Discussiones Mathematicae
General Algebra and Applications 23 (2003) 101–114

ON LATTICE-ORDERED MONOIDS

Milan Jasem

Department of Mathematics, Faculty of Chemical Technology
Slovak Technical University
Radlinského 9, 812 37 Bratislava, Slovak Republic

e-mail: milian.jasem@stuba.sk

Abstract

In the paper lattice-ordered monoids and specially normal lattice-ordered monoids which are a generalization of dually residuated lattice-ordered semigroups are investigated. Normal lattice-ordered monoids are metricless normal lattice-ordered autometrized algebras. It is proved that in any lattice-ordered monoid A, $a \in A$ and $na \geq 0$ for some positive integer n imply $a \geq 0$. A necessary and sufficient condition is found for a lattice-ordered monoid A, such that the set I of all invertible elements of A is a convex subset of A and $A^- \subseteq I$, to be the direct product of the lattice-ordered group I and a lattice-ordered semigroup P with the least element 0.

Keywords: lattice-ordered monoid, normal lattice-ordered monoid, dually residuated lattice-ordered semigroup, direct decomposition, polar.

2000 Mathematics Subject Classification: 06F05.

Normal lattice-ordered autometrized algebras were investigated in [4], [10], [11], [12], [13], [19]. Swamy ([16], [17], [18]) introduced and studied dually residuated lattice-ordered semigroups (notation DRl-semigroups) as a common abstraction of Boolean rings and abelian lattice ordered groups (notations l-groups). Swamy and Subba Rao ([20]) investigated isometries in DRl-semigroups. They proved that any isometry fixing zero in a representable DRl-semigroup is an involutory semigroup automorphism. In [5], it was shown that to each weak isometry f fixing zero in a DRl-semigroup G there exists a direct decomposition $G = A \times B$, where A is a DRl-semigroup and B is an l-group, such that $f(x) = x_A + (0 - x_B)$ for each $x \in G$.

Kovář in [6] proved that any *DRl*-semigroup A is the direct product of the l-group of all invertible elements of A and a *DRl*-semigroup with the least element and showed in [8] that conditions (1), (2) and (3) imply the condition (4) in the definition of a *DRl*-semigroup. In [9], he studied the group of zero fixing isometries of a *DRl*-semigroup. Prime ideals in *DRl*-semigroups were investigated by Hansen in [4]. Rachůnek ([14], [15]) proved that *MV*-algebras are in a one-to-one correspondence with special kinds of bounded *DRl*-semigroups. In [11], [12], he studied ideals and polars in *DRl*-semigroups.

Let us review some notions and notations used in the paper.

A system $A = (A; +, \leq)$ is called a *partially ordered semigroup* (*po-semigroup*) if and only if

(1) $(A; +)$ is a semigroup,

(2) $(A; \leq)$ is a partially ordered set,

(3) $a \leq b$ implies $a + x \leq b + x$ and $x + a \leq x + b$ for all $a, b, x \in A$.

A *po*-semigroup $(A; +, \leq)$ is called a *lattice-ordered semigroup* (*l-semigroup*) if and only if

(1) $(A; \leq)$ is a lattice with lattice operations $\vee$ and $\wedge$,

(2) $a + (b \vee c) = (a + b) \vee (a + c)$, $(b \vee c) + a = (b + a) \vee (c + a)$,
$a + (b \wedge c) = (a + b) \wedge (a + c)$,$(b \wedge c) + a = (b + a) \wedge (c + a)$

for each a, b, $c \in A$.

An l-semigroup with zero element 0 is called a *lattice-ordered monoid* (*l-monoid*).

A system $A = (A; +, \leq, -)$ is called a *dually residuated lattice-ordered semigroup* (*DRl-semigroup*) if and only if

(1) $(A; +, \leq)$ is a commutative l-monoid,

(2) for given a, b in A there exists a least $x \in A$ such that $b + x \geq a$, and this x is denoted by $a - b$,

(3) $(a - b) \vee 0 + b \leq a \vee b$ for all a, $b \in A$,

(4) $(a - a) \geq 0$ for each $a \in A$.

Partially ordered semigroup A with a zero element is said to be the *direct product* of its partially ordered subsemigroups P and Q (notation $A = P \times Q$) if the following conditions are fulfilled:

(1) if $a \in P$ and $b \in Q$, then $a + b = b + a$,

(2) each element $c \in A$ can be uniquely represented in the form $c = c_1 + c_2$, where $c_1 \in P$, $c_2 \in Q$,

(3) if $a, b \in A$, $a = a_1 + a_2$, $b = b_1 + b_2$, where a_1, $b_1 \in P$, a_2, $b_2 \in Q$, then $a \geq b$ if and only if $a_1 \geq b_1$ and $a_2 \geq b_2$.

If $A = P \times Q$, then for $x \in A$ we denote by x_P and x_Q the components of x in the direct factors P and Q, respectively.

An element x of an l-monoid A is called *positive* (*negative*) if $x \geq 0$ ($x \leq 0$, resp.). The set of all positive (negative) elements of an l-monoid A will be denoted by A^+ (A^-, resp.). For each element x of a lattice-ordered group G, $|x| = x \vee (-x)$. (Throughout this paper 0 will denote a zero element. We use $\mathbb{N}$ for the set of all positive integers).

