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FINITE ORDERS AND THEIR MINIMAL STRICT COMPLETION LATTICES

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Abstract

Whereas the Dedekind-MacNeille completion $\mathcal{D}(P)$ of a poset P is the minimal lattice L such that every element of L is a join of elements of P, the minimal strict completion $\mathcal{D}(P)^*$ is the minimal lattice L such that the poset of join-irreducible elements of L is isomorphic to P. (These two completions are the same if every element of P is join-irreducible). In this paper we study lattices which are minimal strict completions of finite orders. Such lattices are in one-to-one correspondence with finite posets. Among other results we show that, for every finite poset P, $\mathcal{D}(P)^*$ is always generated by its doubly-irreducible elements. Furthermore, we characterize the posets P for which $\mathcal{D}(P)^*$ is a lower semimodular lattice and, equivalently, a modular lattice.

Keywords: atomistic lattice, join-irreducible element, distributive lattice, modular lattice, lower semimodular lattice, Dedekind-MacNeille completion, strict completion, weak order.

2000 Mathematics Subject Classification: 06A06, 06A11, 06B23, 06C05, 06C10.

1. Introduction

Let P be a finite poset. A lattice L is called a *(join) completion* of P if it has a subposet Q isomorphic to P and such that every element of L is a join of some elements of Q. The best known completion of a poset P is its *Dedekind-MacNeille completion* $\mathcal{D}(P)$ (also called *normal completion.*) The set of all completions of a poset P is a lattice first studied apparently by Robinson and Wolk in [13], then by Dalík ([5], [6]) and Nation and Pogel ([11]). The least element of this lattice is the Dedekind-MacNeille completion of P, whereas its greatest element is the lattice $\mathcal{O}(P)$ of all (order) ideals of P. A lattice L is a *strict (join) completion* of P if it is a completion of P such that the subposet Q is the poset of join-irreducible elements of L. (The dual notion of strict meet completion has been considered by Šešelja and Tepavčević in [14].) The set of all strict completions of P is also a lattice. This lattice was studied by Bordalo and Monjardet in [3]. We denote by $\mathcal{D}(P)^*$ the least element of this lattice and we call it the minimal strict completion of P. In fact the lattice of strict completions is the interval $[\mathcal{D}(P)^*, \mathcal{O}(P)]$ of the lattice of all the completions of P.

It is well known that the correspondence $P \leftrightarrow \mathcal{O}(P)$ between the class of all (finite) posets and the class of all (finite) distributive lattices is one-to-one. On the other hand the correspondence between posets and Dedekind-MacNeille completions is not one-to-one. But a result recalled in Section 2 establishes a one-to-one correspondence between the class of all posets and the class of all minimal strict completions. The purpose of this paper is to study some properties of this correspondence and some related properties on lattices of completions.

In Section 2, we give notations for some notions concerning posets or lattices, useful in this paper and we recall results on the lattices of all completions and of all strict completions. Section 3 starts with characterizing the posets for which the minimal strict completion is the Dedekind-MacNeille completion. This occurs in particular if P is a sum of chains or an ordinal sum of antichains. We study these two cases. In the following Section we first show that the minimal strict completion of a poset is always generated by its doubly-irreducible elements. Then we characterize the posets for which this minimal strict completion is a modular lattice. We finish with results on the behavior of the Dedekind-MacNeille or the minimal strict completions relatively to some operations on posets. In our conclusion we mention some open problems.

N.b.: All definitions, propositions, theorems etc. stated in the paper are numbered from 1 to 17 in their appearance order.

2. Notations and preliminaries

Throughout this paper the terms poset or lattice mean finite poset or lattice. The symbol P is used to denote the set P as well as the poset $(P, \leq)$.

An element j (respectively m) of P is *join-irreducible* (respectively *meet-irreducible*) if $j = \vee A$ (respectively $m = \vee A$) with $A \subseteq P$ implies $j \in A$ (respectively $m \in A$). We denote by $J(P)$ or simply J (resp. $M(P)$ or simply M) the set of join-irreducible (respectively meet-irreducible) elements of P. An element is *join-reducible* (respectively *meet-reducible*) if it is not join-irreducible (respectively meet-irreducible). So, if P has a least element, denoted by 0_P or 0 (respectively a greatest element denoted by 1_P or 1), then $0 = \vee\varnothing$ (respectively $1 = \vee\varnothing$) is join-reducible (respectively meet-reducible).

For $x \in P$, the principal ideal $\{y \in P : y \leq x\}$ generated by x, is denoted by $(x]_P$ or simply by $(x]$ and the ideal $(x] \setminus \{x\}$ is denoted by $(x[_P$ or $(x[$. The following fact will often be useful.

