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COMPLEXITY OF HYPERSUBSTITUTIONS AND LATTICES OF VARIETIES

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Abstract

Hypersubstitutions are mappings which map operation symbols to terms. The set of all hypersubstitutions of a given type forms a monoid with respect to the composition of operations. Together with a second binary operation, to be written as addition, the set of all hypersubstitutions of a given type forms a left-seminearring. Monoids and left-seminearrings of hypersubstitutions can be used to describe complete sublattices of the lattice of all varieties of algebras of a given type. The complexity of a hypersubstitution can be measured by the complexity of the resulting terms. We prove that the set of all hypersubstitutions with a complexity greater than a given natural number forms a sub-left-seminearring of the left-seminearring of all hypersubstitutions of the considered type. Next we look to a special complexity measure, the operation symbol count $op(t)$ of a term t and determine the greatest M-solid variety of semigroups where $M = H_2^{op}$ is the left-seminearring of all hypersubstitutions for which the number of operation symbols occurring in the resulting term is greater than or equal to 2. For every $n \geq 1$ and for $M = H_n^{op}$ we determine the complete lattices of all M-solid varieties of semigroups.

Keywords: hypersubstitution, left-seminearring, complexity of a hypersubstitution, M-solid variety.

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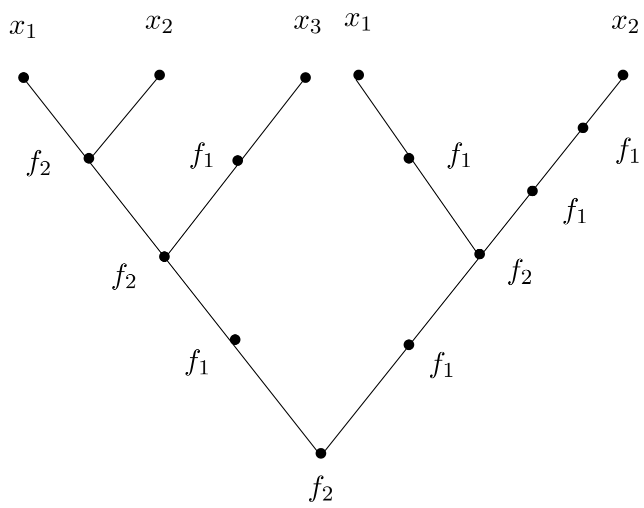
1. Introduction

Let $\tau = (n_i)_{i \in I}$ be a type indexed by a set I, with operation symbols f_i of arity n_i for $n_i \in I\!N$. Let $X = \{x_1, x_2, \ldots\}$ be a countably infinite set of variables. We denote by $W_\tau(X)$ the set of all terms of type τ over the alphabet X.

Terms are often identified with semantic trees to represent the terms. Consider for example the type $\tau = (2, 1)$, with a binary operation symbol f_2 and a unary operation symbol f_1 and a set $X = \{x_1, x_2, x_3\}$ of variables. Then the term $t = f_2(f_1(f_2(f_2(x_1, x_2), f_1(x_3))), f_1(f_2(f_1(x_1), f_1(f_1(x_2)))))$ corresponds to the following semantic tree.



The complexity of a term or a tree plays an important role in Computer Science applications and can be measured in different ways. One can count the number of occurrences of variables in the tree, or one can also count the number of operation symbols. Another method is to compare the lengths of all paths from the root to the leaves in the tree. The length of the longest such path gives the depth while the length of the shortest such path gives the mindepth ([4]). In [7] a general complexity function, called a valuation of terms of type τ into an algebra $I\!N_\tau$ of type τ with the set of all natural numbers $I\!N$ as universe, was considered.

Definition 1.1 [7]. Let a be a fixed natural number and let $I\!N_\tau = (I\!N; (f_i^{I\!N})_{i\in I})$ be an algebra of type τ with base set $I\!N$. Let $v : X \to I\!N$ be a mapping defined by $v(x) = a$ for all $x \in X$. Then v has a unique extension (which we also denote by v) to the set $W_\tau(X)$ of all terms which is a homomorphism from the free algebra $\mathcal{F}_\tau(X) = (W_\tau(X); (\bar{f}_i)_{i\in I})$ into $I\!N_\tau$. This extension homomorphism is called a *valuation of terms of type τ into $I\!N_\tau$* if it satisfies the condition $v(t) \geq v(x)$ for every variable x and every term t. The algebra $I\!N_\tau$ will be called the *valuation algebra* of the valuation v, and a is called the *base value* of the valuation.

