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LOCALLY FINITE M-SOLID VARIETIES OF SEMIGROUPS

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Abstract

An algebra of type τ is said to be *locally finite* if all its finitely generated subalgebras are finite. A class K of algebras of type τ is called locally finite if all its elements are locally finite. It is well-known (see [2]) that a variety of algebras of the same type τ is locally finite iff all its finitely generated free algebras are finite. A variety V is *finitely based* if it admits a finite basis of identities, i.e. if there is a finite set Σ of identities such that $V = Mod\Sigma$, the class of all algebras of type τ which satisfy all identities from Σ. Every variety which is generated by a finite algebra is locally finite. But there are finite algebras which are not finitely based. For semigroup varieties, Perkins proved that the variety generated by the five-element *Brandt-semigroup*

$$B_2^1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is not finitely based ([9], [10]). An identity $s \approx t$ is called a *hyperidentity* of a variety V if whenever the operation symbols occurring in s and in t, respectively, are replaced by any terms of V of the appropriate arity, the identity which results, holds in V. A variety V is called *solid* if every identity of V also holds as a hyperidentity in V. If we apply only substitutions from a set M we speak of *M-hyperidentities* and *M-solid varieties*. In this paper we use the theory of M-solid varieties to prove that a type (2) M-solid variety of the form $V = H_M Mod\{F(x_1, F(x_2, x_3)) \approx F(F(x_1, x_2), x_3)\}$, which consists precisely of all algebras which satisfy the associative law as an

M-*hyperidentity* is locally finite iff the hypersubstitution which maps F to the word $x_1x_2x_1$ or to the word $x_2x_1x_2$ belongs to M and that V is finitely based if it is locally finite.

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1. Preliminaries

First of all we want to give some basic knowledge about hyperidentities and M-solid varieties. Let $\tau = (n_i)_{i\in I}$ be a type of algebras with operation symbols f_i of arity n_i, indexed by some set I. Let $X = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite alphabet of variables and let $X_n = \{x_1, x_2, \ldots, x_n\}$ be an n-element alphabet. We denote by $W_\tau(X_n)$ the set of all n-ary terms of type τ. Let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ be the set of all terms of type τ. Any map $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ which takes each n_i-ary operation symbol to an n_i-ary term is called a *hypersubstitution* of type τ. Each hypersubstitution σ induces a map $\hat{\sigma}$ on the set of all terms which is defined by

(i) $\hat{\sigma}[x] := x$ if $x \in X$ is a variable,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$
for composite terms $f_i(t_1, \ldots, t_{n_i})$.

On the set $Hyp(\tau)$ of all hypersubstitutions of type τ, we define a binary operation $\circ_h : Hyp(\tau) \times Hyp(\tau) \to Hyp(\tau)$ by $\sigma_1 \circ_h \sigma_2 = \hat{\sigma}_1 \circ \sigma_2$, where $\circ$ is the usual composition of functions. Then together with the identity element σ_{id}, mapping each f_i to the term $f_i(x_1, \ldots, x_{n_i})$, we obtain a monoid $(Hyp(\tau); \circ_h, \sigma_{id})$.

For a variety V the equation $s \approx t$ is called a *hyperidentity* in V if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities in V for all hypersubstitutions σ. If $M \subseteq Hyp(\tau)$ is a submonoid of $Hyp(\tau)$ and if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities for all $\sigma \in M$, the identity $s \approx t$ is called an M-*hyperidentity.* The variety V is called *solid* if every identity in V is satisfied as a hyperidentity and M-*solid* if every identity in V is satisfied as an M-hyperidentity. The collection of all M-solid varieties of a given type τ form a complete sublattice $\mathcal{S}_M(\tau)$ of the lattice $\mathcal{L}(\tau)$ of all varieties of type τ with $\mathcal{S}_{M_1}(\tau) \subseteq \mathcal{S}_{M_2}(\tau)$ if $M_2 \subseteq M_1$. To study such complete sublattices can help to get more insight into the lattice $\mathcal{L}(\tau)$.

