fulfilled.

Moreover,

$$\forall X, Y \in |K| \ \Big(K'[(X \otimes Y)F, (X \otimes Y)F] \ni 1_{X \otimes Y}^{(K)} F = \left(1_X^{(K)} \otimes 1_Y^{(K)} \right) F$$

$$= 1_X^{(K)} F \otimes 1_Y^{(K)} F = 1_{XF}^{(K')} \otimes 1_{YF}^{(K')} = 1_{XF \otimes YF}^{(K')} \in K'[XF \otimes YF] \Big),$$

hence

$$\forall X, Y \in |K| \ ((X \otimes Y)F = XF \otimes YF).$$

Now let X and Y be any objects of $|K|$. Then

$$\begin{aligned}
F^* \langle X, Y \rangle &= d_{(X \otimes Y)F}^{(K')} \left(p^{(K)}{}_1^{X,Y} F \otimes p^{(K)}{}_2^{X,Y} F \right) && \text{(by definition)} \\
&= d_{X \otimes Y}^{(K)} F \left(p^{(K)}{}_1^{X,Y} F \otimes p^{(K)}{}_2^{X,Y} F \right) && \text{((sFD))} \\
&= \left(d_{X \otimes Y}^{(K)} \left(p^{(K)}{}_1^{X,Y} \otimes p^{(K)}{}_2^{X,Y} \right) \right) F && \text{((sFM))} \\
&= \left(1_{X \otimes Y}^{(K)} \right) F = 1_{XF \otimes YF}^{(K')} \in Un_{K'}. && \blacksquare
\end{aligned}$$

Proposition 3.9. *If functors $F : \underline{K} \to \underline{K'}$ and $G : \underline{K'} \to \underline{K''}$ between Hoehnke categories $\underline{K}$, $\underline{K'}$, $\underline{K''}$ fulfil the conditions* (sFD), (sFT), and (sFM), *then the functor $FG : \underline{K} \to \underline{K''}$ satifies the same conditions.* ■

Corollary 3.10. *If any functor $F : \underline{K} \to \underline{K'}$ as above fulfils* (sFT) *and* (sFM), *then F is a d-monoidal functor satisfying* (sFI*).

Proof. It remains to prove the validity of (F∗).

$$F^*\langle X, Y\rangle = d^{(K')}_{(X\otimes Y)F}\left(p^{(K)}{}^{X,Y}_{1} F \otimes p^{(K)}{}^{X,Y}_{2} F\right)$$

$$= d^{(K')}_{XF\otimes YF}\left(\left(\left(1^{(K)}_X \otimes t^{(K)}_Y\right) r^{(K)}_X\right) F \otimes \left(\left(t^{(K)}_X \otimes 1^{(K)}_Y\right) l^{(K)}_X\right) F\right)$$

$$= d^{(K')}_{XF\otimes YF}\left(\left(1^{(K)}_X F \otimes t^{(K)}_Y F\right) \otimes \left(t^{(K)}_X F \otimes 1^{(K)}_Y F\right)\right)\left(r^{(K)}_X\right) F \otimes l^{(K)}_X F\right)$$

$$= \left(d^{(K')}_{XF}\left(\left(1^{(K)}_X \otimes t^{(K)}_X\right) F \otimes d^{(K')}_{YF}\left(t^{(K)}_Y \otimes 1^{(K)}_Y\right) F\right) b^{(K')}_{XF,IF,IF,YF}\right.$$
$$\left(r^{(K)}_X F \otimes l^{(K)}_X F\right)$$

$$= \left(\left(r^{(K')}_{XF}\right)^{-1} \otimes \left(l^{(K')}_{YF}\right)^{-1}\right) 1^{(K')}_{(XF\otimes IF)\otimes(IF\otimes YF)}\left(r^{(K)}_X F \otimes l^{(K)}_X F\right)$$

$$= \left(r^{(K')}_{XF}\right)^{-1} r^{(K)}_X F \otimes \left(l^{(K')}_{YF}\right)^{-1} l^{(K)}_X F \in Iso_{K'}.$$ ■

4. Functors between theories, theory morphisms

The following considerations are confined to *dhts*-theories, but it is easily to see that all results are transferable to *dhth∇s*-theories and *dts*-theories, respectively.

Lemma 4.1. *Let F be a d-monoidal functor from a Hoehnke theory $\underline{\mathbf{T}}$ into a Hoehnke theory $\underline{\mathbf{T'}}$ such that all morphisms $\widetilde{F}\langle A, B\rangle$ and i_F are central morphisms only. Then F maps every central morphism $c \in \mathbf{C_T}$ to a central morphism $cF \in \mathbf{C_{T'}}$.*

Proof. Every functor maps unit morphisms to unit morphism. Any d-monoidal functor fulfils the conditions (FA), (FR), and (FL) and since i_F

and every $\widetilde{F}\langle A, B\rangle$ are central morphisms, all images $a_{A,B,C}F$, r_AF, l_AF, $(a^{-1}_{A,B,C})F$, $(r^{-1}_A)F$, $(l^{-1}_A)F$ are central morphisms in $\underline{\mathbf{T'}}$.

Therefore, the images of all morphisms of $\mathbf{C}^{(0)}_{\mathbf{T}}$ are central morphisms in $\underline{\mathbf{T'}}$.

Assuming that all morphisms of $\mathbf{C}^{(n)}_{\mathbf{T}}$ for any $n \in \mathbb{N}$ are mapped by F to central morphisms in $\underline{\mathbf{T'}}$ one has

$$\forall \varphi \in \mathbf{C}^{(n+1)}_{\mathbf{T}} \setminus \mathbf{C}^{(n)}_{\mathbf{T}}\ \exists \varphi_1, \varphi_2 \in \mathbf{C}^{(n)}_{\mathbf{T}}\ (\varphi F = (\varphi_1\varphi_2)F =$$

$$(\varphi_1 F)(\varphi_2 F) \in \mathbf{C}_{\mathbf{T'}} \vee\ \varphi F = (\varphi_1 \otimes \varphi_2)F =$$

$$(\widetilde{F}\langle \operatorname{dom}\varphi_1, \operatorname{dom}\varphi_2\rangle)^{-1}(\varphi_1 F \otimes \varphi_2 F)\widetilde{F}\langle \operatorname{cod}\varphi_1, \operatorname{cod}\varphi_2\rangle) \in \mathbf{C}_{\mathbf{T'}}),$$

hence $\forall \varphi \in \mathbf{C}_{\mathbf{T}}\ (\varphi F \in \mathbf{C}_{\mathbf{T'}})$. ■

Observe that especially strict d-monoidal functors map central morphisms to central morphisms.

Theorem 4.2. *Let $\underline{\mathbf{T}}$ be a J-sorted Hoehnke theory. Then the function Φ as defined in* 2.5 *induces a d-monoidal functor $\Phi : \underline{\mathbf{T}} \to \underline{\mathbf{T}}$ relative to $\widetilde{\Phi}$ and i_Φ such that*

$$\forall X, Y \in |\mathbf{T}|\ (\widetilde{\Phi}\langle X, Y\rangle := (c^{-1}_X \otimes c^{-1}_Y)c_{X\otimes Y}) \quad \textit{and} \quad i_\Phi := 1_I.$$

Proof. The object mapping is given by the function $\Phi : |\mathbf{T}| \to |\mathbf{T}|$, namely

$$X\Phi = \begin{cases} X, & \text{for all} \quad X \in J \cup \{I, O\}, \\ I, & \text{for all} \quad X \in \langle I\rangle, \\ \bigotimes\limits_{j=1}^{n} A_j & \text{for all} \quad X \in |\mathbf{T}| \setminus \langle I\rangle^\circ, \end{cases}$$

where $A_1, ..., A_n \in J$ are exactly the factors appearing in X in this sequence independently of brackets.

Using the uniquely determined central morphisms $c_X \in \mathbf{C}_{\mathbf{T}}[X, X\Phi]$ define a morphism mapping by

$$\mathbf{T}[X, Y] \ni \varphi \mapsto \varphi\Phi := c^{-1}_X \varphi c_Y \in \mathbf{T}[X\Phi, Y\Phi].$$

Then the functor conditions are fulfilled, since

$$\forall \varphi \in \mathbf{T}\ ((\mathrm{dom}\varphi)\Phi = \mathrm{dom}(\varphi\Phi), (\mathrm{cod}\varphi)\Phi = \mathrm{cod}(\varphi\Phi)) \text{ by definition,}$$

$$\forall X \in |\mathbf{T}|\ (1_X\Phi = c_X^{-1} 1_X c_X = 1_{X\Phi},$$

$$\forall X, Y, P \in |\mathbf{T}|\ \forall \varphi \in \mathbf{T}[X, Y]\ \forall \psi \in \mathbf{T}[Y, P]$$

$$\big((\varphi\psi)\Phi = c_X^{-1}\varphi\psi c_P = c_X^{-1}\varphi c_Y c_Y^{-1}\psi c_P = (\varphi\Phi)(\psi\Phi)\big).$$

By Theorem 3.6, it is sufficient to prove the conditions (F$*$), (FI*), and (FM*) for the functor Φ.

Ad (F$*$): Let X and Y be arbitrary objects of $\mathbf{T}$. Then, by definition,

$$\begin{aligned}\Phi^*\langle X, Y\rangle &= d_{(X\otimes Y)\Phi}(p_1^{X,Y}\Phi \otimes p_1^{X,Y}\Phi) = d_{(X\otimes Y)\Phi}(c_{X,Y}^{-1}p_1^{X,Y}c_X \otimes c_{X,Y}^{-1}p_2^{X,Y}c_Y)\\ &= c_{X,Y}^{-1}d_{(X\otimes Y)}(p_1^{X,Y} \otimes p_2^{X,Y})(c_X \otimes c_Y) = c_{X,Y}^{-1}(c_X \otimes c_Y) \in \mathbf{C_T} \subseteq Iso_{\mathbf{T}}.\end{aligned}$$

Ad (FI*): Because of $I\Phi = I$, $t_{I\Phi} = t_I = 1_I \in Iso_{\mathbf{T}}$.