Now we want to give some examples of such valuations of terms. The operation symbol count of a term, denoted by $op(t)$ is inductively defined by

(i) $op(t) = 0$ if t is a variable,

(ii) $op(t) = 1 + \sum_{j=1}^{n_i} op(t_j)$ if t is a composite term $t = f_i(t_1, \ldots, t_{n_i})$.

In this case the operations $f_i^{I\!N}$ are defined by $f_i^{I\!N}(a_1, \ldots, a_{n_i}) = 1+\sum_{j=1}^{n_i} a_j$, with $op(t) = 0$ if t is a variable.

The *minimum depth of a term* t, denoted by $mindepth(t)$, is the length of the shortest path from the root to a vertex in the tree, and is defined inductively by

(i) $mindepth(t) = 0$ if t is a variable,

(ii) $mindepth(t) = 1 + min\{mindepth(t_j) \mid 1 \leq j \leq n_i\}$ if t is a composite term $t = f_i(t_1, \ldots, t_{n_i})$.

For *mindepth* we use the operations $f_i^{I\!N}$ defined by $f_i(a_1, \ldots, a_{n_i}) = 1 + min\{a_1, \ldots, a_{n_i}\}$, with $mindepth(t) = 0$ if t is a variable.

The *depth of a term* t, denoted by $depth(t)$, is inductively defined by

(i) $depth(t) = 0$ if t is a variable,

(ii) $depth(t) = 1 + max\{depth(t_j) \mid 1 \leq j \leq n_i\}$ if t is a composite term $t = f_i(t_1, \ldots, t_{n_i})$.

In this case $f_i^{\mathbb{N}}$ is defined by $f_i(a_1, \ldots, a_{n_i}) = 1 + max\{a_1, \ldots, a_{n_i}\}$, with $depth(t) = 0$ if t is a variable.

In all our examples the operations $f_i^{\mathbb{N}}$ of the valuation algebra are monotone, meaning that the following condition (OC) is satisfied:

(OC) If $a_j \leq b_j$ for $1 \leq j \leq n_i$ and f_i is an n_i- ary operation symbol of type τ, then for the corresponding operation $f_i^{\mathbb{N}}$ we have $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) \leq f_i^{\mathbb{N}}(b_1, \ldots, b_{n_i})$.

We denote the superposition of the term s with the terms $t_1, \ldots, t_n$ by $s(t_1, \ldots, t_n)$. It was proved in [7] that for valuations satisfiying (OC) the following condition is satisfied.

Lemma 1.2. *Let v be a valuation of terms of type τ into $\mathbb{N}_\tau$ which satisfies (OC). Then for any n-ary term s and m-ary terms $t_1, \ldots, t_n$ we have $v(s(t_1, \ldots, t_n)) \geq v(s)$.*

Hypersubstitutions are defined as mappings from the set $\{f_i \mid i \in I\}$ of operation symbols to the set $W_\tau(X)$ of all terms of type τ which preserve the arity, i.e. n_i-ary operation symbols are mapped to terms which use at most the variables $x_1, \ldots, x_{n_i}$.

Hypersubstitutions were introduced to make precise the concept of a hyperidentity and generalizations to M-hyperidentities.

Any hypersubstitution can be uniquely extended to a map $\hat{\sigma}$ on $W_\tau(X)$ which is inductively defined by the following steps:

(i) $\hat{\sigma}[x] := x$ if $x \in X$,

(ii) $\hat{\sigma}[t] := \sigma(f)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ if $t = f(t_1, \ldots, t_{n_i})$.

An identity $s \approx t$ of type τ is called a *hyperidentity* of a variety V of type τ if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity for every hypersubstitution σ.