This is particularly interesting if one studies subvariety lattices of given varieties, for instance, the lattice $\mathcal{L}(SEM)$ of all subvarieties of the variety SEM of all semigroups. Then $\mathcal{S}_M(SEM)$ is the complete sublattice of all M-solid varieties of $\mathcal{L}(SEM)$. For every M-solid variety V of type τ there exists a set Σ of equations such that V is precisely the class of all algebras of type τ which satisfy all equations from Σ as M-hyperidentities. This class is called the *M-hypermodel class determined by* Σ, and is denoted by $V = H_M Mod\Sigma$. If we define an operator $\chi_M : \mathcal{P}(W_\tau(X)^2) \to \mathcal{P}(W_\tau(X)^2)$, where $\mathcal{P}$ denotes the formation of the power set, by

$$\chi_M[\Sigma] := \{\hat{\sigma}[s] \approx \hat{\sigma}[t] \mid s \approx t \in \Sigma, \sigma \in M\},$$

then V is the class of all algebras of type τ satisfying every equation from $\chi_M[\Sigma]$ as identity, i.e. $H_M Mod\Sigma = Mod\chi_M[\Sigma]$.

It is interesting to remark that there are varieties V having a finite hyperidentity basis Σ but no finite basis for their identities. As an example, we consider the type $\tau = (2,1)$ with a binary operation symbol to be indicated by juxtaposition and a unary operation symbol f. Let Σ be the following finite set of equations:

$$\Sigma = \{(x_1x_2)x_3 \approx x_1(x_2x_3),\ x_1x_2x_3x_4 \approx x_1x_3x_2x_4,\ x_2x_1^2x_2$$

$$\approx x_1x_2^2x_1,\ x_2f(x_1)x_1^2x_2 \approx x_1x_2f(x_1)x_2x_1\}.$$

We consider the monoid M of hypersubstitutions of the form $\sigma^{(k)}$, for any natural number k, where $\sigma^{(k)}$ maps the binary operation symbol to itself and the unary operation symbol f to x_1^k, for $k \geq 1$, together with the identity hypersubstitution σ_{id}. Then the set

$$\chi_M[\Sigma] = \{(x_1x_2)x_3 \approx x_1(x_2x_3),\ x_1x_2x_3x_4 \approx x_1x_3x_2x_4,\ x_2x_1^2x_2 \approx x_1x_2^2x_1\} \cup$$

$$\cup \{x_2x_1^kx_1^2x_2 \approx x_1x_2x_1^kx_2x_1 \mid k \geq 1\}$$

does not have a finite basis, as shown by Perkins in [10]. Another similar example was given by Paseman in [8] using the type $\tau = (1,1,1)$.

The variety $V_{HS} = HMod\{F(x_1, F(x_2, x_3)) \approx F(F(x_1, x_2), x_3)\}$ consisting of all semigroups satisfying the associative law as hyperidentity is the greatest solid variety of semigroups and the variety $H_M Mod\{F(x_1, F(x_2, x_3))$ $\approx F(F(x_1, x_2), x_3)\}$ is the greatest M-solid variety of semigroups for a submonoid M of $Hyp(2)$. In [12] it was proved that V_{HS} is finitely based by identities and that $V_{HS} = \{x_1(x_2x_3) \approx (x_1x_2)x_3,\ x_1x_2x_1x_3x_1x_2x_1 \approx x_1x_2x_3x_2x_1,\ (x_1^2x_2)^2x_3 \approx x_1^2x_2^2x_3,\ x_1x_2^2x_2^3 \approx x_1(x_2x_3^2)^2, x_1^2 \approx x_2^4\}$. In a similar way for the monoid $Pre = Hyp(2)\setminus\{\sigma_{x_1}, \sigma_{x_2}\}$, where σ_{x_1} maps F to x_1 and σ_{x_2} maps F to x_2, of all Pre-hypersubstitutions, it was proved that $H_{Pre}Mod\{F(x_1, F(x_2, x_3)) \approx F(F(x_1, x_2), x_3)\}$ has the set $\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ x_1x_2x_1x_3x_1x_2x_1 \approx x_1x_2x_3x_2x_1,\ x_1^2 \approx x_2^2,\ x_1^3 \approx x_2^3\}$ as finite basis for all its identities.