Kovář showed in [10] (Theorem 1) that the set I of all invertible elements of a normal lattice-ordered autometrized algebra is an l-group. Analogous assertion is valid for l-monoids.

Theorem 1. *The set I of all invertible elements of an l-monoid A is an l-group and a sublattice of A.*

Proof. Clearly, I is a group. Let $a, b \in I$. Then we have $a \vee b + (-a) \wedge (-b) = [a \vee b + (-a)] \wedge [a \vee b + (-b)] = [0 \vee (b-a)] \wedge [(a-b) \vee 0] \geq 0, a \vee b + (-a) \wedge (-b) = [a + (-a) \wedge (-b)] \vee [b + (-a) \wedge (-b)] = [0 \wedge (a-b)] \vee [(b-a) \wedge 0] \leq 0$. Thus $a \vee b + (-a) \wedge (-b) = 0$. Analogously, we obtain $(-a) \wedge (-b) + a \vee b = 0$. Hence $-(a \vee b) = (-a) \wedge (-b)$. Similarly, we can prove that $-(a \wedge b) = (-a) \vee (-b)$. Therefore, $a \vee b$, $a \wedge b \in I$. Hence I is a sublattice of A. ■

As a consequence of Theorem 1, we obtain:

Corollary 1. *The set I of all invertible elements of a lattice-ordered autometrized algebra A is an l-group.* ■

Lemma 1. *Let A be an l-monoid, a, b, $c \in A$.*

If $a \wedge b = 0$ and $a \wedge c = 0$, then $a \wedge (b + c) = 0$.

If $a \vee b = 0$ and $a \vee c = 0$, then $a \vee (b + c) = 0$.

The proof is the same as in the case of l-groups. See [1], p. 294. ■

Remark 1. Choudhury showed in [2] (p. 72) that $a + b = a \vee b + a \wedge b$ for each elements a, b of a commutative l-semigroup.

Theorem 2. *Let A be an l-monoid. Let I be the set of all invertible elements of A, and let $P = \{y \in A \ : \ y \wedge |z| = 0$ for each $z \in I\}$, where $|z|$ is the absolute value of z in I. Then*

(i) $P \subseteq A^+$, 0 *is the least element of* P, $I \cap P = \{0\}$,

(ii) P *is a convex subset of* A,

(iii) P *is a sublattice of* A *and an* l-*semigroup.*

Proof. (i): Clearly, $0 \in P \subseteq A^+$. Hence 0 is the least element of P. Let $a \in I \cap P$. Then $a = a \wedge |a| = 0$.

(ii): Let a, $b \in P$, $x \in A$, and $a \geq x \geq b$. Then $0 = a \wedge |z| \geq x \wedge |z| \geq b \wedge |z| = 0$ for each $z \in I$. Thus $x \wedge |z| = 0$ for each $z \in I$ and hence $x \in P$.

(iii): Let a, $b \in P$. By Lemma 1, $a + b \in P$. Hence P is a subsemigroup of A. Since $a \geq 0$, $b \geq 0$, we have $a + b \geq b$, $a + b \geq a$. Thus $a + b \geq a \vee b \geq a \wedge b \geq 0$. From the convexity of P, it follows that $a \vee b$, $a \wedge b \in P$. ■

Theorem 3. *Let A be a commutative l-monoid. Let each negative element of A be invertible and the set I of all invertible elements of A be a convex subset of A. Let P be as in Theorem* 2. *Then A is the direct product of the l-group I and the l-semigroup P with the least element* 0 *if and only if A satisfies the following condition:*

(C) *For each $a \in A^+ \setminus I$ the set $M_a = \{a \wedge x \ : \ x \in I^+\}$ has the greatest element.*

Proof. Let the set I of all invertible elements in A be a convex subset of A and let $A^- \subseteq I$. By Theorem 1, I is an l-group. By Theorem 2, P is an l-monoid with the least element 0.

Suppose that A satisfies the condition (C). Assume that $a \in A^+ \setminus I$. Since $0 \leq a \wedge x \leq x$ for each $x \in I^+$, from the convexity of I it follows that $a \wedge x \in I^+$ and hence $M_a \subseteq I^+$. Let a_1 be the greatest element of M_a, $a_2 = a + (-a_1)$. Then $0 \leq a_1 \leq a$, $0 \leq a_2 \leq a$. Hence $a = a_1 + a_2$, where $a_1 \in I^+$. Now we prove that $a_2 \in P$. Let $b \in I$, $d = |b| \vee a_1$. Then $d + a_1 \in I^+$. Then (C) yields $a \wedge (d + a_1) \leq a_1$. This implies $a_2 \wedge d = [a + (-a_1)] \wedge [d + a_1 + (-a_1)] = [a \wedge (d + a_1)] + (-a_1) \leq 0$. Clearly, $0 \leq a_2 \wedge d$. Thus $a_2 \wedge d = 0$. Then $0 \leq |b| \leq d$ yields $0 = 0 \wedge a_2 \leq |b| \wedge a_2 \leq d \wedge a_2 = 0$. Hence $a_2 \wedge |b| = 0$. Therefore, $a_2 \in P$.

If $a \in I^+$, then we can write $a = a + 0$ and hence each $a \in A^+$ can be written in the form $a = a_1 + a_2$, where $a_1 \in I^+$, $a_2 \in P$.