Lemma 1. *For all $x, y \in P$, $(x[\cap(y[= (x] \cap (y]$ if x and y are incomparable, and $(x[\cap(y[= (y[$ if $y \leq x$.* ■

Let P, Q be two disjoint posets:

- The symbol $\mathcal{O}(P)$ denotes the lattice of all order ideals of P and $\mathcal{D}(P)$ its *Dedekind-MacNeille* or *normal completion*: $\mathcal{D}(P) = \{(x] : x \in P\} \cup \{\cap(x_i] : x_i \in P\} \cup \{P\}$.
- The symbol $P + Q$ denotes the *cardinal* or *direct sum* of P and Q, i.e. the disjoint union of P and Q.
- The symbol $P \times Q$ denotes the *direct product* of P and Q.
- The symbol $P \oplus Q$ denotes the *ordinal sum* of P and Q (every element of P is put below every element of Q).
- The symbol $P \oplus' Q$ denotes the *reduced ordinal sum* of a poset P admitting a maximum element 1_P and a poset Q admitting a minimum element 0_Q, these two elements being identified.

We now recall the definitions and/or the notations for some types of posets. The poset formed by n incomparable elements, i.e. the *antichain* of size n, will be denoted by A_n and A_1 will be also denoted by $\mathbf{1}$. The poset formed by n comparable elements, i.e. the *chain* of size n, will be denoted by C_n.

A poset P is a *weak order* if it does not contain the direct sum of a singleton and of a 2-element chain as subposet. Equivalently, a weak order is the ordinal sum of antichains and we denote it by $A_{n_1} \oplus ... \oplus A_{n_k} ... \oplus A_{n_m}$. The sets A_{n_k} are called the *levels* of the weak order.

A poset P is *series-parallel* if it does not contain the poset called N and represented on Figure 1 as a subposet. A poset P is N-*free* if it does not contain the poset represented on Figure 1 as a convex subposet.

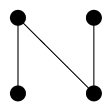

Figure 1

Finally, a poset P is *chain-antichain complete* if every maximal chain in P meets every maximal antichain in P.

Definitions 2. Let P be a poset. A lattice L is a *completion* of a poset P if it has a subposet Q isomorphic to P and such that every element of L is a join of some elements of Q.

A completion L of P is *strict* if Q is the poset of join-irreducible elements of L.

A strict completion L of P is *minimal* if it has minimal cardinality among all strict completions of P.

Obviously a strict completion is a completion. We use the term completion or strict completion instead of the more precise terms join completion used by Nation and Pogel in [11], or strict join completion used by Bordalo and Monjardet in [3].

Theorem 3 below summarizes some results proved in [11] for completions, and in [3] for strict completions. It shows in particular that every poset P has a (unique up to isomorphism) minimal strict completion and, moreover, it determines all the strict completions. In fact the strict completions of the poset P form a lattice, denoted by $\mathcal{M}_P$, with the minimal strict completion as minimum element.

We must first recall the definitions of some classes of lattices. A lattice is *atomistic* if its join-irreducible elements are its atoms, i.e. the elements covering the least element of the lattice.

A lattice is *lower semimodular* if x covered by $x \vee y$ implies $x \wedge y$ covered by y. A lattice is *locally lower distributive* if it is lower semimodular and does not contain the modular lattice M_3 with 3 atoms and 5 elements as a sublattice.

One defines a dependency relation D on the set $J(L)$ of join-irreducible elements of a lattice L by: aDb if there exists $p \in L$ with $a \leq b \vee p$ and $a \nleq c \vee p$ for $c < p$. A lattice is *lower bounded* if its dependence relation has no cycles. (See [8] for equivalent definitions and properties of lower bounded lattices.) Recall in particular that an atomistic and lower bounded lattice is locally lower distributive.

The Boolean lattice with n atoms will be denoted by $\mathbf{2}^n$.

Theorem 3. *For any given poset P there exists exactly one (up to isomorphism) minimal strict completion of P, denoted by $\mathcal{D}(P)^*$. This completion is a meet subsemilattice of the lattice $\mathcal{O}(P)$ of all order ideals of P and it consists of the following ideals: $\mathcal{D}(P)^* = \{P\} \cup \{(x] : x \in P\} \cup \{\cap(x_i] : x_i \in P\} \cup \{(x[: x \in P\} = \mathcal{D}(P) \cup \{(x[: x \in P\}$.*

A lattice L is a strict completion of a poset P (respectively a completion of P) if and only if it is isomorphic to a meet subsemilattice of the lattice $\mathcal{O}(P)$ containing $\mathcal{D}(P)^*$ (respectively to a meet subsemilattice of the lattice $\mathcal{O}(P)$ containing $\mathcal{D}(P)$).