A hypersubstitution of type $\tau = (2)$ is called *regular*, if F is mapped to a binary term containing both variables x_1 and x_2. The set Reg of all regular hypersubstitutions of type $\tau = (2)$ forms also a submonoid of the monoid Hyp of all hypersubstitutions and in [4] it was proved that $H_{Reg}Mod\{F(x_1, F(x_2, x_3)) \approx F(F(x_1, x_2), x_3)\}$ is finitely based by identities. Therefore, in a very natural way the following problems arise:

For which monoids M of hypersubstitutions of type $\tau = (2)$ is $H_MMod\{F(x_1, F(x_2, x_3)) \approx F(F(x_1, x_2), x_3)\}$ finitely based by identities ?

We will show that this problem is closely connected with the following one:

For which monoids M of hypersubstitutions of type $\tau = (2)$ is $H_MMod\{F(x_1, F(x_2, x_3)) \approx F(F(x_1, x_2), x_3)\}$ locally finite ?

2. Locally finite M-solid varieties

Let V be a variety of type τ and let IdV be the set of all identities valid in V. Then a hypersubstitution $\sigma \in Hyp(\tau)$ is called V-*proper* if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$ for every $s \approx t \in IdV$. Let $P(V)$ be the set of all V-proper hypersubstitutions. Then $P(V)$ is a submonoid of $Hyp(\tau)$. Every variety is $P(V)$-solid and $P(V)$ is the greatest submonoid M of $Hyp(\tau)$ such that V is M-solid. A variety is solid if and only if $P(V) = Hyp(\tau)$. For checking whether an identity is satisfied as a hyperidentity in a variety V, not all hypersubstitutions are important. In [11] the following binary relation on $Hyp(\tau)$ was introduced:

Definition 2.1. Let V be a variety of type τ. Two hypersubstitutions σ_1 and σ_2 of type τ are called V-*equivalent* if and only if $\sigma_1(f_i) \approx \sigma_2(f_i) \in IdV$ for all $i \in I$. In this case, we write $\sigma_1 \sim_V \sigma_2$.

Clearly, $\sim_V$ is an equivalence relation on $Hyp(\tau)$ and on its submonoids. If V is M-solid then the restriction of $\sim_V$ to $M \times M$ is a congruence relation on M. The following property is easy to check:

If $s \approx t \in IdV$, $\sigma_1 \sim_V \sigma_2$ and $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$, then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$.

In other words, the monoid $P(V)$ is a union of equivalence classes with respect to the relation $\sim_V$.

As a consequence, to check whether $s \approx t$ is satisfied as an M-hyperidentity in a variety V we can restrict our checking to one representative from each equivalence class of the quotient set $M/_{\sim_V}$. Such sets of representatives can be selected by a choice function $\Phi : M/_{\sim_V} \to M$. From the class of the identity hypersubstitution $[\sigma_{id}]_{\sim V}$ one has to choose σ_{id}. We will use the notation $M_\Phi^N(V)$ and call the elements of this set *normal form hypersubstitutions*. In [1], it was proved that $Mod\chi_M[\Sigma] = Mod\chi_{M_\Phi^N(V)}[\Sigma]$ for every variety V, for every set Σ of equations, and every choice function Φ.

Now we come back to varieties of semigroups and prove:

Theorem 2.2. *If V is an M-solid locally finite variety of semigroups which is definable by a finite set Σ of M-hyperidentities, then V is finitely based by identities.*

Proof. As an M-solid variety, V is the class of all semigroups which satisfy all equations from Σ as M-hyperidentities, i.e. $V = H_M Mod\Sigma$. Using the equations $H_M Mod\Sigma = Mod\chi_M[\Sigma] = Mod\chi_{M_\Phi^N(V)}[\Sigma]$, we see that $\chi_{M_\Phi^N(V)}[\Sigma]$ is a basis for the set of all identities satisfied in V. Clearly, the cardinality of M is equal to the cardinality of all images of f under hypersubstitutions from M, i.e. $|M| = |\{\sigma(f) \mid \sigma \in M\}|$. From this and from $\sigma_1 \sim_V \sigma_2$ iff $\sigma_1(f) \approx \sigma_2(f) \in IdV$, we obtain $|M_\Phi^N(V)| = |M/_{\sim_V}| = |\{[\sigma(f)]_{IdV} \mid \sigma \in M\}| \leq |F_V(\{x,y\})|$. Since V is locally finite, $F_V(\{x,y\})$ is finite and then $M_\Phi^N(V)$ is also finite. Therefore, $\chi_{M_\Phi^N}[\Sigma]$ is finite and V is finitely based by identities. ∎