Let $g \in A$. In view of Remark 1, we have $g = g \wedge 0 + g \vee 0$. Then $g \vee 0 = (g \vee 0)_1 + (g \vee 0)_2$, where $(g \vee 0)_1 \in I$, $(g \vee 0)_2 \in P$. Let $g_1 = g \wedge 0 + (g \vee 0)_1$, $g_2 = (g \vee 0)_2$. Since $g \wedge 0 \in I$, we have $g_1 \in I$. Thus $g = g_1 + g_2$, where $g_1 \in I$, $g_2 \in P$.

Let $g = g_1 + g_2 = h_1 + h_2$, where $h_1 \in I$, $h_2 \in P$. Thus $g_2 = h_1 + h_2 + (-g_1)$. Then $g_2 \vee h_2 = [h_1 + h_2 + (-g_1)] \vee h_2 = [h_1 + (-g_1)] \vee 0 + h_2 \in P$. Since $h_2 = g_1 + g_2 + (-h_1)$, we have $g_2 \vee h_2 = g_2 \vee [g_1 + g_2 + (-h_1)] = 0 \vee [g_1 + (-h_1)] + g_2 \in P$. Hence $2(g_2 \vee h_2) = [h_1 + (-g_1)] \vee 0 + h_2 + 0 \vee [g_1 + (-h_1)] + g_2 = g_2 + h_2 + (h_1 - g_1) \vee 0 \vee (g_1 - h_1) = g_2 + h_2 + |h_1 + (-g_1)| \in P$. Since $h_1 + (-g_1) \in I$ and $2(g_2 \vee h_2) \in P$, we get $0 = 2(g_2 \vee h_2) \wedge |h_1 + (-g_1)| = [g_2 + h_2 + |h_1 + (-g_1)|] \wedge |h_1 + (-g_1)| = |h_1 + (-g_1)| + [(g_2 + h_2) \wedge 0] = |h_1 + (-g_1)|$. This implies $h_1 = g_1$. Then clearly $g_2 = h_2$. Therefore, each $g \in A$ is uniquely represented in the form $g = g_1 + g_2$, where $g_1 \in I$, $g_2 \in P$. Clearly, if $g \in A^+$, then g is uniquely represented in the form $g = g_1 + g_2$, where $g_1 \in I^+$, $g_2 \in P$.

Let f, $h \in A$, $f \geq h$, $f = f_1 + f_2$, $h = h_1 + h_2$, where f_1, $h_1 \in I$, f_2, $h_2 \in P$. From $f_1 + f_2 \geq h_1 + h_2$, we get $f_1 - h_1 + f_2 \geq h_2 \geq 0$. This and $f_1 - h_1 \in I$, $f_2 \in P$ yield $f_1 - h_1 \geq 0$. Hence $f_1 \geq h_1$. Since $|f_1 - h_1| \wedge h_2 = 0$ and $h_2 \leq f_1 - h_1 + f_2 \leq |f_1 - h_1| + f_2$, we get $h_2 = h_2 \wedge (f_2 + h_2) \leq (|f_1 - h_1| + f_2) \wedge (f_2 + h_2) = (|f_1 - h_1| \wedge h_2) + f_2 = f_2$. Therefore, $A = I \times P$.

Let $A = I \times P$, $a \in A^+$. Then $a = a_I + a_P$, where $a_I \in I^+$, $a_P \in P$. Since $a \geq a_I$, we have $a \wedge a_I = a_I \in M_a$. Let $x \in I^+$. From the convexity of I and $x \geq a \wedge x \geq 0$, it follows that $a \wedge x \in I$. Then $a \geq a \wedge x$ implies $a_I \geq (a \wedge x)_I = a \wedge x$. Therefore, a_I is the greatest element of M_a. ∎

A commutative l-monoid A is called a *normal l-monoid* if for each a, $b \in A$ such that $a \leq b$ there exists $x \in A^+$ such that $a + x = b$.

Remark 2. In the definition of the normal l-monoid it suffices to require for each a, $b \in A$ such that $a \leq b$ the existence $x \in A$ such that $a + x = b$, since then there exists also a positive element $y \in A$ such that $a + y = b$. In fact, if we put $y = x \vee 0$, then we get $a + y = a + x \vee 0 = (a + x) \vee a = b \vee a = b$.

Theorem 4. *Let A be a normal l-monoid, a, $b \in A$, $a \leq b$, $S_{ab} = \{x \in A; a + x = b\}$. Then S_{ab} is a sublattice of A.*

Proof. Let a, b, x, $y \in A$, $a \leq b$, $a + x = b$, $a + y = b$. Then $a + x \vee y = (a + x) \vee (a + y) = b$, $a + x \wedge y = (a + x) \wedge (a + y) = b$. ∎

As a consequence of Theorem 2, we obtain:

Corollary 2. *If A is a normal l-monoid and I, P are as in Theorem* 2, *then P is a normal l-monoid.*

Proof. Let a, $b \in P$, $a \leq b$. Then there exists $x \in A^+$ such that $a + x = b$. Since $0 \leq a$, we get $0 \leq x \leq a + x = b$. By Theorem 2 (ii), $x \in P$. The rest also follows by Theorem 2. ∎