Ad (FM*): For all objects X_1, X_2, Y_1, Y_2 and all morphisms $\varphi_i \in \mathbf{T}[X_i, Y_i]$, $i \in \{1, 2\}$, the equation

$$\begin{aligned}(\varphi_1 \otimes \varphi_2)\Phi^*\langle Y_1, Y_2\rangle &= c_{X_1\otimes X_2}^{-1}(\varphi_1 \otimes \varphi_2)c_{Y_1\otimes Y_2}c_{Y_1\otimes Y_2}^{-1}(c_{Y_1} \otimes c_{Y_2})\\ &= c_{X_1\otimes X_2}^{-1}(\varphi_1 c_{Y_1} \otimes \varphi_2 c_{Y_2})\\ &= c_{X_1\otimes X_2}^{-1}(c_{X_1} \otimes c_{X_2})\left(c_{X_1}^{-1}\varphi_1 c_{Y_1} \otimes c_{X_2}^{-1}\varphi_2 c_{Y_2}\right)\\ &= \Phi^*\langle X_1, X_2\rangle(\varphi_1\Phi \otimes \varphi_2\Phi)\end{aligned}$$

is valid. Therefore, $(\Phi, \widetilde{\Phi}, i_\Phi)$ with $\widetilde{\Phi} := (\Phi^*)^{-1}$ and $i_\Phi := 1_I$ is a d-monoidal functor from $\underline{\mathbf{T}}$ into $\underline{\mathbf{T}}$. ■

The functor Φ shall be called the canonical functor of $\underline{\mathbf{T}}$.

Corollary 4.3. *Let $\underline{\mathbf{T}}$ be a J-sorted dhts-theory. Then the canonical functor of $\underline{\mathbf{T}}$ possesses the following properties:*

(1) $\forall X \in |\mathbf{T}|\ ((X\Phi)\Phi = X\Phi)$,

(2) $\forall X \in |\mathbf{T}|\ ((t_X)\Phi = t_{X\Phi})$,

(3) $\forall X \in |\mathbf{T}|\ ((r_X)\Phi = 1_{X\Phi} = (l_X)\Phi)$,

(4) $\forall X \in |\mathbf{T}|\ (d_X\Phi\Phi^*\langle X, X\rangle = d_{X\Phi})$,

(5) $\forall X \in |\mathbf{T}|\ (\nabla_X\Phi = \Phi^*\langle X, X\rangle\nabla_{X\Phi})$,

(6) $\forall X \in |\mathbf{T}|\ \big(\Phi^*\langle X, I\rangle = (r_{X\Phi})^{-1}, \Phi^*\langle I, X\rangle = (l_{X\Phi})^{-1}\big)$,

(7) $\forall X \in |\mathbf{T}|\ ((c_X)\Phi = 1_{X\Phi} = (1_X)\Phi = c_{X\Phi})$,

(8) $\forall \varphi \in \mathbf{T}\Big(\mathrm{dom}\varphi = \overset{n}{\underset{j=1}{\otimes}} A_j \wedge \mathrm{cod}\varphi = \overset{m}{\underset{k=1}{\otimes}} B_k \wedge A_j, B_k \in J \Rightarrow \varphi\Phi = \varphi\Big)$,

(9) $\forall \varphi \in \mathbf{T}\ ((\varphi\Phi)\Phi = \varphi\Phi)$.

Proof. Ad (1): $(X\Phi)\Phi = \left(\overset{n}{\underset{j=1}{\otimes}} A_j\right)\Phi = \overset{n}{\underset{j=1}{\otimes}} A_j = X\Phi.$

Ad (2): $(t_X)\Phi = c_X^{-1}t_Xc_I = t_{X\Phi}$ since $c_X \in Iso_T\ \wedge\ c_I = 1_I$.

Ad (3): The assertion is a special case of (7).

Ad (4): $d_X\Phi = c_X^{-1}d_Xc_{X\otimes X} = d_{X\Phi}\left(c_X^{-1} \otimes c_X^{-1}\right)c_{X\otimes X} = d_{X\Phi}(\Phi^*\langle X, X\rangle)^{-1}$

$$\Rightarrow d_X\Phi\Phi^*\langle X, X\rangle = d_{X\Phi}).$$

Ad (5): $\nabla_X\Phi = (c_{X\otimes X})^{-1}\nabla_Xc_X = (c_{X\otimes X})^{-1}(c_X \otimes c_X)\nabla_{X\Phi} = \Phi^*\langle X, X\rangle\nabla_{X\Phi}$.

Ad (6): $\Phi^*\langle X, I\rangle \in \mathbf{C}_T[X\Phi, X\Phi \otimes I]$ and $r_{X\Phi} \in \mathbf{C}_T[X\Phi \otimes I, X\Phi]$,

hence $\Phi^*\langle X, I\rangle = (r_{X\Phi})^{-1}$ by the coherence principle.

Ad (7): $c_X \in \mathbf{C}_T[X, X\Phi] \Rightarrow (c_X)\Phi \in \mathbf{C}_T[X\Phi, (X\Phi)\Phi] = \mathbf{C}_T[X\Phi, X\Phi] \ni 1_{X\Phi}$

$$\Rightarrow (c_X)\Phi = 1_{X\Phi} = 1_X\Phi.$$

$$c_{X\Phi} \in \mathbf{C}_T[X\Phi, X\Phi\Phi] = \mathbf{C}_T[X\Phi, X\Phi] \text{ shows } c_{X\Phi} = 1_{X\Phi}.$$

Ad (8): $\varphi\Phi = c_{X\Phi}^{-1}\varphi c_{Y\Phi} = \varphi$, where $X = \underset{j=1}{\overset{n}{\otimes}} A_j = X\Phi \ \wedge \ Y =$

$$= \underset{k=1}{\overset{m}{\otimes}} B_k = Y\Phi.$$

Ad (9): $(\varphi\Phi)\Phi = \left(c_X^{-1}\varphi c_Y\right)\Phi = \left(c_X^{-1}\right)\Phi(\varphi)\Phi(c_Y)\Phi = \varphi\Phi.$ ■

Definition 4.4. Let $\underline{\mathbf{T}}$ be a J-sorted Hoehnke theory and let Φ be the canonical d-monoidal functor of $\underline{\mathbf{T}}$. Then define a binary relation $\varkappa$ for objects and morphisms of $\mathbf{T}$ as follows:

$$(X, Y) \in \varkappa :\Leftrightarrow X\Phi = Y\Phi,$$

$$(\varphi_1, \varphi_2) \in \varkappa :\Leftrightarrow \varphi_1\Phi = \varphi_2\Phi.$$

Theorem 4.5. *The relation $\varkappa$ defined by the canonical d-monoidal functor Φ of a J-sorted Hoehnke theory $\underline{\mathbf{T}}$ as above is a "generalized" congruence on $\underline{\mathbf{T}}$.*

Proof. Concidering small categories as many-sorted total algebras, a congruence ρ is defined as a family of equivalence relationes on the isolated morphism sets, i.e. $(\varphi, \psi) \in \rho \Rightarrow \mathrm{dom}\varphi = \mathrm{dom}\psi \ \wedge \ \mathrm{cod}\varphi = \mathrm{cod}\psi$.

That is not true for the relation $\varkappa$, since only
$\forall \varphi, \psi \in \mathbf{T}((\varphi, \psi) \in \varkappa \Rightarrow (\mathrm{dom}\varphi)\Phi = (\mathrm{dom}\psi)\Phi \wedge (\mathrm{cod}\varphi)\Phi = (\mathrm{cod}\psi)\Phi)$,
because of

$$(\varphi, \psi) \in \varkappa \Rightarrow \varphi\Phi = \psi\Phi \Rightarrow c_{\mathrm{dom}\varphi}^{-1}\varphi c_{\mathrm{cod}\varphi} = c_{\mathrm{dom}\psi}^{-1}\varphi c_{\mathrm{cod}\psi}$$

$$\Rightarrow (\mathrm{dom}\varphi)\Phi = (\mathrm{dom}\psi)\Phi \ \wedge \ (\mathrm{cod}\varphi)\Phi = (\mathrm{cod}\psi)\Phi.$$

Moreover, the relation $\varkappa$ is not compatible with the morphism composition in the strong sense.

By definition, the relation $\varkappa$ is reflexive, symmetric, and transitive for objects and morphisms, respectively.

The relation is compatible with $\otimes$-operation of morphisms and objects, respectively, because of the following argumentation.

Using of (FM*) and Corollary 4.3 (5) one has for morphisms:

$$(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in \varkappa \Rightarrow (\varphi_1 \otimes \psi_1)\Phi = \Phi^*\langle X_1, P_1\rangle(\varphi_1\Phi \otimes \psi_1\Phi)(\Phi^*\langle Y_1, Q_1\rangle)^{-1}$$

$$= \Phi^*\langle X_1, P_1\rangle(\varphi_2\Phi \otimes \psi_2\Phi)(\Phi^*\langle Y_1, Q_1\rangle)^{-1}$$

$$= c^{-1}_{X_1 \otimes P_1}\left(c_{X_1}c^{-1}_{X_2} \otimes c_{P_1}c^{-1}_{P_2}\right)(\varphi_2 \otimes \psi_2)\left(c_{Y_2}c^{-1}_{Y_1} \otimes c_{Q_2}c^{-1}_{Q_1}\right)c_{Y_1 \otimes Q_1}$$

$$\Rightarrow (\varphi_1 \otimes \psi_1)\Phi = ((\varphi_1 \otimes \psi_1)\Phi)\Phi$$

$$= \left(c^{-1}_{X_1 \otimes P_1}\left(c_{X_1}c^{-1}_{X_2} \otimes c_{P_1}c^{-1}_{P_2}\right)(\varphi_2 \otimes \psi_2)\left(c_{Y_2}c^{-1}_{Y_1} \otimes c_{Q_2}c^{-1}_{Q_1}\right)c_{Y_1 \otimes Q_1}\right)\Phi$$

$$= (\varphi_2 \otimes \psi_2)\Phi$$

$$\Rightarrow (\varphi_1 \otimes \psi_1, \varphi_2 \otimes \psi_2) \in \varkappa.$$