Using this extension we can define a binary operation $\circ_h$ on the set $Hyp(\tau)$ of all hypersubstitutions of type τ by $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$, where $\circ$ is the usual composition of operations. Clearly, this makes $(Hyp(\tau); \circ_h, \sigma_{id})$ a monoid with the identity σ_{id} which maps every operation symbol f_i to a so-called fundamental term $f_i(x_1, \ldots, x_{n_i})$. A second binary operation $+$ can be defined on $Hyp(\tau)$ by $(\sigma_1 + \sigma_2)(f_i) = \sigma_2(f_i)(\sigma_1(f_i), \ldots, \sigma_1(f_i))$ for every $i \in I$. It was proved in [1] that $(Hyp(\tau), \circ_h, +)$ is a *left-seminearring.*

If M is any submonoid of $(Hyp(\tau), \circ_h)$, then an identity $s \approx t$ of a variety V is called an *M-hyperidentity* in V if for every $\sigma \in M$ the equation

$\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity in V. Here we remark that not every submonoid of $(Hyp(\tau), \circ_h)$ is closed under addition.

A variety V is said to be *solid* if every identity is satisfied as a hyperidentity or M-*solid* if every identity of V is an M-hyperidentity in V. The set of all solid varieties of type τ forms a complete sublattice of the lattice of all varieties of type τ and if $M_1 \subseteq M_2$, then for the lattices $S_{M_1}(\tau), S_{M_2}(\tau)$ of M-solid varieties we have $S_{M_2}(\tau) \subseteq S_{M_1}(\tau)$.

2. Complexity of hypersubstitutions

Since hypersubstitutions of type τ map operation symbols to terms, we can use the definition of a valuation of a term of type τ into an algebra $\mathbb{N}_\tau$ to define the value of a hypersubstitution.

Definition 2.1. Let v be a valuation of terms of type τ into an algebra $\mathbb{N}_\tau = (\mathbb{N}; (f_i^{\mathbb{N}})_{i \in I})$ and let σ be a hypersubstitution of type τ. Then we define $v(\sigma) = min\{v(\sigma(f_i)) \mid i \in I\}$.

Example 2.2. For a hypersubstitution σ which maps each f_i to a variable we have $v(\sigma) = min\{v(x_i)) \mid i \in I\} = min\{a\} = a$. For the hypersubstitution σ_{id} we have $v(\sigma_{id}) = min\{v(\sigma_{id}(f_i(x_1, \ldots, x_{n_i}))) \mid i \in I\} = min\{f_i^{\mathbb{N}}(a, \ldots, a) \mid i \in I\}$.

We denote by $Pre(\tau)$ the set of all *prehypersubstitutions*, i.e. hypersubstitutions which do not map any operation symbol to a variable. From the definition one can easily derive some properties of the valuation of a hypersubstitution.

Proposition 2.3. *For any two hypersubstitutions $\sigma_1, \sigma_2 \in Hyp(\tau)$ and for every valuation v which satisfies the condition (OC) we have $v(\sigma_1 + \sigma_2) \geq v(\sigma_2)$ and if $\sigma_2 \in Pre(\tau)$, then $v(\sigma_1 \circ_h \sigma_2) \geq v(\sigma_1)$.*

Proof. Let f_i be an arbitrary operation symbol. For the operation $+$, we have $v((\sigma_1 + \sigma_2)(f_i)) = v(\sigma_2(f_i)(\sigma_1(f_i), \ldots, \sigma_1(f_i))) \geq v(\sigma_2(f_i))$ by Lemma 1.2 and then $v(\sigma_1 + \sigma_2) = min\{v((\sigma_1 + \sigma_2)(f_i)) \mid i \in I\} \geq min\{v(\sigma_2(f_i)) \mid i \in I\} = v(\sigma_2)$.