Lemma 2. *Each negative element of a normal l-monoid A is invertible.*

Proof. If $a \in A$, $a \leq 0$, then there exists $x \in A^+$ such that $a + x = 0$. ∎

Theorem 5. *The set I of all invertible elements of a normal l-monoid A is a convex subset of A.*

The proof is similar to the proof of Theorem 2 of [10]. ∎

By Theorem 3, Corollary 2, Lemma 2, and Theorem 5, we get the following theorem:

Theorem 6. *Let A be a normal l-monoid. Let I and P be as in Theorem* 2. *Then A is the direct product of the l-group I and the normal l-emigroup P with the least element* 0 *if and only if A satisfies condition* (C). ∎

The following example shows that there exists a commutative l-monoid H such that each negative element of H is invertible, the set I of all invertible elements of H is a convex subset of H and H satisfies the condition (C), but H is not a normal l-monoid.

Example. Let $(B, \leq_1)$ be the interval $\langle 0, 1\rangle$ of real line with the natural order. Let $x \oplus y = y \oplus x = 1$ for each x, $y \in (0, 1\rangle$ and let $0 \oplus z = z \oplus 0 = z$ for each $z \in \langle 0, 1\rangle$. Then $(B, \oplus, \leq_1)$ is a commutative l-monoid with the least element 0, but not a normal l-monoid. Let $(\mathbb{Z}, +, \leq)$ be the additive group of all integers with the natural order. Then the direct product H of $(\mathbb{Z}, +, \leq)$ and $(B, \oplus, \leq_1)$ has the above mentioned properties.

Theorem 7. *Each DRl-semigroup A is a normal l-monoid and satisfies the condition* (C) *from Theorem* 3. *Moreover,* $0 - (0 - a)$ *is the greatest element of $M_a = \{a \wedge x \ : \ x \in I^+\}$ for each $a \in A^+ \setminus I$, where I is the set of all invertible elements of A.*

Proof. Let A be a DRl-semigroup. Let $x, y \in A$, $x \leq y$. From Lemma 8 of [16] and Corollary of [16], p. 107, it follows that $x + (y - x) = y$, $y - x \geq 0$. Hence A is a normal l-monoid. Let $a \in A^+ \setminus I$. Since $a \geq 0$, from Lemmas 1 and 3 of [16], we get $0 - a \leq 0$ and $0 \leq 0 - (0 - a)$. By Lemma 1.2 of [5], $0 - (0 - a)$ is the inverse of $0 - a$ and hence $0 - a$, $0 - (0 - a) \in I$. In view of Lemma 13 of [16], $0 - (0 - a) \leq a$. Then $a \wedge (0 - (0 - a)) = 0 - (0 - a)$. Hence $0 - (0 - a) \in M_a$. Let $x \in I^+$. From the convexity of I and $x \geq a \wedge x \geq 0$ we obtain $a \wedge x \in I^+$. By Lemma 1.1 (i) of [5], $0 - (0 - (a \wedge x)) = a \wedge x$. Since $a \geq a \wedge x$, in view of Lemma 3 of [16], we have $0 - (a \wedge x) \geq 0 - a$ and $0 - (0 - a) \geq 0 - (0 - (a \wedge x)) = a \wedge x$. Hence $0 - (0 - a)$ is the greatest element of M_a. ■

The following example shows that there exists a normal l-monoid satisfying (C) which is not a DRl-semigroup.

Example. Let $(\mathbb{R}, +, \leq)$ be the additive group of all real number with the natural order. Let $(G, \leq)$ be the interval $<0, 1>$ of real line with the natural order. Let $G^\infty = G \cup \{\infty\}$. We put $x \leq_1 y$ if $x \leq y$, $x, y \in \langle 0, 1\rangle$ and $x \leq_1 \infty$ for each $x \in G^\infty$. Further, for each $x, y \in \langle 0, 1\rangle$ we put $x \oplus y = y \oplus x = x + y$ if $x + y \leq 1$ and $x \oplus y = y \oplus x = \infty$ if $x + y > 1$. Let $x \oplus \infty = \infty \oplus x = \infty$ for each $x \in G^\infty$. Then $(G^\infty, \oplus, \leq_1)$ is a normal l-monoid with the least element 0, but not a DRl-semigroup because $\infty - 1$ there does not exist in G^∞. Then the direct product of $(\mathbb{R}, +, \leq)$ and $(G^\infty, \oplus, \leq_1)$ is a normal l-monoid satisfying (C), but not a DRl-semigroup.

Hence Theorems 3 and 6 generalize The Representation Theorem 12 of Kovař in [6].