The set $\mathcal{M}_P$ of all strict completions (respectively the set of all completions) of a poset P is isomorphic to the interval $[\mathcal{D}(P)^*, \mathcal{O}(P)]$ (respectively to the interval $[\mathcal{D}(P), \mathcal{O}(P)]$) of all the meet subsemilattices of $\mathcal{O}(P)$ containing $\mathcal{D}(P)^*$ (respectively, containing $\mathcal{D}(P)$). It is a lower bounded and lower semimodular lattice. ∎

In view of this theorem, henceforth we will look upon each completion of a poset P as a meet subsemilattice of $\mathcal{O}(P)$. So each element of such a completion is considered as an order ideal of P. Note also that, since these completions are closed by the intersection operation and contain P, they are *Moore families* (*closure systems*) on the set P. In fact the above intervals $[\mathcal{D}(P)^*, \mathcal{O}(P)]$ and $[\mathcal{D}(P), \mathcal{O}(P)]$) are intervals in the lattice $\mathcal{M}$ of all Moore families on P (a lattice studied among others in [4]).

Definition 4. We say that a lattice L is a *minimal strict completion* if it is a *minimal strict completion* of some poset P.

As we said in the introduction, Theorem 3 implies that the map which assigns to each poset its mimimal strict completion is a one-to-one correspondence between all posets and all lattices which are minimal strict completions. Its inverse map assigns to every minimal strict completion L its poset $J(L)$ of join-irreducible elements.

3. Posets for which the minimal strict completion is the Dedekind-MacNeille completion

By Theorem 3, the two lattices of all strict completions of the poset P and of all the completions of P are the same if and only if $\mathcal{D}(P)^* = \mathcal{D}(P)$, an equality characterized below.

Proposition 5. *Let P be a poset. Then $\mathcal{D}(P)^* = \mathcal{D}(P)$ if and only if every element of P is join-irreducible.*

Proof. By definitions, $\mathcal{D}(P)^* = \mathcal{D}(P)$ if and only if every ideal $(x[$ is a principal ideal or an intersection of such ideals. If x is join-reducible, it is the join of $p > 1$ elements x_i covered by x and so $(x[= \cup\{(x_i] : i = 1, ..., p\}$ is not a principal ideal. Moreover, $(x[$ cannot be an intersection of principal ideals, since any such ideal containing $(x[$ must contain x. Conversely, if x is join-irreducible, either there exists a unique y covered by x or there exists $p + 1 (\geq 3)$ elements $y_1, ..., y_p, t$ such that, for every $i, y_i < x$ and $y_i < t$ or x is a minimal (not minimum) element of P. In the first case $(x[= (y]$, in the second $(x[= (x] \cap t]$ and in the third $(x[= (x] \cap (z]$, where z is another minimal element of P. ■

Remark. It follows from Theorem 3 and Proposition 5 that, among all posets having the same Dedekind-MacNeille completion, there exists at most one with the property that every element is join-irreducible.

We consider now some particular posets P for which $\mathcal{D}(P)^* = \mathcal{D}(P)$, i.e. for which every element of P is join-irreducible. The simplest case occurs, when P is A_n, the n-element antichain. (In this case $J(P) = M(P) = P$). The poset P is the n-element antichain A_n if and only if its minimal strict completion is isomorphic to the modular lattice M_n with n atoms and $n+2$ elements, and also if and only if its lattice of order ideals is the Boolean lattice 2^n with n atoms. We need the definition of a certain significant class of Moore families: let $A \subseteq A_n$ and $x \notin A$. We define $\mathcal{F}_{A,x}$ as $\{X \subseteq A_n : A \nsubseteq X$ or $x \in X\}$.

Proposition 6. i) *The lattice $\mathcal{M}_{A_n}$ of all strict completions of the antichain A_n is lower bounded, atomistic with $2^n - (n+2)$ atoms, and, therefore, lower locally distributive. Its least element is the Moore family on A_n containing all the singletons of A_n, the sets $\varnothing$, and A_n.*

ii) *The meet-irreducible elements of $\mathcal{M}_{A_n}$ are the Moore families $\mathcal{F}_{A,x}$ with $A \subseteq A_n$, where $|A| \geq 2$ and $x \notin A$. In particular its coatoms are the*

n Moore families $\mathcal{F}_{(A_n \setminus \{x\}),x} = \mathbf{2}^n \setminus (A_n \setminus \{x\})$ (where x is any element of A_n).

iii) *If L is an atomistic lattice with n atoms, then there exists a strict completion in $\mathcal{M}_{A_n}$ which is isomorphic to L.*

Proof. i) The fact that $\mathcal{M}_{A_n}$ is lower bounded is a consequence of Theorem 3. Since $\mathcal{D}(P)^*$ is isomorphic to M_n, it is the Moore family on A_n containing all the singletons of A_n (and A_n and $\varnothing$). Then the elements of $\mathcal{M}_{A_n}$ are the Moore families containing all the singletons of A_n. So the atoms of $\mathcal{M}_{A_n}$ are the Moore families containing all the singletons, the sets $\varnothing, A_n$ and another arbitrary subset of A_n. Thus there are $2^n-(n+2)$ atoms. It is obvious that every element of $\mathcal{M}_{A_n}$ is the join of the atoms which it contains. Then $\mathcal{M}_{A_n}$ is atomistic and since it is lower bounded it is lower locally distributive.