For the operation $\circ_h$, since $\sigma_2 \in Pre(\tau)$, we have σ_2 does not map f_i to a variable. Then $v((\sigma_1 \circ_h \sigma_2)(f_i)) = v(\hat{\sigma}_1[\sigma_2(f_i)]) = v(\sigma_1(f_k)(\hat{\sigma}_1[t_1], \ldots, \hat{\sigma}_1[t_{n_k}])) \geq v(\sigma_1(f_k))$ by Lemma 1.2. Here we have used the fact that σ_2

does not map f_i to a variable, so we can write $\sigma_2(f_i) = f_k(t_1, \ldots, t_{n_k})$ for some index $k \in I$ and some terms $t_1, \ldots, t_{n_k}$. Then

$$v(\sigma_1 \circ_h \sigma_2) = min\{(\sigma_1 \circ_h \sigma_2)(f_i) \mid i \in I\} \geq min\{(v(\sigma_1(f_i)) \mid \ i \in I\} = v(\sigma_1).$$

■

If v is a valuation of hypersubstitutions, then for every given natural number n we consider the following set of hypersubstitutions:

$$H_n^v = \{\sigma \in Hyp(\tau) \mid v(\sigma) \geq n\}.$$

We list some properties of H_n^v.

Proposition 2.4. *Let v be a valuation of hypersubstitutions of type τ which satisfies the condition (OC). Then for every $n \in I\!N$, H_n^v has the following properties:*

(i) $H_n^v = Hyp(\tau)$ *if and only if* $0 \leq n \leq a$ *(where a is the base value of v),*

(ii) $H_n^v \subseteq Pre(\tau)$ *if and only if* $n > a$.

Proof. (i) Assume that $n > a$. Then $H_n^v \neq Hyp(\tau)$ since hypersubstitutions which map one of the operation symbols to a single variable, have the value a. Assume that $0 \leq n \leq a$. Let σ be an element from $Hyp(\tau)$. For all $i \in I$, we have $v(\sigma(f_i)) \geq a \geq n$ and this means $v(\sigma) \geq n$ and $\sigma \in H_n^v$. Altogether, we have $Hyp(\tau) = H_n^v$.

(ii) Assume that $0 \leq n \leq a$. By (i) we have $H_n^v = Hyp(\tau)$ which is not contained in $Pre(\tau)$. If $n > a$ and $\sigma \in H_n^v$, then for all $i \in I$ we have $v(\sigma(f_i)) \geq n > a$. Thus $\sigma(f_i)$ is not a variable and $\sigma \in Pre(\tau)$. ■

Then we can prove:

Theorem 2.5. *Let v be a valuation of hypersubstitutions of type τ which satisfies the condition (OC). Then for every $n \in I\!N$ the set H_n^v forms a sub-left-seminearring of $(Hyp(\tau); \circ_h, +)$.*

Proof. Let a be the base value of v. If $0 \leq n \leq a$, then by Proposition 2.4 (i), $H_n^v = Hyp(\tau)$. If $n > a$, by Proposition 2.4 (ii), $H_n^v \subseteq Pre(\tau)$. By Proposition 2.3, H_n^v forms a sub-left-seminearring of $(Hyp(\tau); \circ_h, +)$. ■

Clearly, from $n_1 \leq n_2$ we get $H^v_{n_1} \supseteq H^v_{n_2}$ and therefore we have a chain $H^v_a = Hyp(\tau) \supseteq H^v_{a+1} \supseteq \cdots$ of left-seminearrings. To each left-seminearring we form the reduct to $\circ_h$, add the identity hypersubstitution σ_{id} and consider the H^v_n-solid varieties. Then we get a chain of complete lattices: $S_{H^v_a}(\tau) \subseteq S_{H^v_{a+1}}(\tau) \subseteq \cdots$ of M-solid varieties for the monoids $H^v_n, n \geq a$.

3. APPLICATIONS TO VARIETIES OF SEMIGROUPS

As a special valuation we consider the operation symbol count *op* defined in the introduction. The valuation *op* satisfies the condition (OC) and therefore for every $n \in \mathbb{N}$ the set H^{op}_n forms a sub-left-seminearring of the left-seminearring $(Hyp(\tau); \circ_h, +)$. We add the identity hypersubstitution and denote by H^{op}_n, for short, the left-seminearring $(H^{op}_n; \circ_h, +, \sigma_{id})$.

Assume now that the type is $\tau = (2)$. Clearly, H^{op}_1 is the set $Pre(2)$ of all pre-hypersubstitutions. We determine the greatest H^{op}_2-solid variety of semigroups. We denote by σ_t for a term $t \in W_\tau(\{x, y\})$ the hypersubstitution which maps the binary operation symbol to the term t.