Theorem 8. *Let A be a finite normal l-monoid with the least element* 0*. Then A is a DRl-semigroup and $a + (b - a) = a \vee b$ for each a, $b \in A$.*

Proof. Let a, $b \in A$. Since $a \leq a \vee b$, there exists $a_1 \in A$ such that $a + a_1 = a \vee b \geq b$. Let $a_1, \ldots, a_n$ be all elements of A such that $a + a_i \geq b$, $i = 1, \ldots, n$. Then $a + a_1 \wedge \cdots \wedge a_n = (a + a_1) \wedge \cdots \wedge (a + a_n) \geq b$. Hence $a_1 \wedge \cdots \wedge a_n = b - a$. Since $a_1 \geq a_1 \wedge \cdots \wedge a_n = b - a$, we have $a + (b - a) \leq a + a_1 = a \vee b$. From $b - a \geq 0$, we obtain $a + (b - a) \geq a$. Then $a + (b - a) \geq a \vee b$. Therefore, $a \vee b = a + (b - a) = a + (b - a) \vee 0$. ■

Theorem 9. *In any finite normal l-monoid,* 0 *is the least element.*

Proof. If x is an element of a finite normal l-monoid A, then from Theorem 1 and Lemma 1, it follows that $x \wedge 0$ is an element of the l-group I of all invertible elements of A. If $x \wedge 0 < 0$, then I is infinite, a contradiction. Hence $x \wedge 0 = 0$. Therefore, 0 is the least element of A. ■

From Theorems 8 and 9, we immediately obtain:

Theorem 10. *Any finite normal l-monoid is a DRl-semigroup.* ■

Let x be an element of an l-group G. Then $x^+ = x \vee 0$ is called the positive part of x and $x^- = x \wedge 0$ is called the negative part of x. (See Birkhof [1], p. 293, Fuchs [3], p. 75.) We use the same definition x^+ and x^- also for an element x of an l-monoid A.

Remark 3. For the negative part x^- of an element x in an l-group, formula $x^- = (-x) \vee 0$ was used in [6]. Hansen defined in [4] the negative part x^- of an element x of a DRl-semigroup by the formula $x^- = (0 - x) \vee 0$. The negative part x^- of x is a positive element in these cases.

The elements x^+ and x^- in an l-monoid A have analogous properties as in an l-group.

Theorem 11. *Let A be an l-monoid, x, $y \in A$. Then*

(i) $x = x^+$ *if and only if* $x \geq 0$,

(ii) $x = x^-$ *if and only if* $x \leq 0$,

(iii) $(x + y)^+ \leq x^+ + y^+$, $(x + y)^- \geq x^- + y^-$.

The proof is obvious. ■

Lemma 3. *Let A be a normal l-monoid. Let $x \in A$, and let b, $c \in A^+$ such that $x \wedge 0 + b = x$, $x \wedge 0 + c = 0$. Then $b = x \vee 0 = x + c$, $b + c = b \vee c$, $b \wedge c = 0$.*

Proof. Let $x \in A$, b, $c \in A^+$, $x \wedge 0 + b = x$, $x \wedge 0 + c = 0$. In view of Remark 1, we have $b = b+0 = b+x\wedge 0+c = x+c = x\vee 0+x\wedge 0+c = x\vee 0$. Further we get $b + c = x \vee 0 + c = b \vee c$, $b \wedge c = (x + c) \wedge c = (x \wedge 0) + c = 0$. ■

Theorem 12. *Let A be a normal l-monoid, $x \in A$, $n \in \mathbb{N}$. Then $n(x^-) = (nx)^-$, $n(x^+) = (nx)^+$.*

Proof. Let $x \in A$, $n \in N$. Let $b, c \in A^+$ such that $x \wedge 0 + b = x$, $x \wedge 0 + c = 0$. By Lemma 3, $b \wedge c = 0$. From Lemma 1 and Remark 1, it follows that $nb \wedge nc = 0$, $nb + nc = nb \vee nc$. Then $(nx)^- = (nx) \wedge (n0) = [n(x \wedge 0) + nb] \wedge [n(x \wedge 0) + nc] = n(x \wedge 0) + nb \wedge nc = n(x \wedge 0) = n(x^-)$. In view of Lemma 3, we have $(nx)^+ = (nx) \vee (n0) = [n(x \wedge 0) + nb] \vee [n(x \wedge 0) + nc] = n(x \wedge 0) + nc \vee nb = n(x \wedge 0) + nc + nb = n(x \wedge 0 + c) + nb = nb = n(x \vee 0) = n(x^+)$. ■

Remark 4. Kovař showed in [7] (p. 16) that for any element x of an l-monoid A the following assertions are valid:

(i) $a = a \wedge 0 + a \vee 0 = a \vee 0 + a \wedge 0$,

(ii) $n(a \wedge 0) = na \wedge (n-1)a \wedge \cdots \wedge a \wedge 0$, where $n \in \mathbb{N}$.

Then, clearly, $a + a \wedge 0 = a \wedge 0 + a$, $a + a \vee 0 = a \vee 0 + a$, $n(a \vee 0) = na \vee (n-1)a \vee \cdots \vee a \vee 0$ for any $a \in A$, $n \in \mathbb{N}$.

Lemma 4. *Let A be an l-monoid, $a \in A$, and $n \in \mathbb{N}$. Then:*

(i) *If $2a \geq 0$, then $a \geq 0$;*

(ii) *If $na \geq 0$, then $(n+1)a \geq 0$.*

Proof. (i): Let $a \in A$, and $2a \geq 0$. Then $2(a \wedge 0) = 2a \wedge a \wedge 0 = a \wedge 0$. Hence $a = a \wedge 0 + a \vee 0 = 2(a \wedge 0) + a \vee 0 = a \wedge 0 + a$. Further, we have $2(a \vee 0) = 2a \vee a \vee 0 = 2a \vee a = a + a \vee 0$. Then $a = a \wedge 0 + a \vee 0 = (a + a \vee 0) \wedge (a \vee 0) = 2(a \vee 0) \wedge (a \vee 0) = a \vee 0$. Therefore, $a \geq 0$.