3. Zimin-Words

M. Sapir discovered ([13]) that there exists a property of semigroup varieties such that every finitely generated variety has this property and any finitely based variety satisfying this property must satisfy an identity of a specific form. This identity has the form $Z_n \approx w$ where the left hand side is called a Zimin word and is defined inductively by:

$$Z_1 := x_1,$$

$$Z_{n+1} := Z_n x_{n+1} Z_n.$$

Then the sequence of Zimin words can be produced by the hypersubstitution $\sigma_{x_1x_2x_1}$ which maps a binary operation symbol f to the word $x_1x_2x_1$(or by $\sigma_{x_2x_1x_2}$ mapping f to the word $x_2x_1x_2$).

Definition 3.1. Let f be a binary operation symbol. Then we define

$$f^1(x_1) := x_1,$$

$$f^n(x_1, x_2, \ldots, x_n) := f(f^{n-1}(x_1, \ldots, x_{n-1}), x_n).$$

Proposition 3.2. *For every $n \geq 1$, we have*

$$\hat{\sigma}_{x_1x_2x_1}[f^n(x_1, x_2, \ldots, x_n)] = Z_n.$$

Proof. For $n = 1$, we have $\hat{\sigma}_{x_1x_2x_1}[x_1] = x_1 = Z_1$.

Assume that $\hat{\sigma}_{x_1x_2x_1}[f^{n-1}(x_1, x_2, \ldots, x_{n-1})] = Z_{n-1}$. Then $\hat{\sigma}_{x_1x_2x_1}[f^n(x_1, x_2, \ldots, x_n)] = \hat{\sigma}_{x_1,x_2,x_1}[f(f^{n-1}(x_1, \ldots, x_{n-1}), x_n)] = \sigma_{x_1x_2x_1}(f)(\hat{\sigma}_{x_1x_2x_1}[f^{n-1}(x_1, \ldots, x_{n-2})], x_n]) = Z_{n-1}x_nZ_{n-1} = Z_n$. ■

Dually, if we apply the hypersubstitution $\sigma_{x_2x_1x_2}$ to the terms $f^{(n)}(x_1, x_2, \ldots, x_n)$ inductively defined by $f^{(1)}(x_1) := x_1, f^{(n)}(x_1, \ldots, x_n) := f(x_1, f^{(n-1)}(x_2, \ldots, x_n))$, then we obtain a sequence of words from which we get the Zimin words by exchange of variables.

Then we have

Proposition 3.3. *If $\sigma_{x_1x_2x_1}(\sigma_{x_2x_1x_2})$ is a proper hypersubstitution of a variety V of semigroups, then for every $n \geq 3$ there is a term u_n such that $Z_n \approx u_n$ is an identity in V.*

Proof. Clearly, for every $n \geq 3$ the equations $f^n(x_1, \ldots, x_n) \approx f^{(n)}(x_1, \ldots, x_n)$ are identities in V, since these equations are consequences of the associative law. Then using $\sigma_{x_1x_2x_1}$, we obtain all Zimin words on the left hand side, and if $\sigma_{x_2x_1x_2}$ is a proper hypersubstitution of V, we obtain the Zimin words on the right hand side after exchanging of variables. ■

We recall the following definitions:

Definition 3.4. A semigroup $\mathcal{S}$ is said to be *periodic* if for every $a \in S$ there exist two different numbers m_a and n_a such that $a^{m_a} = a^{n_a}$. A variety V of semigroups is called *periodic* if every member of V is periodic. A *zero* 0 in a semigroup $\mathcal{S}$ is an element from S with $x0 = 0x = 0$ for every $x \in S$. A semigroup $\mathcal{S}$ with zero is called a *nil-semigroup* if for any $a \in S$ there is a natural number n with $a^n = 0$.