Concerning the object relation one obtains

$$(X_1, X_2) \in \varkappa \ \wedge \ (Y_1, Y_2) \in \varkappa \Rightarrow X_1\Phi = X_2\Phi \ \wedge \ Y_1\Phi = Y_2\Phi$$

$$\Rightarrow 1_{X_1}\Phi = 1_{X_2}\Phi \ \wedge \ 1_{Y_1}\Phi = 1_{Y_2}\Phi$$

$$\Rightarrow (1_{X_1}, 1_{X_2}) \in \varkappa \ \wedge \ (1_{Y_1}, 1_{Y_2}) \in \varkappa$$

$$\Rightarrow (1_{X_1} \otimes 1_{Y_1}, 1_{X_2} \otimes 1_{Y_2}) \in \varkappa$$

$$\Rightarrow (1_{X_1 \otimes Y_1}, 1_{X_2 \otimes Y_2}) \in \varkappa$$

$$\Rightarrow 1_{X_1 \otimes Y_1}\Phi = 1_{X_2 \otimes Y_2}\Phi$$

$$\Rightarrow 1_{(X_1 \otimes Y_1)\Phi} = 1_{(X_2 \otimes Y_2)\Phi}$$

$$\Rightarrow (X_1 \otimes Y_1)\Phi = (X_2 \otimes Y_2)\Phi$$

$$\Rightarrow (X_1 \otimes Y_1, X_2 \otimes Y_2) \in \varkappa.$$

The relation $\varkappa$ is, as already mentioned, reflexive, therefore it preserves all morphisms wich are determined by constant operation symbols.

For the morphism composition:

Let $\varphi_i \in \mathbf{T}[X_i, Y_i], \psi_i \in \mathbf{T}[P_i, Q_i]$ for $i \in \{1, 2\}$ be arbitrary morphisms of $\underline{\mathbf{T}}$. Then, for $Y_1 = P_1$, i. e. φ_1 is composable with ψ_1,

$$(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in \varkappa \Rightarrow (\varphi_1\psi_1)\Phi = (\varphi_1\Phi)(\psi_1\Phi)$$

$$= (\varphi_2\Phi)(\psi_2\Phi) = c_{X_2}^{-1}\varphi_2 c_{Y_2} c_{P_2}^{-1}\psi_2 c_{Q_2},$$

therefore, by Corollary 4.3 (7) and (5),

$$(\varphi_1\psi_1)\Phi = ((\varphi_1\psi_1)\Phi)\Phi = \left(c_{X_2}^{-1}\varphi_2 c_{Y_2} c_{P_2}^{-1}\psi_2 c_{Q_2}\right)\Phi = (\varphi_2 c_{Y_2,P_2}\psi_2)\Phi,$$

hence $(\varphi_1\psi_1, \varphi_2 c_{Y_2,P_2}\psi_2) \in \varkappa$.

Observe that especially φ_2 and ψ_2 have not to be composable in general, but there is a central morphism c such that there exists the compositum $\varphi_2 c\psi_2$. ∎

Remark. It is easy to verify that the generating central morphisms $1,\ a,\ a^{-1},\ r,\ r^{-1},\ l,\ l^{-1}$ of any J-sorted theory $\underline{\mathbf{T}}$ fulfil even the following conditions:

$$\forall X, Y, P \in |\mathbf{T}|\ ((1_{X\otimes(Y\otimes P)}, 1_{(X\otimes Y)\otimes P)}) \in \varkappa),$$

$$\forall X, Y, P \in |\mathbf{T}|\ ((a_{X,Y,P}, 1_{X\otimes(Y\otimes P)}) \in \varkappa \wedge ((a_{X,Y,P})^{-1}, 1_{(X\otimes Y)\otimes P)}) \in \varkappa),$$

$$\forall X \in |\mathbf{T}|\ ((1_{X\otimes I}, 1_X), (1_{I\otimes X}), 1_X) \in \varkappa),$$

$$\forall X \in |\mathbf{T}|\ ((r_X, 1_{X\otimes I}), \left((r_X)^{-1}, 1_X\right), ((l_X), 1_{I\otimes X}), \left((l_X)^{-1}, 1_X\right) \in \varkappa).$$

Theorem 4.6. *To every J-sorted Hoehnke theory*

$$\underline{\mathbf{T}} \in |Th^{\circ}_{dht}(J)|$$

there exists in a natural manner a J-sorted strict Hoehnke theory

$$\underline{\mathbf{T_s}} \in |sTh^{\circ}_{dht}(J)|.$$

Proof. The canonical d-monoidal functor $\Phi : \underline{\mathbf{T}} \to \underline{\mathbf{T}}$ related to any J-sorted Hoehnke theory $\underline{\mathbf{T}}$ induces the "generalized" congruence $\varkappa$.

Construct a new category $\mathbf{T_s}$ by using the knowledge about $\underline{H^{\circ}}, \underline{S^{\circ}}$ and the functions W and W^*.

$|\mathbf{T_s}| := S^{\circ} \quad (:= S)$,

$\mathbf{T_s} := \{[\varphi]_{\varkappa} \mid \varphi \in \mathbf{T}\}$, where $[\varphi]_{\varkappa} = \{\varphi' \in \mathbf{T} \mid \varphi\Phi = \varphi'\Phi\}$,

$\mathrm{dom}^{(\mathbf{T_s})}[\varphi]_{\varkappa} := \left(\mathrm{dom}^{(\mathbf{T})}\varphi\right) W^*$, $\mathrm{cod}^{(\mathbf{T_s})}[\varphi]_{\varkappa} := \left(\mathrm{cod}^{(\mathbf{T})}\varphi\right) W^*$,

$1_A^{(\mathbf{T_s})} := \left[1_{AW}^{(\mathbf{T})}\right]_{\varkappa}$,

$[\varphi]_{\varkappa} \cdot_{(\mathbf{T_s})} [\psi]_{\varkappa} := [\varphi c_{Y,P} \psi]_{\varkappa}$, where $Y\Phi = (\mathrm{cod}\varphi)\Phi = (\mathrm{dom}\psi)\Phi = P\Phi$

$$(\Leftrightarrow YW^* = (\mathrm{cod}\varphi)W^* = (\mathrm{dom}\psi)W^*) = PW^*),$$

$A \otimes_{(\mathbf{T_s})} B = (AW \otimes_{(\mathbf{T})} BW)W^*$ (by (W4)),
$[\varphi]_{\varkappa} \otimes_{(\mathbf{T_s})} [\psi]_{\varkappa} := [\varphi \otimes_{(\mathbf{T})} \psi]_{\varkappa}$,

$a_{A,B,C}^{(\mathbf{T_s})} := \left[a_{AW,BW,CW}^{(\mathbf{T})}\right]_{\varkappa} = [1_{AW\otimes(BW\otimes CW)}^{(\mathbf{T})}]_{\varkappa}$,

$r_A^{(\mathbf{T_s})} := \left[r_{AW}^{(\mathbf{T})}\right]_{\varkappa} = \left[1_{AW}^{(\mathbf{T})}\right]_{\varkappa} = \left[l_{AW}^{(\mathbf{T})}\right]_{\varkappa} =: l_A^{(\mathbf{T_s})}$,

$s_{A,B}^{(\mathbf{T_s})} := \left[s_{AW,BW}^{(\mathbf{T})}\right]_{\varkappa}$, $d_A^{(\mathbf{T_s})} := \left[d_{AW}^{(\mathbf{T})}\right]_{\varkappa}$, $t_A^{(\mathbf{T_s})} := \left[t_{AW}^{(\mathbf{T})}\right]_{\varkappa}$, $\nabla_A^{(\mathbf{T_s})} := \left[\nabla_{AW}^{(\mathbf{T})}\right]_{\varkappa}$,

$o^{(\mathbf{T_s})} := \left[o^{(\mathbf{T})}\right]_{\varkappa}$.

Obviously, $(S^{\circ}; \otimes, I, O)$ is an algebra of type $(2, 0, 0)$ with an associative binary operation, a unit element I, and a zero element O.

Moreover, $(|\mathbf{T_s}|, \mathbf{T_s}, \cdot, \mathrm{dom}, \mathrm{cod}, 1)$ is a small category, since $|\mathbf{T_s}|$ is a set and

$$[\varphi]_\varkappa \in \mathbf{T_s}[A,B] \Rightarrow \varphi \in \mathbf{T}[X,Y] \ \wedge \ A = XW^*, \ B = YW^* \ \Rightarrow 1_A[\varphi]_\varkappa$$

$$= [1_X]_\varkappa[\varphi]_\varkappa = [1_X c_{X,X}\varphi]_\varkappa = [\varphi]_\varkappa = [\varphi c_{Y,Y} 1_Y]_\varkappa = [\varphi]_\varkappa[1_Y]_\varkappa = [\varphi]_\varkappa 1_B,$$

$$[\varphi]_\varkappa \in \mathbf{T_s}[A,B], [\psi]_\varkappa \in \mathbf{T_s}[B,C], [\chi]_\varkappa \in \mathbf{T_s}[C,D]$$

$$\Rightarrow [\varphi]_\varkappa([\psi]_\varkappa[\chi]_\varkappa) = [\varphi]_\varkappa[\psi c_{P,Q}\chi]_\varkappa = [\varphi c_{X,Y}\psi c_{P,Q}\chi]_\varkappa$$

$$= [\varphi c_{X,Y}\psi]_\varkappa[\chi]_\varkappa = ([\varphi]_\varkappa[\psi]_\varkappa)[\chi]_\varkappa.$$

$(\mathbf{T_s}; \otimes, I, 1, 1, 1, s)$ is a symmetric strictly monoidal category since the defining conditions are fulfilled. Observe that to every morphism $\rho \in \mathbf{T_s}[A,B]$ there is a morphism $\varphi \in \mathbf{T}[X,Y]$ such that $A = XW^*, B = YW^*, \rho = [\varphi]_\varkappa$.