ii) It is clear that the meet-irreducible elements of $\mathcal{M}_{A_n}$ are the meet-irreducible elements of $\mathcal{M}$, the lattice of all Moore families on A_n, that belong to $\mathcal{M}_{A_n}$. It is known (see, e.g., [4]) that the meet-irreducible elements of M are the Moore families $\mathcal{F}_{A,x}$ with $\varnothing \subset A \subseteq A_n$ and $x \notin A$. In particular its coatoms are the families $\mathcal{F}_{(A_n \setminus \{x\}),x}$. Since $\mathcal{F}_{\{y\},x} = \{X \subseteq A_n : y \in X$ implies $x \in X\}$ does not contain $\{y\}$, it follows that $\mathcal{F}_{A,x}$ belongs to $\mathcal{M}_{A_n}$ if and only if $|A| \geq 2$.

iii) Obvious. ■

Remark. $|\mathcal{M}_{A_n}|$ is the number of Moore families on a set of cardinality n containing all the singletons of this set. This number is known up to $n = 6$ (L. Nourine, [12]).

Corollary 7. *The cardinality of an atomistic lattice L with n atoms takes all the values of the interval $[n+2, 2^n]$.*

Proof. By Proposition 6 iii) above, such a lattice is isomorphic to an arbitrary Moore family of $\mathcal{M}_{A_n}$. As the lattice $\mathcal{M}_{A_n}$ is lower locally distributive, it contains a Moore family of cardinality k for every $k \in [n+2, 2^n]$. ■

The following result gives a restricted converse condition.

Proposition 8. *Let L be a lattice. For every $n > 2$, if $|J(L)| = n$ and $|L| = 2^n - 1$, then L is atomistic.*

Proof. Suppose $J(L)$ is not an antichain. We consider two cases.

Case a): Assume that $J(L)$ has a maximum element x. Since $|J(L)| = n > 2$, there exist $a, b \in J(L)$ with $a, b < x$. So, the sets $\{x\}$ and $\{x, a\}$ are not ideals of $J(L)$, and $|L| \leq |\mathcal{O}((J(L))| \leq 2^n - 2$, a contradiction.

Case b): If $J(L)$ has not a maximum element, it has several maximal elements. Let $\{m_1, ..., m_k\}$ be the maximal elements of $J(L)$ which are not minimal. If $k \geq 2$, then we argue that each $\{m_i\}, i = 1, ..., k$, is not an ideal, so $|L| \leq |\mathcal{O}((J(L))| \leq 2^n - k$, another contradiction. If $k = 1$, then there must exist an element x incomparable to m_1. Then $\{m_1\}$ and $\{x, m_1\}$ are not ideals of $J(L)$, and we obtain the same contradiction as in case a). ■

We consider now two generalizations of the case where P is an antichain. We begin with the case where P is an ordinal sum of antichains of cardinality greater than one.

Proposition 9. *Let $P = A_{n_1} \oplus ... \oplus A_{n_k} \oplus ... \oplus A_{n_m}$ be a weak order with $n_k \geq 2$ for every $k = 1, ..., m$. Then $\mathcal{D}(P) = \mathcal{D}(P)^*$ is a series-parallel modular lattice such that no two consecutive levels have both cardinality* 1.

Proof. It is clear that every element of such a weak order P is join-irreducible (and also meet-irreducible). Thus $\mathcal{D}(P)^* = \mathcal{D}(P)$. The lattice $\mathcal{D}(P)$ is obtained from P by adding an element a_1 below A_{n_1}, an element $a_{k,k+1}$ between two consecutive levels A_{n_k} and $A_{n_{k+1}}$, and an element $a_{n_{m+1}}$ above A_{n_m}. Then $\mathcal{D}(P)$ is the ordinal sum of the antichains A_{n_k} and of the antichains formed by the added elements. So it is series-parallel and it is obvious that it satisfies the required condition. ■

The second generalization, where P is a sum of chains, has been already studied in [3]. Recall the result:

Proposition 10. *If P is a sum of m chains of cardinalities $n_1, ..., n_m$, then $\mathcal{D}(P) = \mathcal{D}(P)^*$ is the lattice $1 \oplus P \oplus 1$ and $\mathcal{M}_P$ is an atomistic lower bounded lattice with $\prod\{n_i + 1 : i = 1, ..., m\} - \sum\{n_i : i = 1, ..., m\} - 2$ atoms.* ■

Remark. For such a sum of chains, $\mathcal{D}(P) = \mathcal{D}(P)^*$ is a modular lattice if and only if each chain has cardinality 1.