Theorem 3.1. *The variety* $V = Mod\{x(yz) \approx (xy)z,\ x^3 \approx x^5,\ xyxzxyx \approx xyzyx,\ (x^2y)^2z \approx x^2y^2z,\ x(yz^2)^2 \approx xy^2z^2\}$ *is the greatest* H^{op}_2*-solid variety of semigroups.*

Proof. We denote by f the binary operation symbol. The following hypersubstitutions belong to H^{op}_2: $\sigma_{f(x,f(y,x))}, \sigma_{f(f(x,x),y))}, \sigma_{f(x,f(y,y))}$ and so does the identity hypersubstitution $\sigma_{id} = \sigma_{f(x,y)}$, since we assume that H^{op}_2 is a left-seminearring with identity. Applying these hypersubstitutions to the associative law we get $x(yz) \approx (xy)z, xyxzxyx \approx xyzyx, (x^2y)^2z \approx x^2y^2z, x(yz^2)^2 \approx xy^2z^2$ and applying $\sigma_{f(x,f(x,x))}$ which also belongs to H^{op}_2 to the associative law we get the equation $x^9 \approx x^3$. From $xyxzxyx \approx xyzyx$ by identification of all variables with x one obtains $x^7 \approx x^5$ and then $x^3 \approx x^5$.

The greatest H^{op}_2-solid variety of semigroups is the class of all semigroups which satisfy the associative law as a H^{op}_2-hyperidentity. We denote this class by $H_{H^{op}_2}ModAss$. Our calculations so far show that $H_{H^{op}_2}ModAss \subseteq V$.

To prove the converse inclusion we use a result of [3]. In this paper all elements of the two-generated free algebra with respect to the variety V_{HR} were calculated, where $V_{HR} = Mod\{x(yz) \approx (xy)z, xyxzxyx \approx xyzyx, (x^2y)^2z \approx x^2y^2z, x(yz^2)^2 \approx xy^2z^2\}$.

The algebra $F_{V_{HR}}(\{x,y\})$ consists exactly of the following elements, where congruence classes with respect to IdV_{HR} are for short denoted by their representatives:

$xyxyx, xy^2xyx, xy^2xy, xy^3xy, xy^4xy, xyx^2y, xyx^4y, x^2y^2xy, x^2y^3xy, x^3y^2xy,$

$x^3y^3xy, xyx^2y^2, xyx^2y^3, xyx^3y^2, xyx^3y^3, x^2yx^2y, xy^2xy^2, xyxy, x^2yxy,$

$x^3yxy, xyxy^2, xyxy^3, x^2yxy^2, xy^2x^2y, xyx, x^2yx, x^3yx, x^4yx, x^5yx, x^2yx^2,$

$x^2yx^3, xyx^2, xyx^3, xyx^4, xyx^5, x^2y^2x, x^3y^2x, x^3y^3x, x^2y^3x, xy^2x^2, xy^2x^3,$

$xy^3x^2, xy^3x^3, xy^2x, xy^3x, xy^4x, x^2y^2x^2, x^2y^5x^2, x^2y^2x^3, x^2y^3x^2, xy, x^2y, x^3y,$

$x^4y, x^5y, xy^2, xy^3, xy^4, xy^5, x^2y^2, x^2y^3, x^2y^4, x^3y^2, x^3y^3, x, x^2, x^3, x^4, x^5, x^6$

and all terms arising from these terms by exchanging x and y.

Since our variety V is a subvariety of V_{HR}, the set of all elements (classes) in $F_V(\{x,y\})$ is a subset of the set of all elements (classes) of $F_{V_{HR}}(\{x,y\})$. It was proved in [3] that every hypersubstitution σ_t, where t is one of the terms listed before and containing both variables x and y, preserves the associative identity in V_{HR}. Since we now have a subset of this set and the identities in V_{HR} form a subset of the identities in V, all hypersubstitutions σ_t, where $t \in F_V(\{x,y\})$ preserve the associative law in V. We have only to consider the hypersubstitutions σ_t where t is a term built up only by x. Because of $op(\sigma) \geq 2$, we have only to consider σ_{x^3} and σ_{x^4} since $x^3 \approx x^5$ is an identity in V. Applying these hypersubstitutions to the associative law gives $x^9 \approx x^3$ and $x^{16} \approx x^4$. Both equations are consequences of $x^5 \approx x^3$ and therefore satisfied in V. This shows $V \subseteq H_{H_2^{op}}ModAss$, and therefore V is the greatest H_2^{op}-solid variety of semigroups. ∎

It is easy to get some conditions under which a H_2^{op}-solid variety of semigroups is solid.