Ad (F1): $\forall \rho, \rho' \in \mathbf{T_s}$ $(\mathrm{dom}\,(\rho \otimes \rho') = \mathrm{dom}\,([\varphi]_\varkappa \otimes [\varphi']_\varkappa)$

$$= \mathrm{dom}\,[\varphi \otimes \varphi']_\varkappa = (\mathrm{dom}\,(\varphi \otimes \varphi'))W^*$$

$$= ((\mathrm{dom}\,\varphi) \otimes (\mathrm{dom}\,\varphi'))W^* = (\mathrm{dom}\,\varphi)W^* \otimes (\mathrm{dom}\,\varphi')W^*$$

$$= (\mathrm{dom}\,[\varphi]_\varkappa) \otimes (\mathrm{dom}\,[\varphi']_\varkappa) = \mathrm{dom}\,\rho \otimes \mathrm{dom}\,\rho').$$

Ad (F2): The assertion $\forall \rho, \rho' \in \mathbf{T_s}$ $(\mathrm{cod}\,(\rho \otimes \rho') = \mathrm{cod}\,\rho \otimes \mathrm{cod}\,\rho')$ will be proved in the same manner.

Ad (F3): $\forall A, B \in |\mathbf{T_s}|$ $(1_{A\otimes B} = [1_{(A\otimes B)W}]_\varkappa = [1_{AW\otimes BW}]_\varkappa$

$$= [1_{AW} \otimes 1_{BW}]_\varkappa = [1_{AW}]_\varkappa \otimes [1_{BW}]_\varkappa = 1_A \otimes 1_B),$$

since $\underline{\mathbf{T}}$ is a symmetric monoidal category and for all $A, B \in S^\circ$ one has $(A \otimes B)W\Phi = (A \otimes B)WW^*W = (A \otimes B)W = (AWW^* \otimes BWW^*)W = (AW \otimes BW)\Phi$.

Ad (F4): $\forall A, B, C, A', B', C' \in |\mathbf{T_s}|$ $\forall \rho \in \mathbf{T_s}[A,B]$

$$\forall \sigma \in \mathbf{T_s}[B,C] \ \forall \rho' \in \mathbf{T_s}[A',B'] \forall \sigma' \in \mathbf{T_s}[B',C']$$

$$\begin{aligned}((\rho\otimes\rho')(\sigma\otimes\sigma') &= ([\varphi]_\varkappa\otimes[\varphi']_\varkappa)([\psi]_\varkappa\otimes[\psi']_\varkappa)\\
&= [\varphi\otimes\varphi']_\varkappa[\psi\otimes\psi']_\varkappa\\
&= [(\varphi\otimes\varphi')c_{Y\otimes Y',P\otimes P'}(\psi\otimes\psi')]_\varkappa\\
&= [(\varphi\otimes\varphi')(c_{Y,P}\otimes c_{Y',P'})(\psi\otimes\psi')]_\varkappa\\
&= [\varphi c_{Y,P}\psi\otimes\varphi' c_{Y',P'}\psi']_\varkappa\\
&= [\varphi c_{Y,P}\psi]_\varkappa\otimes[\varphi' c_{Y',P'}\psi']_\varkappa\\
&= [\varphi]_\varkappa[\psi]_\varkappa\otimes[\varphi']_\varkappa[\psi']_\varkappa\\
&= \rho\sigma\otimes\rho'\sigma').\end{aligned}$$

Ad (M1), (M2), (M3): The conditions are trivially fulfilled since a and r consist of unit morphisms only.

Ad (M4): $\forall A,B\in|\mathbf{T_s}|\ \Big(s^{(\mathbf{T_s})}_{A,B}s^{(\mathbf{T_s})}_{B,A}=\Big[s^{(\mathbf{T})}_{AW,BW}\Big]_\varkappa\Big[s^{(\mathbf{T})}_{BW,AW}\Big]_\varkappa$

$$=\Big[s^{(\mathbf{T})}_{AW,BW}c_{BW\otimes AW,BW\otimes AW}s^{(\mathbf{T})}_{BW,AW}\Big]_\varkappa$$

$$=\Big[s^{(\mathbf{T})}_{AW,BW}s^{(\mathbf{T})}_{BW,AW}\Big]_\varkappa=\Big[1^{(\mathbf{T})}_{AW\otimes BW}\Big]_\varkappa=1^{(\mathbf{T_s})}_{(AW\otimes BW)W^*}=1^{(\mathbf{T_s})}_{A\otimes B}\Big).$$

Ad (M5): $\forall A\in|\mathbf{T_s}|\ \Big(s^{(\mathbf{T_s})}_{A,I}l^{(\mathbf{T_s})}_{A}=\Big[s^{(\mathbf{T})}_{AW,IW}\Big]_\varkappa\Big[l^{(\mathbf{T})}_{AW}\Big]_\varkappa=\Big[s^{(\mathbf{T})}_{AW,IW}l^{(\mathbf{T})}_{AW}\Big]_\varkappa$

$$=\Big[r^{(\mathbf{T})}_{AW}\Big]_\varkappa=r^{(\mathbf{T_s})}_{A}=1^{(\mathbf{T_s})}_{A}\Big).$$

Ad (M6): $\forall A,B,C,A',B',C'\in|\mathbf{T_s}|\ \forall\rho\in\mathbf{T_s}[A,A']$

$$\forall\sigma\in\mathbf{T_s}[B,B']\forall\tau\in\mathbf{T_s}[C,C']$$

$$\Big(a^{(\mathbf{T_s})}_{A,B,C}((\rho\otimes\sigma)\otimes\tau)=\Big[a^{(\mathbf{T})}_{X,Y,P}\Big]_\varkappa(([\varphi]_\varkappa\otimes[\psi]_\varkappa)\otimes[\chi]_\varkappa)$$

$$= \left[a^{(\mathbf{T})}_{X,Y,P} c_{(X\otimes Y)\otimes P,(X\otimes Y)\otimes P}((\varphi \otimes \psi) \otimes \chi)\right]_{\varkappa}$$

$$= \left[(\varphi \otimes (\psi \otimes \chi))a^{(\mathbf{T})}_{X',Y',P'}\right]_{\varkappa}$$

$$= \left[(\varphi \otimes (\psi \otimes \chi))c_{X'\otimes(Y'\otimes P'),X'\otimes(Y'\otimes P')}a^{(\mathbf{T})}_{X',Y',P'}\right]_{\varkappa}$$

$$= \Big([\varphi]_{\varkappa} \otimes ([\psi]_{\varkappa} \otimes [\chi]_{\varkappa})\Big) \left[a^{(\mathbf{T})}_{X',Y',P'}\right]_{\varkappa}$$

$$= (\rho \otimes (\sigma \otimes \tau))a^{(\mathbf{T_s})}_{A',B',C'}\Big).$$

Ad (M7): $\forall A, A' \in |\mathbf{T_s}|\ \forall \rho \in \mathbf{T_s}[A, A']\ \ \Big(r^{(\mathbf{T_s})}_{A}\rho = \left[r^{(\mathbf{T})}_{AW}]_{\varkappa}[\varphi\right]_{\varkappa}$

$$= \left[r^{(\mathbf{T})}_{AW} c_{AW,X}\varphi\right]_{\varkappa} \qquad \text{(by } XW^* = AWW^* = A\text{)}$$

$$= \left[\Big(c_{AW,X}\varphi \otimes 1^{(\mathbf{T})}_{I}\Big)\, r^{(\mathbf{T})}_{X'}\right]_{\varkappa}$$

$$= \left[\Big(c_{AW,X}\varphi \otimes 1^{(\mathbf{T})}_{I}\Big)\, c_{X'\otimes I,X'\otimes I} r^{(\mathbf{T})}_{X'}\right]_{\varkappa}$$

$$= \left[c_{AW,X}\varphi \otimes 1^{(\mathbf{T})}_{I}\right]_{\varkappa} \left[r^{(\mathbf{T})}_{X'}\right]_{\varkappa}$$

$$= \Big([c_{AW,X}\varphi]_{\varkappa} \otimes \left[1^{(\mathbf{T})}_{I}\right]_{\varkappa}\Big) \left[r^{(\mathbf{T})}_{X'}\right]_{\varkappa}$$

$$= \Big([\varphi]_{\varkappa} \otimes \left[1^{(\mathbf{T})}_{I}\right]_{\varkappa}\Big) \left[r^{(\mathbf{T})}_{X'}\right]_{\varkappa}$$

$$= \Big(\rho \otimes 1^{(\mathbf{T_s})}_{I}\Big)\, r^{(\mathbf{T_s})}_{A'}\Big) \qquad \text{(by } X'W^* = A'WW^* = A'\text{).}$$

Ad (M8): $\forall A, B \in |\mathbf{T_s}|\ \forall \rho \in \mathbf{T_s}[A, A'], \sigma \in \mathbf{T_s}[B, B']$

$$\Big(s^{(\mathbf{T_s})}_{A,B}(\sigma \otimes \rho) = \left[s^{(\mathbf{T})}_{AW,BW}\right]_{\varkappa} ([\psi]_{\varkappa} \otimes [\varphi]_{\varkappa}\Big)$$

$$= \left[s^{(\mathbf{T})}_{AW,BW}\right]_{\varkappa} [\psi \otimes \varphi]_{\varkappa}$$

$$= \left[s^{(\mathbf{T})}_{AW,BW} c_{BW\otimes AW,Y\otimes X}(\psi \otimes \varphi)\right]_{\varkappa}$$

$$= \left[c_{AW\otimes BW,X\otimes Y} s^{(\mathbf{T})}_{X,Y}(\psi \otimes \varphi)\right]_{\varkappa}$$

$$= \left[c_{AW\otimes BW,X\otimes Y}(\varphi\otimes\psi)s^{(\mathbf{T})}_{X',Y'}\right]_{\varkappa}$$

$$= \left[(\varphi\otimes\psi)c_{X'\otimes Y',X'\otimes Y'}s^{(\mathbf{T})}_{X',Y'}\right]_{\varkappa}$$

$$= [\varphi\otimes\psi]_{\varkappa}\left[s^{(\mathbf{T})}_{X',Y'}\right]_{\varkappa}$$

$$= ([\varphi]_{\varkappa}\otimes[\psi]_{\varkappa})\left[s^{(\mathbf{T})}_{X',Y'}\right]_{\varkappa}$$

$$= (\rho\otimes\sigma)s^{(\mathbf{T_s})}_{A',B'},$$

where

$$XW^* = AWW^* = A, \quad X'W^* = A'WW^* = A',$$

$$YW^* = BWW^* = B, \quad Y'W^* = B'WW^* = B'. \qquad \blacksquare$$

Theorem 4.7. *Let $\underline{\mathbf{T}} \in |Th^{\circ}_{dht}(J)|$ be a J-sorted Hoehnke theory. Then there exists in a natural manner a strictly d-monoidal functor Ψ into the corresponding strict Hoehnke theory $\underline{\mathbf{T_s}} \in |sTh^{\circ}_{dht}(J)|$.*