4. More about minimal strict completions

If P contains at least one join-reducible element, then $\mathcal{D}(P) \subset \mathcal{D}(P)^*$, and the following result shows that the completions of P contained between $\mathcal{D}(P)$ and $\mathcal{D}(P)^*$ form a Boolean lattice under the inclusion order:

Proposition 11. *Let P be a poset and s the number of its join-reducible elements. In the lattice of all completions of P, the interval $[\mathcal{D}(P), \mathcal{D}(P)^*]$ is isomorphic to the Boolean lattice $\mathbf{2}^s$.*

Proof. First note that $\{\mathcal{D}(P)^* = \mathcal{D}(P) \cup \{(x[: x \in P/J(P)\}$. (Indeed, if $x \in J(P)$, then $(x[$ is a principal ideal or an intersection of such ideals). We only have to prove that, for every $\mathcal{A} \subseteq \mathcal{I} = \{(x[: x \in P/J(P)\}, \mathcal{D}(P) \cup \mathcal{A}$ is a completion of P, i.e., by Theorem 3, that it is closed under the intersection operation. Let I, J be in $\mathcal{D}(P) \cup A$. If I and $J \in \mathcal{D}(P)$, then $I \cap J \in \mathcal{D}(P)$. If $I = (x] \in \mathcal{D}(P)$ or $I = (x[\in A$ and $J = (y[\in A$ with I incomparable to J (thus x incomparable to y), then $I \cap J = (x] \cap (y]$ by Lemma 1. Thus $I \cap J \in \mathcal{D}(P)$. If $I = \cap\{(x_i]\ : i = 1, ..., r\} \in \mathcal{D}(P)$ and $J = (y[\in A, I \cap J = \cap\{(x_i]\ : i = 1, ..., r\} \cap (y[= \cap\{(x_i] \cap (y[: i = 1, ..., r\}$ is the intersection of r elements belonging to $\mathcal{D}(P)$, then $I \cap J \in \mathcal{D}(P)$. ∎

Remark. Proposition 11 shows that $[\mathcal{D}(P), \mathcal{D}(P)^*] \oplus' [\mathcal{D}(P)^*, \mathcal{O}(P)] = 2^s \oplus' [\mathcal{D}(P)^*, \mathcal{O}(P)] \subseteq [\mathcal{D}(P), \mathcal{O}(P)]$. In general, the inclusion is strict. In particular, there exist posets P admitting maximal chains of completions of P from $\mathcal{D}(P)$ to $\mathcal{O}(P)$ which contain only $\mathcal{O}(P)$ as a strict completion. For instance, this is the case for the poset shown on Figure 2.

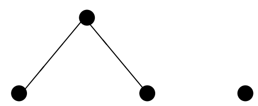

Figure 2

It is well known that the join- and the meet-irreducible elements of the Dedekind-MacNeille completion $\mathcal{D}(P)$ of a poset P are the same (up to isomorphism) as the join- and the meet-irreducible elements of P. The following result characterizes these irreducible elements in $\mathcal{D}(P)^*$.

Proposition 12. *Let I be an ideal of $\mathcal{D}(P)^*$. Then I is join-irreducible whenewer I is a principal ideal $(x]$ for some $x \in P$. It is join-reducible whenever I is an ideal $(x[$, for some join-reducible x in P, or if I is an intersection of principal ideals but is not a principal ideal itself. It is meet-irreducible if I is a principal ideal $(x]$ for some meet-irreducible x in P, or if I is an ideal $(x[$ for some join-reducible x in P. It is meet-reducible if I is a principal ideal $(x]$ for some meet-reducible x in P, or if I is an intersection* *of principal ideals which is not a principal ideal. It is doubly-irreducible if I is a principal ideal $(x]$ for some meet-irreducible x in P.*

Proof. The first assertion follows from the definition of $\mathcal{D}(P)^*$ and implies the second. The ideal $(x[$ cannot be an intersection of principal ideals (see proof of Proposition 5), just as $(x]$ if x is meet-irreducible in P. The two last assertions immediately follow. ■

Note that doubly irreducible elements do not always occur in lattices (as witnessed by the Boolean lattices $\mathbf{2}^s$, for $s > 2$). However, since every poset has at least one meet-irreducible element, Proposition 12 yields the following:

Corollary 13. *For every poset P, we have $J(\mathcal{D}(P)^*) \cap M(\mathcal{D}(P)^*) \neq \varnothing$.* ■

Moreover, not only minimal strict completions possess doubly-irreducible elements, but they are generated by them.