For periodic semigroups we have

Proposition 3.5 ([6]). *A finitely based periodic semigroup variety V is locally finite iff all groups in V and all nil-semigroups in V are locally finite.* ■

In [6] (see also [7], §7) sufficient conditions for groups to be locally finite are given.

Proposition 3.6. *Every group satisfying the identity $x \approx x^3$ (or $x \approx x^2,\ x \approx x^4,\ x \approx x^5,\ x \approx x^7$) is locally finite.* ■

We denote by $F_{SEM}(X_n)$ the free semigroup generated by X_n. In [6], the following result is proved:

Theorem 3.7. *Let V be a variety of semigroups given by a (possibly infinite) set Σ of identities. Assume that the number of variables occurring in words of Σ is n. Then the following conditions are equivalent:*

(i) *All nil-semigroups from V are locally finite;*

(ii) *V satisfies a non-trivial identity with one side equal to Z_{n+1};*

(iii) *There exists an identity $s \approx t \in \Sigma$ and a substitution*

$$\Phi : X_n \longrightarrow F_{SEM}(X_n)$$

such that Z_{n+1} contains a value $\bar{\Phi}(s)$ or $\bar{\Phi}(t)$ where $\bar{\Phi}(s) \neq \bar{\Phi}(t)$. ■

Now we prove:

Theorem 3.8. *Let V be a variety of semigroups and assume that $\sigma_{x_1x_2x_1}$ (or $\sigma_{x_2x_1x_2}$) is a proper hypersubstitution of V. Then V is locally finite.*

Proof. Since $\sigma_{x_1x_2x_1}$(or $\sigma_{x_2x_1x_2}$) is a proper hypersubstitution of V, for every $n \geq 3$ the identities $\hat{\sigma}_{x_1x_2x_1}[f^n(x_1, \ldots, x_n)] \approx \sigma_{x_1x_2x_1}[f^{(n)}(x_1, \ldots, x_n)]$ are satisfied in V. For $n = 3$ this gives $x_1x_2x_1x_3x_1x_2x_1 \approx x_1x_2x_3x_2x_1$. For $\sigma_{x_2x_1x_2}$ we obtain the identity $x_3x_2x_3x_1x_3x_2x_3 \approx x_3x_2x_1x_2x_3$ and by exchange x_1 and x_3 we obtain the first identity. Therefore V is a subvariety of the variety

$$V_1 := Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ x_1x_2x_1x_3x_1x_2x_1 \approx x_1x_2x_3x_2x_1\}.$$

We prove that the last variety is locally finite. The variety V_1 is finitely based and periodic since by identification of variables from the second identity of the basis we obtain $x_1^7 \approx x_1^5$. We can apply Proposition 3.5 and show that all groups and all nil-semigroups in V_1 are locally finite. If $\mathcal{G}$ is a group in V_1, then x_1^4 has an inverse and by multiplication of $x_1^7 \approx x_1^5$ with this inverse we obtain $x_1^3 \approx x_1$. Then by Proposition 3.6, $\mathcal{G}$ is locally finite. Now we apply Theorem 3.7. In the basis of V_1 we have words containing the three variables x_1, x_2, x_3. From $x_1x_2x_1x_3x_1x_2x_1 \approx x_1x_2x_3x_2x_1$ we obtain the identity $x_1x_2x_1x_3x_1x_2x_1x_4x_1x_2x_1x_3x_1x_2x_1 \approx x_1x_2x_3x_2x_1x_4x_1x_2x_3x_2x_1$ which is also satisfied in V_1. The left hand side is the Zimin word Z_4 and therefore every nil-semigroup from V_1 is locally finite. By Proposition 3.5 the variety V_1 is locally finite and the variety V, as a subvariety of V_1, is also locally finite. ■

Now Theorem 2.2 and Theorem 3.8 give the following result:

Corollary 3.9. *Let V be an M-solid variety of semigroups and assume that $\sigma_{x_1x_2x_1}$*(or *$\sigma_{x_2x_1x_2}$) is a proper hypersubstitution of V. If V is definable by a finite set Σ of M-hyperidentities, then V is finitely based by identities.* ■