(ii): Let $a \in A$, $n \in \mathbb{N}$, and $na \geq 0$. Then $n(a \wedge 0) = na \wedge (n-1)a \wedge \cdots \wedge a \wedge 0 = (n-1)a \wedge \cdots \wedge a \wedge 0 = (n-1)(a \wedge 0)$. Hence $(n+1)a = (n+1)(a \vee 0) + a \wedge 0 + n(a \wedge 0) = (n+1)(a \vee 0) + a \wedge 0 + (n-1)(a \wedge 0) = a \vee 0 + n(a \vee 0) + n(a \wedge 0) = a \vee 0 + na \geq 0$. ■

The following theorem generalizes Lemmas 16 and 17 of paper [16] by Swamy.

Theorem 13. *Let A be an l-monoid, $a \in A$, and $n \in \mathbb{N}$. Then:*

(i) *If $na \geq 0$, then $a \geq 0$;*

(ii) *If $na \leq 0$, then $a \leq 0$;*

(iii) *If $na = 0$, then $a = 0$.*

Proof. (i): We prove this statement by induction on n. The statement is valid for $n = 1$. Suppose that the statement is valid for all $k \in \mathbb{N}$, such that $k \leq n$.

Let $(n+1)a \geq 0$. If $n+1$ is an even number, then $n+1 = 2m$, where $m \in \mathbb{N}$, and $m \leq n$. In view of Lemma 4 (i), from $0 \leq (n+1)a = 2(ma)$, we obtain $0 \leq ma$. Hence $0 \leq a$. If $n+1$ is an odd number, then $n+2$ is an even number and $n+2 = 2s$, where $s \in \mathbb{N}$, and $s \leq n$. By Lemma 4 (ii), $0 \leq (n+2)a = 2(sa)$. Then, from Lemma 4 (i), it follows that $0 \leq sa$. Therefore, $0 \leq a$.

Assertion (ii) can be proved dually.

(iii): It follows from (i) and (ii). ■

Theorem 14. *Let S be an l-monoid, and $S = A \times B$. Then:*

(i) *A, B are convex sublattices of S,*

(ii) *$(x \wedge y)_A = x_A \wedge y_A$, $(x \wedge y)_B = x_B \wedge y_B$ for each x, $y \in S$,*

(iii) *$(x \vee y)_A = x_A \vee y_A$, $(x \vee y)_B = x_B \vee y_B$ for each x, $y \in S$,*

(iv) *if S is normal, then A and B are normal l-monoids.*

Proof. (i): Let u, $v \in A$, $z \in S$, and $u \leq z \leq v$. Then $0 = u_B \leq z_B \leq v_B = 0$. Thus $z_B = 0$ and hence $z = z_A \in A$. Let x, $y \in A$. Since $x \wedge y \leq x$, y, we have $(x \wedge y)_A \leq x_A = x$, $(x \wedge y)_A \leq y_A = y$. Thus $(x \wedge y)_A \leq x \wedge y \leq x$. From the convexity of A, we get $x \wedge y \in A$. Similarly, $x \vee y \in A$. Analogously, we can show that B is a convex sublattice of S.

(ii) and (iii) are obvious.

(iv): Let x, $y \in A$, $x \leq y$. Then there exists $z \in S^+$, such that $x + z = y$. Hence $x_A + z_A = y_A$, $x_B + z_B = y_B$. Since $x_B = y_B = 0$, we have $z_B = 0$. Therefore, $z = z_A \in A^+$. By (i), A is a lattice. The rest is obvious. Similarly, B is a normal l-monoid. ■

If A is a commutative l-monoid and $A = I \times P$, where I and P are as in Theorem 2, then A is called a decomposable l-monoid.

Theorem 15. *Let A be a decomposable l-monoid, $x \in A$. Then:*

(i) $x^+ = (x_I)^+ + x_P$,

(ii) $x^- = (x_I)^-$,

(iii) $x^+ \wedge (-x^-) = 0$.

Proof. (i): In view of Theorem 14, we have $x^+ = x \vee 0 = x_I \vee 0 + x_P \vee 0 = (x_I)^+ + x_P$.

The proof of (ii) is analogous.

(iii): Since $a^+ \wedge (-a)^+ = 0$ for any element a of an l-group (see [1], p. 295), in view of (i), (ii), Theorem 1 and 14, we have $x^+ \wedge (-x^-) = [(x_I)^+ + x_P] \wedge [-(x_I)^-] = (x_I)^+ \wedge [-(x_I)^-] + x_P \wedge 0 = (x_I)^+ \wedge [-(x_I \wedge 0)] = (x_I)^+ \wedge (-x_I)^+ = 0$ for each $x \in A$. ■

The absolute value $|x|$ of an element x in an l-group G is defined by the formula $|x| = (-x) \vee x$. We cannot use this definition for decomposable l-monoids. But we can define the absolute value $|x|$ of an element x in a decomposable l-monoid analogously as in [6]: $|x| = |x_I| + x_P$, where $|x_I|$ is the absolute value of x_I in the l-group I. A such defined absolute value has analogous properties as the absolute value in an l-group.