Proposition 14. *Every minimal strict completion $\mathcal{D}(P)^*$ is generated by its doubly-irreducible elements.*

Proof. Let I be a join-irreducible element of $\mathcal{D}(P)^*$ which is meet-reducible. By Proposition 12, $I = (x]$ with x meet-reducible in P. So, we can write $x = \wedge\{a_i : i = 1, ..., r\}$ as an infimum in P of meet-irreducible elements. Then $I = (x] = \cap\{(a_i] : i = 1, ..., r\}$, where each $(a_i]$ is doubly-irreducible in $\mathcal{D}(P)^*$, again by Proposition 12. ■

Remarks. Proposition 14 is obviously equivalent to the fact that every lattice which is a minimal strict completion has a unique minimal set of generators.

It is also obvious that atomistic lattices M_n of cardinality $n + 2$ are also generated by their doubly-irreducible elements. These lattices are isomorphic to the atoms of the lattice $\mathcal{M}_{A_n}$ of the strict completions of the antichain A_n.

We would like to determine when the lattice $\mathcal{D}(P)^*$ has some "classical" properties. It is clear that $\mathcal{D}(P)^*$ is distributive if and only if $\mathcal{D}(P)^* = \mathcal{O}(P)$, a case which has been characterized in [3]. Recall that the *width* of a poset P is the maximum cardinality $w(P)$ of its antichains. Then $\mathcal{D}(P)^* = \mathcal{O}(P)$ if and only if P is an ordinal sum of singletons or 2-elements antichains, i.e. if and only if P is a weak order such that $w(P) \leq 2$. As shown in the following theorem this result can be generalized: $\mathcal{D}(P)^*$ is lower semimodular (or equivalently modular) if and only if P is a weak order.

Theorem 15. *Let P be a partial order. The following conditions are equivalent.*

1) *P is a weak order,*
2) *$\mathcal{D}(P)^*$ is a weak order,*
3) *$\mathcal{D}(P)^*$ is an ordinal sum of m antichains A_i, where $|A_1| = |A_m| = 1$, and where $|A_i| > 1$ implies $|A_{i+1}| = 1$,*
4) *$\mathcal{D}(P)^*$ is a N-free modular lattice,*
5) *$\mathcal{D}(P)^*$ is a chain-antichain complete modular lattice,*
6) *$\mathcal{D}(P)^*$ is a series-parallel modular lattice,*
7) *$\mathcal{D}(P)^*$ is a modular lattice which does not contain a lattice isomorphic to $C_2 \times C_3$,*
8) *$\mathcal{D}(P)^*$ is a modular lattice,*
9) *$\mathcal{D}(P)^*$ is a lower semimodular lattice.*

Proof. 1) $\Rightarrow$ 2): Let $P = A_{n_1} \oplus ... \oplus Ab_{n_k} \oplus ... \oplus A_{n_m}$ be a weak order. The lattice $\mathcal{D}(P)^*$ is obtained from P by adding an element a_1 below A_{n_1}, if $n_1 > 1$, an element $a_{n_{m+1}}$ above A_{n_m} and in some cases an element $a_{k,k+1}$ between two consecutive levels A_{n_k} and $A_{n_{k+1.}}$ Then $\mathcal{D}(P)^*$ is the ordinal sum of the antichains A_{n_k} and of the antichains formed by the added elements and so it is a weak order.

2) $\Rightarrow$ 3): Obvious, since $\mathcal{D}(P)^*$ is a lattice.

3) $\Rightarrow$ 1): By definition, P is isomorphic to the join-irreducible elements of $\mathcal{D}(P)^* = A_{n_1} \oplus ... \oplus A_{n_k} \oplus ... \oplus A_{n_m}$. Then P is obtained from $\mathcal{D}(P)^*$ by deleting the levels A_{n_k} such that $n_k = 1$ and $n_{k-1} > 1$. So P is the ordinal sum of the other levels and is a weak order.

3) $\Leftrightarrow$ 4) $\Leftrightarrow$ 5) $\Leftrightarrow$ 6) $\Leftrightarrow$ 7): See [9] and [1], where it is proved that a lattice is series-parallel if and only if it does not contain a sub-lattice isomorphic to the lattice $C_2 \times C_3$.

7) $\Rightarrow$ 8) $\Rightarrow$ 9): Obvious.

9) $\Rightarrow$ 1): We first prove that if $\mathcal{D}(P)^*$ is a lower semimodular lattice, P is a series-parallel poset. To do that we assume that P is not series-parallel and we show that $\mathcal{D}(P)^*$ contains a sublattice isomorphic to the lattice N_5 with a join-irreducible element x (see Figure 3). But, by Lemma 7 in [2], such a sublattice cannot occur in a lower semimodular lattice. We call such a sublattice a forbidden N_5.