Proof. Defining $X\Psi := XW^*$, $\varphi\Psi := [\varphi]_{\varkappa}$ one obtains for arbitrary objects X, Y, P and morphisms $\varphi \in T[X,Y]$, $\psi \in T[Y,P]$

$$\left(\mathrm{dom}^{(T)}\varphi\right)\Psi = X\Psi = XW^* = \mathrm{dom}^{(T_s)}[\varphi]_{\varkappa} = \mathrm{dom}^{(T_s)}(\varphi\Psi),$$

$$\left(\mathrm{cod}^{(T)}\varphi\right)\Psi = Y\Psi = YW^* = \mathrm{cod}^{(T_s)}[\varphi]_{\varkappa} = \mathrm{cod}^{(T_s)}(\varphi\Psi),$$

$$1^{(T)}_X\Psi = \left[1^{(T)}_X\right]_{\varkappa} = 1^{(T_s)}_{XW^*} = 1^{(T_s)}_{X\Psi},$$

$$(\varphi\cdot_{(T)}\psi)\Psi = [\varphi\cdot_{(T)}\psi]_{\varkappa} = [\varphi]_{\varkappa}\cdot_{(T_s)}[\psi]_{\varkappa} = (\varphi\Psi)\cdot_{(T_s)}\psi\Psi),$$

hence Ψ is a functor.

By Lemma 3.8, it is sufficient to show (sFD), (sFT), and (sFM).

Ad (sFD): $\quad d^{(\mathbf{T})}_X\Psi = \left[d^{(\mathbf{T})}_X\right]_{\varkappa} = \left[d^{(\mathbf{T})}_{XW^*W}\right]_{\varkappa} = d^{(\mathbf{T_s})}_{XW^*} = d^{(\mathbf{T_s})}_{X\Psi}.$

Ad (sFT): $t_X^{(\mathbf{T})}\Psi = \left[t_X^{(\mathbf{T})}\right]_{\varkappa} = \left[t_{XW^*W}^{(\mathbf{T})}\right]_{\varkappa} = t_{XW^*}^{(\mathbf{T_s})} = t_{X\Psi}^{(\mathbf{T_s})}$.

Ad (sFM): $(\varphi \otimes \psi)\Psi = [\varphi \otimes \psi]_{\varkappa} = [\varphi]_{\varkappa} \otimes [\psi]_{\varkappa} = \varphi\Psi \otimes \psi\Psi$.

Therefore, $\Psi : \underline{T} \to \underline{T_s}$ is a strictly d-monoidal functor. ∎

The converse question is also positively answered by the following theorem:

Theorem 4.8. *Let $\underline{\mathbf{T}}_s \in |sT^\circ_{dht}(J)|$ be a strict J-sorted Hoehnke theory. Then there corresponds to $\underline{\mathbf{T_s}}$ in a natural way a J-sorted Hoehnke theory $\underline{\mathbf{T}} \in |T^\circ_{dht}(J)|$.*

Proof. Take $|\mathbf{T}| = H^\circ$ $(|T| = H)$, where $(H^\circ; \otimes, I, O)$ $((H; \otimes, I))$ is the free $\mathcal{G}^\circ$-algebra (free $\mathcal{G}$-algebra) freely generated by J.

Defining $\mathbf{T}[X,Y] := \{(X,\varphi,Y) \mid \varphi \in \mathbf{T_s}[XW^*, YW^*]\}$ for arbitrary $X, Y \in H^\circ$ $(X, Y \in H)$ one obtains obviously $\mathbf{T}[X,Y] \cup \mathbf{T}[X',Y'] = \emptyset$ if $X \neq X'$ or $Y \neq Y'$ and, by definition, $\mathrm{dom}^{(\mathbf{T})}(X,\varphi,Y) = X$, $\mathrm{cod}^{(\mathbf{T})}(X,\varphi,Y) = Y$ and $1_X^{(\mathbf{T})} = (X, 1_{XW^*}^{(\mathbf{T_s})}, X)$.

Morphisms (X,φ,Y) and (P,ψ,Q) are composable for $Y = P$ defined by

$$(X,\varphi,Y) \cdot_{(\mathbf{T})} (Y,\psi,Q) := (X, \varphi \cdot_{(\mathbf{T_s})} \psi, Q).$$

Then

$$1_X^{(\mathbf{T})} \cdot_{(\mathbf{T})} (X,\varphi,Y) = (X, 1_{XW^*}^{(\mathbf{T_s})}, X) \cdot_{(\mathbf{T})} (X,\varphi,Y) = (X, 1_{XW^*}^{(\mathbf{T_s})}\varphi, Y) = (X,\varphi,Y),$$

$$(X,\varphi,Y) \cdot_{(\mathbf{T})} 1_Y^{(\mathbf{T})} = (X,\varphi,Y) \cdot_{(\mathbf{T})} (Y, 1_{YW^*}^{(\mathbf{T_s})}, Y) = (X, \varphi 1_{YW^*}^{(\mathbf{T_s})}, Y) = (X,\varphi,Y),$$

$$(X,\varphi,Y) \cdot_{(\mathbf{T})} ((Y,\psi,P) \cdot_{(\mathbf{T})} (P,\chi,Q)) = (X, \varphi(\psi\chi), Q)$$

$$= (X, (\varphi\psi)\chi, Q) = ((X,\varphi,Y) \cdot_{(\mathbf{T})} (Y,\psi,P)) \cdot_{(\mathbf{T})} (P,\chi,Q),$$

hence one has a category.

By the agreements

$$(X_1,\varphi_1,Y_1) \otimes_{(\mathbf{T})} (X_2,\varphi_2,Y_2) := (X_1 \otimes_{(\mathbf{T_s})} X_2, \varphi_1 \otimes_{(\mathbf{T_s})} \varphi_2, Y_1 \otimes_{(\mathbf{T_s})} Y_2),$$

$a^{(\mathbf{T})}_{X,Y,P} := (X \otimes (Y \otimes P), 1^{(T_s)}_{XW^* \otimes YW^* \otimes PW^*}, (X \otimes Y) \otimes P),$

$r^{(\mathbf{T})}_{X} := \Big(X \otimes I, 1^{(\mathbf{T_s})}_{XW^*}, X\Big),$

$l^{(\mathbf{T})}_{X} := \Big(I \otimes X, 1^{(\mathbf{T_s})}_{XW^*}, X\Big),$

$s^{(\mathbf{T})}_{X,Y} := \Big(X \otimes Y, s^{(\mathbf{T_s})}_{XW^* \otimes YW^*}, Y \otimes X\Big),$

$d^{(\mathbf{T})}_{X} := \Big(X, d^{(\mathbf{T_s})}_{XW^*}, X \otimes X\Big),$

$t^{(\mathbf{T})}_{X} := \Big(X, t^{(\mathbf{T_s})}_{XW^*}, I\Big),$

$o^{(\mathbf{T})} := \big(I, o^{(\mathbf{T_s})}, O\big)$

one obtains a *dhts*-category $(\mathbf{T}, \otimes_{(\mathbf{T})}, I, a^{(\mathbf{T})}, r^{(\mathbf{T})}, l^{(\mathbf{T})}, s^{(\mathbf{T})}, t^{(\mathbf{T})}, o^{(\mathbf{T})})$, i.e. a Hoehnke theory in $|T^{\circ}_{dht}(J)|$, since the validity of the defining axioms obviously carries over from $\underline{\mathbf{T_s}}$ into $\underline{\mathbf{T}}$. ■

Remark. If $\underline{\mathbf{T_s}} \in |sT^{\circ}_{dhth\nabla}(J)|$ is even any strict J-sorted Hoehnke theory with halfdiagonalinversions, then one obtains by the additional agreement

$$\nabla^{(\mathbf{T})}_{X} := \Big(X \otimes X, \nabla^{(\mathbf{T_s})}_{XW^*}, X\Big)$$

a $dhth\nabla s$-category $(\mathbf{T}, \otimes_{(\mathbf{T})}, I, a^{(\mathbf{T})}, r^{(\mathbf{T})}, l^{(\mathbf{T})}, s^{(\mathbf{T})}, t^{(\mathbf{T})}, \nabla^{(\mathbf{T})}, o^{(\mathbf{T})})$, i.e. a Hoehnke theory in $|T^{\circ}_{dhth\nabla}(J)|$.

Definition 4.9. Let $\underline{\mathbf{T}}$ and $\underline{\mathbf{T}'}$ be J-sorted Hoehnke theories in $|Th^{\circ}_{dht}(J)|$ and $|sTh^{\circ}_{dht}(J)|$, respectively.

Then a d-monoidal functor $F : \underline{\mathbf{T}} \to \underline{\mathbf{T}'}$ is called *theory morphism*, if, in addition, the conditions

(Th1) $\quad \forall X \in |\mathbf{T}| \ (XF = X),$

(sF$*$) $\quad \forall X, Y \in |\mathbf{T}| \ (\widetilde{F}\langle X, Y\rangle \in Un_{K'})$

are fulfilled.

Lemma 4.10. *Every theory morphism $F : \underline{\mathbf{T}} \to \underline{\mathbf{T}}'$ has the properties* (sFD), (sFT), (sFM), (sFI*).

Conversely, any functor $F : \underline{\mathbf{T}} \to \underline{\mathbf{T}}'$ is a theory morphism between J-sorted Hoehnke theories $\underline{\mathbf{T}}$ and $\underline{\mathbf{T}}'$, whenever F satifies (Th1), (sFD), (sFT), and (sFM).