Theorem 16. *Let A be a decomposable l-monoid, x, $y \in A$, $n \in N$. Then*

(i) $|x| = 0$ *if and only if* $x = 0$,

(ii) *if* $x \geq 0$, *then* $|x| = x$,

(iii) *if* $x \leq 0$, *then* $|x| = -x$,

(iv) $|x| = x^+ - x^- = x^+ \vee (-x^-)$,

(v) $n|x| = |nx|$,

(vi) $|x| + |y| \geq |x + y|$,

(vii) $|x| + |y| \geq |x| \vee |y| \geq |x \vee y|, |x| \vee |y| \geq |x \wedge y|$.

Proof. (i): Let $|x| = 0$. Then $|x_I| + x_P = 0$ implies $x_I = 0$, $x_P = 0$. Hence $x = 0$. If x =0, then $x_I = 0$, $x_P = 0$. Hence $|x| = 0$.

(ii): Let $x \geq 0$. Then $x_I \geq 0$. Thus $|x| = |x_I| + x_P = x_I + x_P = x$.

(iii): Let $x \leq 0$. Then $x_I \leq 0$, $x_P = 0$. Hence $|x| = |x_I| + x_P = -x_I = -x$.

(iv): In view of Theorem 15 and Theorem 7 of [1] (p. 295), we have $x^+ - (x^-) = (x_I)^+ + x_P - [(x_I)^-] = |x_I| + x_P = |x|$. By Remark 1 and Theorem 15 (iii), $x^+ - (x^-) = x^+ \vee (-x^-) + x^+ \wedge (-x^-) = x^+ \vee (-x^-)$.

(v): By Theorem 8 of [1] (p. 296), we get $n|x| = n(|x_I| + x_P) = n|x_I| + nx_P = |nx_I| + (nx)_P = |(nx)_I| + (nx)_P = |nx|$.

(vi): If a, b are elements of a commutative l-group, then $|a| + |b| \geq |a + b|$ (see [3], p. 76). Let $x, y \in A$. Then, we have $|x| + |y| = |x_I| + x_P + |y_I| + y_P = |x_I| + |y_I| + (x + y)_P \geq |x_I + y_I| + (x + y)_P = |(x + y)_I| + (x + y)_P = |x + y|$.

(vii): Let x, $y \in A$. Since x_P, $y_P \geq 0$, we get $x_P + y_P \geq x_P \vee y_P$. By assertion M) of [3] (p. 76), $|a| + |b| \geq |a| \vee |b| \geq |a \vee b|$ for any elements a, b of an l-group. In view of Theorem 14, we obtain $|x| + |y| = |x_I| + x_P + |y_I| + y_P \geq |x_I| \vee |y_I| + x_P \vee y_P = (|x_I| + x_P) \vee (|y_I| + y_P) = |x| \vee |y| = [x_I \vee (-x_I)] \vee [y_I \vee (-y_I)] + x_P \vee y_P \geq [x_I \vee y_I] \vee [(-x_I) \wedge (-y_I)] + x_P \vee y_P = |x_I \vee y_I| + x_P \vee y_P = |(x \vee y)_I| + (x \vee y)_P = |x \vee y|$. Further, we have $|x| \vee |y| = [x_I \vee (-x_I)] \vee [y_I \vee (-y_I)] + x_P \vee y_P \geq [x_I \wedge y_I] \vee [(-x_I) \vee (-y_I)] + x_P \vee y_P \geq |x_I \wedge y_I| + x_P \wedge y_P = |(x \wedge y)_I| + (x \wedge y)_P = |x \wedge y|$. ∎

Let A be a decomposable l-monoid, and $B \subseteq A$. Then

$$B^{\perp} = \{x \in A \ : \ |x| \wedge |y| = 0 \ \text{ for each } \ y \in B\}$$

is called the *polar* of the set B.

Remark 5. In a decomposable l-monoid A, the l-semigroup P is the polar of the l-group I. In fact, if $x \in I^{\perp}$, then $0 = |x| \wedge |x_I| = (|x_I| + x_P) \wedge |x_I| = |x_I| + x_P \wedge 0 = |x_I|$. Thus $x_I = 0$ and hence $x = x_P \in P$. If $z \in P$, then $0 = z \wedge |y| = |z| \wedge |y|$ for each $y \in I$. Therefore, $P = I^{\perp}$.

Theorem 17. *Let A be a decomposable l-monoid, $B \subseteq A$. Then $B^{\perp}$ is an l-monoid and a convex sublattice of A.*

Proof. Let x, $y \in B^{\perp}$. Then $|x| \wedge |z| = 0$, $|y| \wedge |z| = 0$ for each $z \in B$. By Lemma 1, $(|x| + |y|) \wedge |z| = 0$ for each $z \in B$. Clearly, $0 \in B$. In view of Theorem 16 (vi) and (vii), we obtain $0 = (|x| + |y|) \wedge |z| \geq |x + y| \wedge |z| \geq 0$, $0 = (|x| + |y|) \wedge |z| \geq |x \vee y| \wedge |z| \geq 0$, $0 = (|x| + |y|) \wedge |z| \geq |x \wedge y| \wedge |z| \geq 0$ and hence $(|x + y|) \wedge |z| = 0$, $|x \vee y| \wedge |z| = 0$, $|x \wedge y| \wedge |z| = 0$ for each $z \in B$. Therefore, $x + y$, $x \vee y$, $x \wedge y \in B^{\perp}$. Thus $B^{\perp}$ is an l-monoid and a sublattice of A.