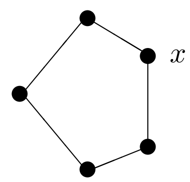

Figure 3

Assume that P is not series-parallel, i.e. that it contains a subposet Q isomorphic to the poset N (Figure 1). We set $Q = \{x, y, z, s\}$ with $z > x, s > x, s > y, x \parallel y, z \parallel y$ and $z \parallel s$. Note that one can assume that s is a minimal element greater than x and y.

We consider two cases:

Case 1. $s = x \vee y$.
We claim that the ideals $(z], (s[$ and $(s]$ generate a forbidden sublattice N_5 in $\mathcal{D}(P)^*$. First note that since $z \parallel y$ and $z \parallel s$, one has $(z] \parallel (s[$ and $(z] \cap (s[= (z] \cap (s]$. Now let $I = (z] \vee (s[$. We prove that $I = (z] \vee (s]$, i.e. that $s \in I$. If $I = P$, it is obvious. If not, then either there exists $t \in P$ such that $I = (t]$ or $(t[$, or I is the intersection of ideals $(t_i]$. In the first case, $t > x, y$, and, since $s = x \vee y$, $s = t$ is impossible, one obtains $t > s$, i.e. $s \in I$. In the second case, one has $t_i > s$, for every i, for the same reason. So, $s \in I$. We have proved that the ideals $(z], (s[$ and $(s]$ generate a sublattice N_5. Since $(s]$ is join-irreducible in $\mathcal{D}(P)^*$ our claim is proved.

Case 2. $x \vee y$ does not exist.

As above we have $(z] \cap (s[= (z] \cap (s]$. We consider $I = (z] \vee (s[$ and assume that I does not contain s. (If not, then $I = (z] \vee (s]$ and, as in Case 1, one obtains a forbidden N_5 in $\mathcal{D}(P)^*$). Then there exists $t \in P$ such that $t > z$ and y, and $t \parallel s$. And so, the subset z, y, t, s is (isomorphic to) the poset N, and satisfies $t > z$. Now, repeating the previous reasonning, either one obtains a contradiction, or one obtains a subset z, y, t, s' isomorphic to N and satisfying $s' > s$. Since P is finite, we will finally get a contradiction.

To complete the proof, we have to show that, in the case where $\mathcal{D}(P)^*$ is a lower semimodular lattice, the series-parallel poset P can only be a weak order. We know that series-parallel posets are generated from the poset of cardinality 1 by using iteratively the operations of cardinal and ordinal sums. If we use uniquely the cardinal sum of antichains and the ordinal sum, we get the ordinal sums of antichains, i.e. the weak orders. Assume now that our series-parallel poset is not a weak order and consider in its construction the first time, where we use a cardinal sum of two posets of which at least one is not an antichain. Then we make the cardinal sum of two weak orders of which at least one contains a chain $x > y$. Taking an arbitrary element z in the other weak order, it is clear that x, y, z will generate a forbidden N_5 in $\mathcal{D}(P)$ and, thus, in $\mathcal{D}(P)^*$. ■

Corollary 16. *Let P be a poset.*

a) *The following are equivalent:*
   - $\mathcal{D}(P) = \mathcal{D}(P)^*$ *and* $\mathcal{D}(P)^*$ *is a lower semimodular lattice,*
   - $\mathcal{D}(P) = \mathcal{D}(P)^*$ *and* $\mathcal{D}(P)^*$ *is a modular lattice,*
   - $\mathcal{D}(P)^*$ *is an ordinal sum of* m *antichains* A_i, *where* $|A_1| = |A_m| = 1$, *and where* $|A_i| > 1$ *(respectively* $|A_i| = 1$*) implies* $|A_i + 1| = 1$ $(|A_i + 1| > 1, \textit{respectively})$,
   - P *is a weak order without an element comparable with any other element.*

b) *The following are equivalent:*
   - $\mathcal{D}(P)^* = \mathcal{O}(P)$,
   - $\mathcal{D}(P)^*$ *is a lower locally distributive lattice,*
   - $\mathcal{D}(P)^*$ *is a distributive lattice,*
   - $\mathcal{D}(P)^*$ *is an ordinal sum of singletons or* 2*-elements antichains,*
   - P *is a weak order of width at most* 2.

c) *The following are equivalent*
   - $\mathcal{D}(P) = \mathcal{O}(P)$,
   - P *is an ordinal sum of* 2*-elements antichains.*

Proof. a) This follows from Theorem 15 and Proposition 5, since all the elements of a weak order P are join-irreducible if and only if P has no element comparable with any other element.

b) By definition, $\mathcal{D}(P)^*$ is lower locally distributive if and only if $\mathcal{D}(P)^*$ is lower semimodular and does not contain the modular lattice M_3 with 3 atoms as sublattice. Then by Theorem 15, $\mathcal{D}(P)^*$ is lower locally distributive if and only if P a weak order without M_3, i.e. if and only if P a weak order of width at most 2, and, by Proposition 16 in [3], if and only if $\mathcal{D}(P)^* = \mathcal{O}(P)$, i.e. if and only if $\mathcal{D}(P)^*$ is a distributive lattice, and also if and only if $\mathcal{D}(P)^*$ is an ordinal sum of singletons or 2-elements antichains. (Note that in the paper quoted above there is a misprint: it is written "of width 2" instead "of width ≤ 2").