Proof. The assertion is an immediate consequence of Lemma 3.8 and Corollary 3.10. ∎

Theorem 4.11. *All J-sorted Hoehnke theories together with the corresponding theory morphisms form a category $Th^{\circ}_{dht}(J)$ and $sTh^{\circ}_{dht}(J)$, respectively, where the composition of theory morphisms is defined by the usual composition of functors.*

Proof. Obviously, dom $(F : \underline{\mathbf{T}} \to \underline{\mathbf{T}}') = \underline{\mathbf{T}}$, cod $(F : \underline{\mathbf{T}} \to \underline{\mathbf{T}}') = \underline{\mathbf{T}}'$.

The identical functor $1_{\underline{\mathbf{T}}} : \underline{\mathbf{T}} \to \underline{\mathbf{T}}$ is a theory morphism with respect to

$$\widetilde{1_{\underline{\mathbf{T}}}} = (\widetilde{1_{\underline{\mathbf{T}}}}\langle X, Y\rangle = 1_{X\otimes Y} \mid X, Y \in H^{\circ}), \quad i_{1_{\underline{\mathbf{T}}}} = 1_I .$$

Let $F : \underline{\mathbf{T}} \to \underline{\mathbf{T}}'$ and $G : \underline{\mathbf{T}}' \to \underline{\mathbf{T}}''$ be theory morphisms. Then, by definition, FG is a functor fulfilling the condition (Th1).

Moreover, because of Lemma 4.10 and Proposition 3.9, FG is a theory morphism.

Trivially, $F1_{\underline{\mathbf{T}}} = F = F1_{\underline{\mathbf{T}'}}$ and $F(GH) = (FG)H$ for every theory morphism F and all composable theory morphisms F, G and H. ∎

Theorem 4.12. *Let $Th^{\circ}_{dht}(J)$ and $sTh^{\circ}_{dht}(J)$ be the categories introduced above. Then there are the functors*

$$\Sigma : Th^{\circ}_{dht}(J) \to sTh^{\circ}_{dht}(J)$$

$$\underline{\mathbf{T}} \mapsto \underline{\mathbf{T}}\Sigma := \underline{\mathbf{T_s}} \text{ (see 4.6)},$$

$$(F : \underline{\mathbf{T}} \to \underline{\mathbf{T}}') \mapsto (F\Sigma : \underline{\mathbf{T_s}} \to \underline{\mathbf{T_s}}') \textit{ defined by}$$

$$XW^* \mapsto XW^*,\ [\varphi]_{\varkappa} \mapsto [\varphi F]_{\varkappa'}$$

and

$$\Pi : sTh^{\circ}_{dht}(J) \to Th^{\circ}_{dht}(J)$$

$$\underline{\mathbf{T_s}} \mapsto \underline{\mathbf{T_s}}\Pi := \underline{\mathbf{T}} \text{ (see 4.7)},$$

$$(F : \underline{\mathbf{T_s}} \to \underline{\mathbf{T_s}}') \mapsto (F\Pi : \underline{\mathbf{T}} \to \underline{\mathbf{T}'}) \textit{ defined by}$$

$$X \mapsto X,\ (X, \varphi, Y) \mapsto (X, \varphi F, Y)$$

such that Σ *is a left-adjoint functor of the functor* Π.

Proof. **a)** The functor property of Σ:
The mapping on objects is well defined by Theorem 4.5. Let F be a theory morphism from a J-sorted theory $\underline{\mathbf{T}}$ into a J-sorted theory $\underline{\mathbf{T}'}$, i.e. $F \in Th^{\circ}_{dht}(J)[\underline{\mathbf{T}}, \underline{\mathbf{T}'}]$. Then $F\Sigma$, defined as above, is a theory morphism too, more precisely,

$$F\Sigma \in sTh^{\circ}(J)[\underline{\mathbf{T}}\Sigma, \underline{\mathbf{T}'}\Sigma].$$

By definition, the mapping $F\Sigma$ respects "dom" and "cod" and one obtains

$$1^{(\mathbf{T}\Sigma)}_{XW^*}(F\Sigma) = \left[1^{(\mathbf{T})}_X\right]_{\varkappa}(F\Sigma) = \left[1^{(\mathbf{T})}_X F\right]_{\varkappa} = \left[1^{(\mathbf{T}')}_X\right]_{\varkappa} = 1^{(\mathbf{T}'\Sigma)}_{XW^*} = 1^{(\mathbf{T}'\Sigma)}_{(XW^*)(F\Sigma)}$$

for all objects $X \in |\mathbf{T}|$.

Now let $[\varphi]_{\varkappa} \in \mathbf{T}\Sigma[XW^*, YW*]$, $[\psi]_{\varkappa} \in \mathbf{T}\Sigma[UW^*, VW*]$ be arbitrary morphisms such that $YW^* = UW^*$. Then

$$([\varphi]_{\varkappa}[\psi]_{\varkappa})(F\Sigma) = [\varphi c_{Y,U}\psi]_{\varkappa})(F\Sigma) = [\varphi F]_{\varkappa'}[c_{Y,U}F]_{\varkappa'}[\psi F]_{\varkappa'}$$

$$= [\varphi F]_{\varkappa'}[c'_{Y,U}]_{\varkappa'}[\psi F]_{\varkappa'} = [\varphi F]_{\varkappa'}[1'_{YW^*,UW^*}]_{\varkappa'}[\psi F]_{\varkappa'}$$

$$= [\varphi F]_{\varkappa'}[\psi F]_{\varkappa'} = [\varphi]_{\varkappa}(F\Sigma)[\psi]_{\varkappa}(F\Sigma).$$

Furthermore, the functor $F\Sigma$ satisfies (Th1) by definition, (sFD) and (sFT) since for all $A \in S^{\circ}$ one has

$$d_A^{(\mathbf{T}\Sigma)}(F\Sigma)=\Big[d_{AW}^{(\mathbf{T})}\Big]_{\varkappa}(F\Sigma)=\Big[d_{AW}^{(\mathbf{T})}F\Big]_{\varkappa'}=\Big[d_{(AW)F}^{(\mathbf{T}')}\Big]_{\varkappa'}=\Big[d_{AW}^{(\mathbf{T}')}\Big]_{\varkappa'}=d_{A(F\Sigma)}^{(\mathbf{T}'\Sigma)}$$

and

$$t_A^{(\mathbf{T}\Sigma)}(F\Sigma)=\Big[t_{AW}^{(\mathbf{T})}\Big]_{\varkappa}(F\Sigma)=\Big[t_{AW}^{(\mathbf{T})}F\Big]_{\varkappa'}=\Big[t_{(AW)F}^{(\mathbf{T}')}\Big]_{\varkappa'}=\Big[t_{AW}^{(\mathbf{T}')}\Big]_{\varkappa'}=t_{A(F\Sigma)}^{(\mathbf{T}'\Sigma)},$$

and (sFM) since for all $\varphi \in \mathbf{T}[X,U]$, $\psi \in \mathbf{T}[Y,V]$ the equation

$$([\varphi]_{\varkappa} \otimes [\psi]_{\varkappa})(F\Sigma) = [(\varphi \otimes \psi)F]_{\varkappa'} = [\varphi F \otimes \psi F]_{\varkappa'}$$

$$= [\varphi F]_{\varkappa'} \otimes [\psi F]_{\varkappa'} = [\varphi]_{\varkappa}(F\Sigma) \otimes [\psi]_{\varkappa'}(F\Sigma)$$

is valid.

b) The functor property of Π:
The mapping on objects $\underline{\mathbf{T_s}}$ is well defined by Theorem 4.7.
Let $(F : \underline{\mathbf{T_s}} \to \underline{\mathbf{T'_s}})$ be a theory morphism. Then $(F\Pi : \underline{\mathbf{T}} \to \underline{\mathbf{T'}})$ defined by

$$X \mapsto X,\ (X,\varphi,Y) \mapsto (X,\varphi F,Y)$$

is a theory morphism too, since the conditions (Th1), (sFD), (sFT), and (sFM) are satisfied.

Ad (Th1): $\forall X \in H^\circ\ (X(F\Pi) = X)$ by definition.

Ad (sFD):

$$\forall X \in H^\circ \Big(d_X^{(\mathbf{T})}(F\Pi) = \Big(X, d_{XW^*}^{(\mathbf{T_s})}, X \otimes X\Big)(F\Pi) = \Big(X, d_{XW^*}^{(\mathbf{T_s})}F, X \otimes X\Big)$$

$$= \Big(X, d_{XW^*F}^{(\mathbf{T'_s})}, X \otimes X\Big) = \Big(X, d_{XW^*}^{(\mathbf{T'_s})}, X \otimes X\Big) = d_X^{(\mathbf{T}')} = d_{X(F\Pi)}^{(\mathbf{T}')}\Big).$$

Ad (sFT): $\forall X \in H^\circ \Big(t_X^{(\mathbf{T})}(F\Pi) = \Big(X, t_{XW^*}^{(\mathbf{T_s})}, I\Big)(F\Pi) = \Big(X, t_{XW^*}^{(\mathbf{T_s})}F, I\Big)$

$$= \Big(X, t_{XW^*F}^{(\mathbf{T'_s})}, I\Big) = \Big(X, t_{XW^*}^{(\mathbf{T'_s})}, I\Big) = t_X^{(\mathbf{T}')} = t_{X(F\Pi)}^{(\mathbf{T}')}\Big).$$

Ad (sFM): $\forall \rho \in \mathbf{T}[X,U],\ \sigma \in \mathbf{T}[Y,V]\ \Big((\rho \otimes \sigma)(F\Pi)$

$$= ((X,\varphi,U) \otimes (Y,\psi,V))(F\Pi)$$

$$= (X \otimes Y, \varphi \otimes \psi, U \otimes V)(F\Pi)$$

$$= (X \otimes Y, (\varphi \otimes \psi)F, U \otimes V)$$

$$= (X \otimes Y, \varphi F \otimes \psi F, U \otimes V)$$

$$= (X, \varphi F, U) \otimes (Y, \psi F, V)$$

$$= (X,\varphi,U)(F\Pi) \otimes (Y,\psi,V)(F\Pi)$$

$$= \rho(F\Pi) \otimes \sigma(F\Pi)\Big).$$

c) It remains to show that Σ is a left-adjoint of Π. We will prove in several steps that for every $\underline{\mathbf{T}} \in |Th^{\circ}_{dht}(J)|$ and every $\underline{\mathbf{T}_{\mathbf{s}}} \in |sTh^{\circ}_{dht}(J)|$ there is an isomorphism between the sets $sTh^{\circ}_{dht}(J)[\underline{\mathbf{T}}\Sigma, \underline{\mathbf{T}_{\mathbf{s}}}]$ and $Th^{\circ}_{dht}(J)[\underline{\mathbf{T}}, \underline{\mathbf{T}_{\mathbf{s}}}\Pi]$.