Let a, $b \in B^{\perp}$, $u \in A$, and $a \geq u \geq b$. Then $a_I \geq u_I \geq b_I$, $a_P \geq u_P \geq b_P$ and hence $a_I - b_I \geq u_I - b_I \geq 0$. By Theorem 16 (ii), $|a_I - b_I| \geq |u_I - b_I| \geq 0$, $|a_P| \geq |u_P| \geq 0$. Since $a \in B^{\perp}$, we get $0 = |a| \wedge |z| = (|a_I| + a_P) \wedge |z| \geq |a_I| \wedge |z| \geq 0$ for each $z \in B$ and hence $a_I \in B^{\perp}$. Similarly, $b_I \in B^{\perp}$. Analogously, a_P, $b_P \in B^{\perp}$. In view of Lemma 1 and Theorem 16 (vi), we obtain $0 = (|a_I| + |b_I|) \wedge |z| = (|a_I| + |-b_I|) \wedge |z| \geq (|a_I - b_I|) \wedge |z| \geq (|u_I - b_I|) \wedge |z| \geq 0$ for each $z \in B$. Then $(|u_I - b_I|) \wedge |z| = 0$ for each $z \in$B and hence $u_I - b_I \in B^{\perp}$. Therefore, $(u_I - b_I) + b_I = u_I \in B^{\perp}$. Further, we have $0 = |a_P| \wedge |z| \geq |u_P| \wedge |z| \geq 0$ for each $z \in B$. Thus $|u_P| \wedge |z| = 0$ for each $z \in B$ and hence $u_P \in B^{\perp}$. Therefore, $u_I + u_P = u \in B^{\perp}$. ∎

Theorem 18. *Let A be a decomposable l-monoid, B, $D \subseteq A$. Then:*

(i) *if $B \subseteq D$, then $B^{\perp} \supseteq D^{\perp}$,*

(ii) $B \subseteq B^{\perp\perp}$,

(iii) $B^{\perp} = B^{\perp\perp\perp}$.

The proofs of (i) and (ii) are obvious. Assertion (iii) follows from (i) and (ii). ∎

References

[1] G. Birkhoff, *Lattice Theory*, Third edition, Amer. Math. Soc., Providence, RI, 1967.

[2] A.C. Choudhury, *The doubly distributive m-lattice*, Bull. Calcutta. Math. Soc. **47** (1957), 71–74.

[3] L. Fuchs, *Partially ordered algebraic systems*, Pergamon Press, New York 1963.

[4] M. Hansen, *Minimal prime ideals in autometrized algebras*, Czech. Math. J. **44** (**119**) (1994), 81–90.

[5] M. Jasem, *Weak isometries and direct decompositions of dually residuated lattice-ordered semigroups*, Math. Slovaca **43** (1993), 119–136.

[6] T. Kovář, *Any DRl-semigroup is the direct product of a commutative l-group and a DRl-semigroup with the least element*, Discuss. Math.–Algebra & Stochastic Methods **16** (1996), 99–105.

[7] T. Kovář, *A general theory of dually residuated lattice-ordered monoids*, Ph.D. Thesis, Palacký Univ., Olomouc 1996.

[8] T. Kovář, *Two remarks on dually residuated lattice-ordered semigroups*, Math. Slovaca **49** (1999), 17–18.

[9] T. Kovář, *On (weak) zero-fixing isometries in dually residuated lattice-ordered semigroups*, Math. Slovaca **50** (2000), 123–125.

[10] T. Kovář, *Normal autometrized lattice-ordered algebras*, Math. Slovaca, **50** (2000), 369–376.

[11] J. Rachůnek, *Prime ideals in autometrized algebras*, Czechoslovak Math. J. **37** (**112**) (1987), 65–69.

[12] J. Rachůnek, *Polars in autometrized algebras*, Czechoslovak Math. J. **39** (**114**) (1989), 681–685.

[13] J. Rachůnek, *Regular ideals in autometrized algebras*, Math. Slovaca **40** (1990), 117–122.

[14] J. Rachůnek, *DRl-semigroups and MV-algebras*, Czechoslovak Math. J. **48** (**123**) (1998), 365–372.

[15] J. Rachůnek, *MV-algebras are categorically equivalent to a class of* $DRl_{1(i)}$*-semigroups*, Math. Bohemica **123** (1998), 437–441.

[16] K.L.N. Swamy, *Dually residuated lattice-ordered semigroups*, Math. Ann. **159** (1965), 105–114.

[17] K.L.N. Swamy, *Dually residuated lattice-ordered semigroups. II*, Math. Ann. **160** (1965), 64–71.

[18] K.L.N. Swamy, *Dually residuated lattice-ordered semigroups. III*, Math. Ann. **167** (1966), 71–74.

[19] K.L.N. Swamy and N.P. Rao, *Ideals in autometrized algebras*, J. Austral. Math. Soc. Ser. A **24** (1977), 362–374.

[20] K.L.N. Swamy and B.V. Subba Rao, *Isometries in dually residuated lattice-ordered semigroups*, Math. Sem. Notes Kobe Univ. **8** (1980), 369–379.

Received 26 November 2002
Revised 16 May 2003
Revised 20 November 2003