As examples we consider the monoids $M = Hyp(2)$ of all type (2) hypersubstitutions, $M = Pre := Hyp \setminus \{\sigma_x, \sigma_y\}$ (σ_x, σ_y are the hypersubstitutions mapping the operation symbol to the variable x and y, respectively) of all Pre-hypersubstitutions, the monoid $M = Reg$ of all regular hypersubstitutions which map the operation symbol to terms containing both variables and $M = H_2^{op}$, the monoid of all hypersubstitutions which map f to a term containing the operation symbol f at least twice. All these monoids contain $\sigma_{x_1x_2x_1}$ and $\sigma_{x_2x_1x_2}$.The greatest M-solid varieties with respect to these monoids are the M-hyper-model classes of the associative law, i.e. $HMod\{F(x_1, F(x_2, x_3)) \approx F(F(x_1, x_2), x_3)\}, H_{Pre}Mod\{F(x_1, F(x_2, x_3)) \approx F(F(x_1, x_2), x_3)\}$, $H_{Reg}Mod\{F(x_1, F(x_2, x_3)) \approx F(F(x_1, x_2), x_3)\}$, $H_{H_2^{op}}Mod\{F(x_1, F(x_2, x_3)) \approx F(F(x_1, x_2), x_3)\}$.

All these varieties fulfil the presumptions of Corollary 3.9 and are finitely based. Finite bases were given in [4], [5] and [3]:
$H_{Pre}Mod\{F(x_1, F(x_2, x_3)) \approx F(F(x_1, x_2), x_3)\} = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1x_2x_1x_3x_1x_2x_1 \approx x_1x_2x_3x_2x_1,\ x_1^2 \approx x_2^2,\ x_1^3 \approx x_2^3\}$,
$H_{Reg}Mod\{F(x_1, F(x_2, x_3)) \approx F(F(x_1, x_2), x_3)\} = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1x_2x_1x_3x_1x_2x_1 \approx x_1x_2x_3x_2x_1,\ (x_1^2x_2)^2x_3 \approx x_1^2x_2^2x_3,\ x_1x_2^2x_2^3 \approx x_1(x_2x_3^2)^2\}$,

$H_{H_2^{op}}Mod\{F(x_1,F(x_2,x_3))\approx F(F(x_1,x_2),x_3)\} = Mod\{x_1(x_2x_3)\approx(x_1x_2)x_3,$ $x_1x_2x_1x_3x_1x_2x_1 \approx x_1x_2x_3x_2x_1,\ (x_1^2x_2)^2x_3 \approx x_1^2x_2^2x_3,\ x_1x_2^2x_2^3 \approx x_1(x_2x_3^2)^2,$ $x_1^3 \approx x_1^5\}$.

Now we prove that varieties of the form $V = H_M Mod\{F(x_1,F(x_2,x_3)) \approx F(F(x_1,x_2),x_3)\}$ also satisfy the converse of Theorem 3.8, i.e. we have:

Theorem 3.10. *Suppose that V is a variety of semigroups for which there is a monoid M of hypersubstitutions such that $V = H_M Mod\{F(x_1,F(x_2,x_3)) \approx F(F(x_1,x_2),x_3)\}$. Then V is locally finite iff $\sigma_{x_1x_2x_1}(\sigma_{x_2x_1x_2})$ is a proper hypersubstitution of V.*

Proof. Because of Theorem 3.8 we have to show that locally finite varieties of the form $H_M Mod\{F(x_1,F(x_2,x_3)) \approx F(F(x_1,x_2),x_3)\}$ admit $\sigma_{x_1x_2x_1}$ or $\sigma_{x_2x_1x_2}$ as proper hypersubstitutions. All nil-semigroups from V are locally finite and we may assume that the condition Theorem 3.7 (iii) is satisfied. Since $H_M Mod\{F(x_1,F(x_2,x_3)) \approx F(F(x_1,x_2),x_3)\} = Mod\chi_M[\{F(x_1,F(x_2,x_3)) \approx F(F(x_1,x_2),x_3)\}]$, the set $\Sigma = \chi_M[\{F(x_1, F(x_2,x_3)) \approx F(F(x_1,x_2),x_3)\}]$ is an identity basis of V. The set $\chi_M[F(x_1, F(x_2,x_3)) \approx F(F(x_1,x_2),x_3)]$ contains only three variables, therefore we have to find an identity $s \approx t \in \chi_M[\{F(x_1,F(x_2,x_3)) \approx F(F(x_1,x_2),x_3)\}]$ such that there is a substitution $\Phi : X_n \longrightarrow F_{SEM}(X_n)$ into the free semigroup and $\Phi(s)$ or $\Phi(t)$ (where $\Phi(s) \neq \Phi(t)$) occur in the Zimin word $Z_4 = x_1x_2x_1x_3x_1x_2x_1x_4x_1x_2x_1x_3x_1x_2x_1$. The identity $s \approx t$ arises from the associative law by hypersubstitution.