1. A functor from a theory $\underline{\mathbf{T}}$ into $\underline{\mathbf{T}}(\Sigma\Pi)$:

Define a mapping $\Theta_{\mathbf{T}}$ on objects and morphisms of any Hoehnke theory by $X\Theta_{\mathbf{T}} := X$ and $\varphi\Theta_{\mathbf{T}} := (X, [\varphi]_{\varkappa}, Y)$ for $\varphi \in \mathbf{T}[X,Y]$. This mappings are well defined and the values are objects and morphisms of $\underline{\mathbf{T}}(\Sigma\Pi)$.

$\Theta_{\mathbf{T}} : \underline{\mathbf{T}} \to \underline{\mathbf{T}}(\Sigma\Pi)$ is a functor, since the object mapping is compatible with "dom" and "cod" and

$$1_X^{(\mathbf{T})}\Theta_{\mathbf{T}} = \Big(X, \Big[1_X^{(\mathbf{T})}\Big]_{\varkappa}, X\Big) = \Big(X, 1_{XW^*}^{(\mathbf{T}(\Sigma))}, X\Big) = 1_X^{((\mathbf{T}\Sigma)\Pi)} = 1_{X\Theta_{\mathbf{T}}}^{(\mathbf{T}(\Sigma\Pi))},$$

$$(\varphi\psi)\Theta_{\mathbf{T}} = (X, [\varphi\psi]_{\varkappa}, U) = (X, [\varphi]_{\varkappa}[\psi]_{\varkappa}, U)$$

$$= (X, [\varphi]_{\varkappa}, Y)(Y, [\psi]_{\varkappa}, U) = (\varphi\Theta_{\mathbf{T}})(\psi\Theta_{\mathbf{T}}).$$

Moreover, $\Theta_{\mathbf{T}} : \underline{\mathbf{T}} \to \underline{\mathbf{T}}(\Sigma\Pi)$ is even a theory morphism because of the validity of (Th1), (sFD), (sFT), and (sFM) as follows:

$$\forall X \in |\mathbf{T}|\ (X\Theta_{\mathbf{T}} = X) \text{ by definition.}$$

$$\forall X \in |\mathbf{T}|\ \Big(d_X^{(\mathbf{T})}\Theta_{\mathbf{T}} = \Big(X, \Big[d_X^{(\mathbf{T})}\Big]_{\varkappa}, X \otimes X\Big) = \Big(X, d_{XW^*}^{(\mathbf{T}\Sigma)}, X \otimes X\Big)$$
$$= d_X^{((\mathbf{T})\Sigma)\Pi)} = d_{X\Theta_{\mathbf{T}}}^{(\mathbf{T}(\Sigma\Pi))}\Big).$$

$$\forall X \in |\mathbf{T}|\ \Big(t_X^{(\mathbf{T})}\Theta_{\mathbf{T}} = \Big(X, \Big[t_X^{(\mathbf{T})}\Big]_{\varkappa}, I\Big) = \Big(X, t_{XW^*}^{(\mathbf{T}\Sigma)}, I\Big) = t_X^{((\mathbf{T})\Sigma)\Pi)}$$
$$= t_{X\Theta_{\mathbf{T}}}^{(\mathbf{T}(\Sigma\Pi))}\Big).$$

$$\forall \varphi \in \mathbf{T}[X,U],\ \psi \in \mathbf{T}[Y,V]\ ((\varphi \otimes \psi)\Theta_{\mathbf{T}} = (X \otimes Y, [\varphi \otimes \psi]_{\varkappa}, U \otimes V)$$
$$= (X, [\varphi]_{\varkappa}, U) \otimes (Y, [\psi]_{\varkappa}, V) = \varphi\Theta_{\mathbf{T}} \otimes \psi\Theta_{\mathbf{T}}).$$

In such a way, every theory morphism $G' \in |sTh^{\circ}_{dht}(J)|$ determines uniquely a theory morphism $G := \Theta_T(G'\Pi) \in Th^{\circ}_{dht}(J)[\underline{\mathbf{T}}, \underline{\mathbf{T}_{\mathbf{s}}}\Pi]$.

2. A construction of a strictly d-monoidal functor $\overline{G} : \underline{\mathbf{T}} \to \underline{\mathbf{T}_{\mathbf{s}}}$:

To every theory morphism $G \in Th^{\circ}_{dht}(J)[\underline{\mathbf{T}}, \underline{\mathbf{T}_{\mathbf{s}}}\Pi]$ there is assigned in a natural manner a strictly d-monoidal functor $\overline{G} : \underline{\mathbf{T}} \to \underline{\mathbf{T}_{\mathbf{s}}}$ as follows:

Let be given any $G \in Th^{\circ}_{dht}(J)[\underline{\mathbf{T}}, \underline{\mathbf{T}_{\mathbf{s}}}\Pi]$. Then

$$XG = X \quad (X \in |\mathbf{T}|) \text{ and}$$

$$\mathbf{T}[X,U] \ni \varphi \mapsto \varphi G = (X, \varphi_G, U) \in \mathbf{T_s}\Pi[X,U],$$

where $\varphi_G \in \mathbf{T_s}[XW^*, UW^*]$.

The agreements

$$H^{\circ} \ni X \mapsto X\Xi := XW^* \in S^{\circ}$$

and

$$\mathbf{T_s}\Pi[X,U] \ni (X,\psi,U) \mapsto (X,\psi,U)\Xi := \psi \in \mathbf{T_s}[XW^*,UW^*]$$

define a functor $\Xi : \underline{\mathbf{T_s}}\Pi \to \underline{\mathbf{T_s}}$ because of:

$$\mathrm{dom}^{(\mathbf{T_s})}((X,\psi,U)\Xi) = \mathrm{dom}^{(\mathbf{T_s})}(\psi) = XW^* = X\Xi = \left(\mathrm{dom}^{(\mathbf{T_s}\Pi)}(X,\psi,U)\right)\Xi,$$

$$\mathrm{cod}^{(\mathbf{T_s})}((X,\psi,U)\Xi) = \mathrm{cod}^{(\mathbf{T_s})}(\psi) = UW^* = U\Xi = \left(\mathrm{cod}^{(\mathbf{T_s}\Pi)}(X,\psi,U)\right)\Xi,$$

$$1_X^{(\mathbf{T_s}\Pi)}\Xi = \left(X, 1_{XW^*}^{(\mathbf{T_s})}, X\right)\Xi = 1_{XW^*}^{(\mathbf{T_s})} = 1_{X\Xi}^{(\mathbf{T_s})},$$

$$((X,\psi_1,U)(U,\psi_2,Y))\Xi = (X,\psi_1\psi_2,Y)\Xi = \psi_1\psi_2 = (X,\psi_1,U)\Xi(U,\psi_2,Y)\Xi.$$

$\Xi : \underline{\mathbf{T_s}}\Pi \to \underline{\mathbf{T_s}}$ is a strictly d-monoidal functor since (sFD), (sFT), and (sFM) are valid:

$$d_X^{(\mathbf{T_s}\Pi)}\Xi = \left(X, d_{XW^*}^{(\mathbf{T_s})}, X\otimes X\right)\Xi = d_{XW^*}^{(\mathbf{T_s})} = d_{X\Xi}^{(\mathbf{T_s})},$$

$$t_X^{(\mathbf{T_s}\Pi)}\Xi = \left(X, t_{XW^*}^{(\mathbf{T_s})}, I\right)\Xi = t_{XW^*}^{(\mathbf{T_s})} = t_{X\Xi}^{(\mathbf{T_s})},$$

$$((X_1,\psi_1,U_1)\otimes(X_2,\psi_2,U_2))\Xi = (X_1\otimes X_2, \psi_1\otimes\psi_2, U_1\otimes U_2)\Xi$$

$$= \psi_1\otimes\psi_2 = (X_1,\psi_1,U_1)\Xi\otimes(X_2,\psi_2,U_2)\Xi.$$

The compositum $\overline{G} := G\Xi$ is strictly d-monoidal functor from $\underline{\mathbf{T}}$ into $\underline{\mathbf{T_s}}$.

3. The induced theory morphism $G' \in sTh^{\circ}_{dht}(J)$:

Let G, Ξ, and $\overline{G}$ be given as above. Then define a mapping G' by $AG' := A$ for all $A \in S^{\circ}$ and $[\varphi]_{\varkappa}G' := \varphi\overline{G} = (\varphi G)\Xi = (X,\varphi_G,U)\Xi = \varphi_G \in \mathbf{T_s}[XW^*,UW^*]$ for all $\varphi \in \mathbf{T}[X,U]$, where φ_G is a well-defined morphism of $\mathbf{T_s}$.