c) Since $\mathcal{D}(P) = \mathcal{O}(P)$ implies $\mathcal{D}(P) = \mathcal{D}(P)^* = \mathcal{O}(P)$, this follows from a) and b) above. ■

Remark. Case c) of the above Corollary is mentionned in [7] where such a weak order is called a *doubled chain*. The authors give several characterizations of the corresponding distributive lattice $\mathcal{D}(P)$ and they call it a *rhombic chain*.

A weak order is an ordinal sum of antichains. We consider now the ordinal sum of arbitrary posets. Recall that $P \oplus' Q$ denotes the reduced ordinal sum of a poset P admitting a maximum element 1_P and a poset Q admitting a minimum element 0_Q. (Note that 1_P and 0_Q are identified).

Proposition 17. *Let P and Q be two disjoint posets. If P has a maximum element and Q a minimum element, then*

$$\mathcal{D}(P \oplus Q) \quad \textit{is isomorphic to } \mathcal{D}(P) \oplus \mathcal{D}(Q), \textit{ and}$$

$$\mathcal{D}(P \oplus Q)^* \ \textit{ is isomorphic to } \ \mathcal{D}(P)^* \oplus' \mathcal{D}(Q)^*.$$

If not, then

$$\mathcal{D}(P \oplus Q) \ \textit{ is isomorphic to } \ \mathcal{D}(P) \oplus' \mathcal{D}(Q), \ \textit{ and}$$

$$\mathcal{D}(P \oplus Q)^* \ \textit{ is isomorphic to } \ \mathcal{D}(P)^* \oplus \mathcal{D}(Q)^*.$$

Proof. Since the proofs of these four results are similar, we just prove the first one. We set $R = P \oplus Q$.

By definition the Dedekind-McNeille completion $\mathcal{D}(P \oplus Q)$ contains the set $P \cup Q$, all the principal ideals $(x]_R$ with $x \in R$, and all the intersections of these ideals. Now if $x \in P$ and $y \in Q$, then $(x]_R = (x]_P \subset (y]_R = (y]_Q \cup P$. Consider $\{x_1, ..., x_r\} \subseteq R$ and the intersection of all the ideals $(x_i]_R$. There are two cases to consider. Either there exist $x_i \in P$ and then this intersection is the intersection of all the $(x_i]_P$ where $x_i \in P$ and so it belongs to $\mathcal{D}(P)$. Or all the $x_i \in Q$ and then $\cap\{(x_i]_R : i = 1, ..., r\} = \cap(P \cup \{(x_i]_Q) : i = 1, ..., r\}) = P \cup (\cap\{(x_i]_Q : i = 1, ..., r\})$. Now it is clear that the map

$$D \mapsto D, \quad \text{if} \quad D \in \mathcal{D}(P)$$

$$D \mapsto D \cup P, \quad \text{if} \quad D \in \mathcal{D}(Q)$$

is an isomorphism from $\mathcal{D}(P) \oplus \mathcal{D}(Q)$ onto $\mathcal{D}(P \oplus Q)$. ∎

Remark. Obviously, the result of Proposition 17 can be extended to an arbitrary number of posets. In particular, one obtains the implication 1) $\Rightarrow$ 2) of Theorem 15.

5. Conclusion

In this paper, we have studied the minimal strict completion $\mathcal{D}(P)^*$ of a poset P. In particular, we have characterized the case where $\mathcal{D}(P)^*$ is equal to the Dedekind-MacNeille completion $\mathcal{D}(P)$. It results, by the construction of $\mathcal{D}(P)^*$, that it has generally many doubly-irreducible elements and, in fact, we have shown that $\mathcal{D}(P)^*$ is always generated by such elements. We have raised the problem to characterize the posets P for which the lattice $\mathcal{D}(P)^*$ has "classical" properties, and we have given the answer for the cases, where $\mathcal{D}(P)^*$ is either atomistic, lower semimodular, or lower locally distributive.

On the other hand, similar questions could be (or have been) raised for the Dedekind-MacNeille completion $\mathcal{D}(P)$, but we don't know if there exist results for this case. Note that the Dedekind-MacNeille completion of the poset P represented on Figure 1 is distributive, as $\mathcal{O}(P)$ is generated by its doubly irreducible elements (see [10]). However, $\mathcal{D}(P)^*$ is not even modular. It would be interesting to know, for which posets P, is $\mathcal{D}(P)$ a modular non distributive lattice.

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