Applying $\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_1x_2}, \sigma_{x_2x_1}$ on both sides of the associative law gives equal words of $F_{SEM}(X_3)$. If we apply a hypersubstitution σ which maps the operation symbol F to a word which contains a power of a word with an exponent > 1 built up by two variables, then the image of the associative law contains also a power with an exponent > 1. Since the extension $\bar{\Phi} : F_{SEM}(X_3) \to F_{SEM}(X_3)$ of the substitution $\Phi : X_3 \to F_{SEM}(X_3)$ is an endomorphism, the image of a word containing a power of a word contains a power of the image: $\bar{\Phi}(uw^lv) = \bar{\Phi}(u)\bar{\Phi}(w)^l\bar{\Phi}(v)$. But Z_4 contains no power of a word. Therefore only the hypersubstitutions $\sigma_{x_1x_2x_1}$ and $\sigma_{x_2x_1x_2}$ can produce the identity $s \approx t$. It follows that $\sigma_{x_1x_2x_1} \in M$ or $\sigma_{x_2x_1x_2} \in M$. But then $\sigma_{x_1x_2x_1}$ or $\sigma_{x_2x_1x_2}$ is a proper hypersubstitution with respect to V.

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As a consequence of Theorem 2.2 and Theorem 3.9, we have

Corollary 3.11. *A variety of semigroups of the form* $V = H_M Mod\{F(x_1, F(x_2, x_3)) \approx F(F(x_1, x_2), x_3)\}$ *is finitely based if* $\sigma_{x_1x_2x_1}$ or $\sigma_{x_2x_1x_2}$ *is a proper hypersubstitution of* V. ■

References

[1] Sr. Arworn, *Groupoids of Hypersubstitutions and G-Solid Varieties*, Shaker-Verlag, Aachen 2000.

[2] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, Berlin-Heidelberg-New York 1981.

[3] Th. Changphas and K. Denecke, *Complexity of hypersubstitutions and lattices of varieties*, Discuss. Math. – Gen. Algebra Appl. 23 (2003), 31–43.

[4] K. Denecke and J. Koppitz, *M-solid varieties of semigroups*, Discuss. Math. - Algebra & Stochastics Methods **15** (1995), 23–41.

[5] K. Denecke, J. Koppitz and N. Pabhapote, *The greatest regular-solid variety of semigroups*, preprint 2002.

[6] O.G. Kharlampovich and M.V. Sapir, *Algorithmic problems in varieties*, Internat. J. Algebra Comput. **5** (1995), 379–602.

[7] A.Yu. Olshanskiĭ, *Geometry of Defining Relations in Groups*, (Russian), Izdat. "Nauka", Moscow 1989.

[8] G. Paseman, *A small basis for hyperassociativity*, preprint, University of California, Berkeley, CA, 1993.

[9] P. Perkins, *Decision Problems for Equational Theories of Semigroups and General Algebras*, Ph.D. Thesis, University of California, Berkeley, CA, 1966.

[10] P. Perkins, *Bases for equational theories of semigroups*, J. Algebra **11** (1969), 298–314.

[11] J. Płonka, *Proper and inner hypersubstitutions of varieties*, p. 106–116 in: *Proceedings of the International Conference: "Summer School on General Algebra and Ordered Sets"*, Palacký University of Olomouc 1994.

[12] L. Polák, *On hyperassociativity*, Algebra Universalis, **36** (1996), 363–378.

[13] M. Sapir, *Problems of Burnside type and the finite basis property in varieties of semigroups*, (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), 319–340, English transl. in Math. USSR-Izv. **30** (1988), 295–314.

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