Because of

$$\varphi_1 \in \mathbf{T}[X_1,U_1] \ \wedge\ \varphi_2 \in \mathbf{T}[X_2,U_2] \ \wedge\ [\varphi_1]_{\varkappa} = [\varphi_2]_{\varkappa} \Rightarrow$$

$$\Rightarrow X_1W^* = X_2W^* := A \ \wedge\ U_1W^* = U_2W^* := B$$

$$\wedge\ c_{X_1}^{-1}\varphi_1 c_{U_1} = c_{X_2}^{-1}\varphi_2 c_{U_2} \in \mathbf{T_s}[AW,BW] \Rightarrow$$

$$\Rightarrow (c_{X_1}^{-1}G)(\varphi_1 G)(c_{U_1}G) = (c_{X_2}^{-1}G)(\varphi_2 G)(c_{U_2}G) \in \mathbf{T_s}\Pi[A,B]$$

$$\Rightarrow \left(AW, 1_A^{(\mathbf{T_s})}, X_1\right)(X_1, (\varphi_1)_G, U_1)\left(U_1, 1_B^{(\mathbf{T_s})}, BW\right)$$

$$= \left(AW, 1_A^{(\mathbf{T_s})}, X_2\right)(X_2, (\varphi_2)_G, U_2)\left(U_2, 1_B^{(\mathbf{T_s})}, BW\right)$$

$$\Rightarrow (X_1, (\varphi_1)_G, U_1) = (X_2, (\varphi_2)_G, U_2)$$

$$\Rightarrow (\varphi_1)_G = (\varphi_2)_G,$$

possibly different representants of the same $\varkappa$-class of morphisms determine identical images, thus $[\varphi_1]\varkappa G' = [\varphi_2]\varkappa G'$.

The mapping G' determines a functor $G' : \underline{\mathbf{T}}\Sigma \to \underline{\mathbf{T_s}}$ since

$$\mathrm{dom}^{(\mathbf{T_s})}([\varphi]_\varkappa G') = \mathrm{dom}^{(\mathbf{T_s})}(\varphi_G) = XW^* = (XW^*)G' = \left(\mathrm{dom}^{(\mathbf{T}\Sigma)}([\varphi]_\varkappa)\right)G',$$

$$\mathrm{cod}^{(\mathbf{T_s})}([\varphi]_\varkappa G') = \mathrm{cod}^{(\mathbf{T_s})}(\varphi_G) = UW^* = (UW^*)G' = \left(\mathrm{cod}^{(\mathbf{T}\Sigma)}([\varphi]_\varkappa)\right)G',$$

$$\left(1_A^{(\mathbf{T}\Sigma)}\right)G' = \left(\left[1_{AW}^{(\mathbf{T})}\right]_\varkappa\right)G' = \left(1_{AW}^{(\mathbf{T})}\right)\overline{G} = \left(1_{AW}^{(\mathbf{T})}\right)(G\Xi) = \left(\left(1_{AW}^{(\mathbf{T})}\right)G\right)\Xi$$

$$= \left(1_{(AW)G}^{(\mathbf{T_s}\Pi)}\right)\Xi = 1_{((AW)G)\Xi}^{(\mathbf{T_s})} = 1_A^{(\mathbf{T_s})} = 1_{AG'}^{(\mathbf{T_s})},$$

$$([\varphi]_\varkappa[\psi]_\varkappa)G' = ([\varphi c_{U,Y}\psi]_\varkappa)G' = (\varphi c_{U,Y}\psi)\overline{G} = (\varphi\overline{G})(c_{U,Y}\overline{G})(\psi\overline{G})$$

$$= \varphi_G\psi_G = ([\varphi]_\varkappa G')([\psi]_\varkappa G').$$

Moreover, G' is even a theory morphism in $sTh_{dht}^{\circ}(J)$ because of the validity of (Th1) by definition and the validity of (sFD), (sFT), and (sFM) as follows:

$$\left(d_A^{(\mathbf{T}\Sigma)}\right)G' = \left[d_A^{(\mathbf{T})}\right]_\varkappa G' = d_A^{(\mathbf{T})}\overline{G} = d_{AW}^{(\mathbf{T_s}\Pi)}\Xi = d_A^{(\mathbf{T_s})} = d_{AG'}^{(\mathbf{T_s})},$$

$$\left(t_A^{(\mathbf{T}\Sigma)}\right)G' = \left[t_A^{(\mathbf{T})}\right]_\varkappa G' = t_A^{(\mathbf{T})}\overline{G} = t_{AW}^{(\mathbf{T_s}\Pi)}\Xi = t_A^{(\mathbf{T_s})} = t_{AG'}^{(\mathbf{T_s})},$$

$$([\varphi]_{\varkappa} \otimes [\psi]_{\varkappa})G' = ([\varphi \otimes \psi]_{\varkappa})G' = (\varphi \otimes \psi)\overline{G} = (\varphi\overline{G}) \otimes (\psi\overline{G}) =$$

$$= [\varphi]_{\varkappa}G' \otimes [\psi]_{\varkappa}G'.$$

By the functor $\Pi : sTh^{\circ} - dht(J) \to Th^{\circ}_{dht}(J)$, $G'\Pi : \underline{\mathbf{T}}(\Sigma\Pi) \to \underline{\mathbf{T_s}}\Pi$ is a theory morphism.

Moreover, this theory morphism has the property

$$G = \Theta_{\mathbf{T}}(G'\Pi).$$

This is a consequence of

$$H^{\circ} \ni X \mapsto X(\Theta_{\mathbf{T}}(G'\Pi)) = (X\Theta_{\mathbf{T}})(G'\Pi) = X(G'\Pi) = X = XG$$

and

$$\mathbf{T}[X, U] \ni \varphi \mapsto \varphi(\Theta_{\mathbf{T}}(G'\Pi)) = (\varphi\Theta_{\mathbf{T}})(G'\Pi) = (X, [\varphi]_{\varkappa}, U)(G'\Pi) =$$

$$= (X, [\varphi]_{\varkappa}G', U) = (X, \varphi_G, U) = \varphi G.$$

Finally, let $L : \underline{\mathbf{T}}\Sigma \to \underline{T_s}$ be a theory morphism such that $\Theta_{\mathbf{T}})(L\Pi) = G$. Then

$$\forall X \in H^{\circ} \ ((XW^*)G' = XW^* = (XW^*)G)$$

and

$$\forall X, U \in H^{\circ} \ \forall \varphi \in \mathbf{T}[X, U] \ ((X, [\varphi]_{\varkappa}G', U) = (X, \varphi\overline{G}, U) = \varphi G =$$

$$= \varphi(\Theta_{\mathbf{T}})(L\Pi)) = (\varphi\Theta_{\mathbf{T}})(L\Pi) = (X, [\varphi]_{\varkappa}, U)(L\Pi) = (X, [\varphi]_{\varkappa}L, U)$$

$$\Rightarrow [\varphi]_{\varkappa}G' = [\varphi]_{\varkappa}L),$$

thus $L = G'$, i.e. G' is the only theory morphism in $sTh^{\circ}_{dht}(J)$ with the property

$$G = \Theta_{\mathbf{T}}(G'\Pi). \qquad \blacksquare$$

$$\begin{tikzcd}
\underline{\mathbf{T}}(\Sigma\Pi) \arrow[dr, "G'\Pi"] & \\
\underline{\mathbf{T}} \arrow[u, "\Theta_{\mathbf{T}}"] \arrow[r, "G"] \arrow[d, "\Psi"'] \arrow[dr, "\overline{G}"] & \underline{\mathbf{T_s}}\Pi \arrow[d, "\Xi"] \\
\underline{\mathbf{T}}\Sigma \arrow[r, "G'"'] & \underline{\mathbf{T_s}}
\end{tikzcd}$$

The diagram illustrates the individual d-monoidal functors and theory morphisms, respectively, which are considered in the proof of the last theorem. This diagram is commutative in all of its parts, namely $G = \Theta_{\mathbf{T}}(G'\Pi)$ was shown above, $\overline{G} = G\Xi$ by definition, and $\overline{G} = \Psi G'$ follows by

$$X(\Psi G') = (X\Psi)G' = (XW^*)G' = XW^* = X\overline{G}$$

and

$$\varphi(\Psi G') = (\varphi\Psi)G' = [\varphi]_{\varkappa}G' = \varphi_G = \varphi\overline{G}.$$

Corollary 4.13. *The theory morphisms* Θ_T, $\underline{\mathbf{T}} \in |Th^{\circ}_{dht}(J)|$ *form a natural transformation* $\Theta : Id_{Th^{\circ}_{dht}(J)} \to \Sigma\Pi$.

Proof. $\Theta = (\Theta_T \mid \underline{T} \in |Th^{\circ}_{dht}(J)|)$ is a natural transformation $\Theta : Id_{Th^{\circ}_{dht}(J)} \to \Sigma\Pi$ because of the commutativity of the following diagram for arbitrary theories and theory morphisms of $Th^{\circ}_{dht}(J)$:

$$(\mathbf{N}) \qquad \begin{tikzcd} \mathbf{T} \arrow[r, "\Theta_{\mathbf{T}}"] \arrow[d, "F"'] & \mathbf{T}(\Sigma\Pi) \arrow[d, "F(\Sigma\Pi)"] \\ \mathbf{T}' \arrow[r, "\Theta_{\mathbf{T}'}"'] & \mathbf{T}'(\Sigma\Pi) \end{tikzcd}$$

Let X be any object of $\underline{\mathbf{T}}$. Then

$$X(F\Theta_{\mathbf{T}'}) = (XF)\Theta_{\mathbf{T}'} = X\Theta_{\mathbf{T}'} = X$$

and

$$X(\Theta_{\mathbf{T}}F(\Sigma\Pi)) = (X\Theta_{\mathbf{T}})((F\Sigma)\Pi) = X.$$

For every morphism $\varphi \in \mathbf{T}[X, U]$ one has

$$\varphi(F\Theta_{\mathbf{T}'}) = (\varphi F)\Theta_{\mathbf{T}'} = (X, [\varphi F]_{\varkappa'}, U)$$

and

$$\varphi(\Theta_{\mathbf{T}}F(\Sigma\Pi) = (\varphi\Theta_{\mathbf{T}})((F\Sigma)\Pi) =$$
$$= (X, [\varphi]_{\varkappa}, U)((F\Sigma)\Pi)) =$$
$$= (X, [\varphi]_{\varkappa}(F\Sigma), U) = (X, [\varphi F]_{\varkappa'}, U),$$

hence

$$\Theta_{\mathbf{T}}F(\Sigma\Pi) = F\Theta_{\mathbf{T}'}.$$

■

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Received 25 July 2003
Revised 21 October, 2003