Proposition 3.2. *Let V be a non-trivial variety of semigroups. Then V is solid if and only if the following conditions hold:*

(i) *V is H_2^{op}-solid,*

(ii) *V is dual solid,*

(iii) *$RB \subseteq V$,*

(iv) *$V \subseteq Mod\{x^2 \approx x^4\}$.*

Proof. If V is solid, then V is clearly H_2^{op}-solid and dual solid, i.e. for every $s \approx t$ we have $\hat{\sigma}_{yx}[s] \approx \hat{\sigma}_{yx}[t] \in IdV$.

The inclusion $RB \subseteq V$ is also clear, since every identity of V is invariant under the application of σ_x as well as under the application of σ_y, i.e. $s \approx t$ is outermost. The application of σ_{x^2} to the associative identity gives $x^2 \approx x^4 \in IdV$ and then $V \subseteq Mod\{x^2 \approx x^4\}$.

We assume now that (i)–(iv) are satisfied.

Note that $Hyp(\tau) = H_2^{op} \bigcup \{\sigma_{id}, \sigma_{yx}\} \bigcup \{\sigma_x, \sigma_y\} \bigcup \{\sigma_{x^2}, \sigma_{y^2}\}$.

If $\sigma \in H_2^{op}$, then, because of (i), $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$ if $s \approx t \in IdV$.

If $\sigma \in \{\sigma_{id}, \sigma_{yx}\}$, then, because of (ii), we have $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$ if $s \approx t \in IdV$.

If $\sigma \in \{\sigma_x, \sigma_y\}$, then, because of (iii), we get $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$ if $s \approx t \in IdV$.

Since $x^2 \approx x^4 \in IdV$ and we get $\hat{\sigma}_{x^2}[s] \approx x^{2p} \approx x^{2q} \approx \sigma_{x^2}[t] \in IdV$. For σ_{y^2} we conclude in a similar way. Altogether, V is solid. ■

If, in addition, the commutative identity is satisfied for arbitrary $n \geq 1$ we can give the following characterization of all H_n^{op}-solid varieties of commutative semigroups.

Theorem 3.3. *Let $n \geq 1, n \in \mathbb{N}, \tau = (2)$ and $H_n^{op} = \{\sigma \in Hyp(2) \mid op(\sigma(f)) \geq n\}$. Let $V_n = Mod(\{(xy)z \approx x(yz), xy \approx yx, x^{n+1} \approx y^{n+1}, x^{n+1}y \approx xy^{n+1}\} \bigcup \{x^a y^b \approx x^b y^a \mid a, b \in \{1, \ldots, n\}, a + b = n + 1\})$. Then a variety of commutative semigroups is H_n^{op}-solid if and only if $V \subseteq V_n$.*

Proof. Let V be H_n^{op}-solid. We have to check that V satisfies all identities of the identity basis of V_n. To do so, we consider the hypersubstitutions $\sigma_{x^{n+1}}, \sigma_{x^{n+1}y} \in H_n^{op}$ and $\sigma_{x^a y^b} \in H_n^{op}$ for $a, b \in \{1, \ldots, n\}$ with $a + b = n + 1$. Since V is H_n^{op}-solid, it has to satisfy the identities $\hat{\sigma}_{x^{n+1}}[xy] \approx \hat{\sigma}_{x^{n+1}}[yx], \hat{\sigma}_{x^{n+1}y}[xy] \approx \hat{\sigma}_{x^{n+1}y}[yx]$ and $\hat{\sigma}_{x^a y^b}[xy] \approx \hat{\sigma}_{x^a y^b}[yx]$. This gives $x^{n+1} \approx y^{n+1}, x^{n+1}y \approx xy^{n+1} \in IdV$ and $x^a y^b \approx x^b y^a \in IdV$ and the identity hypersubstitution gives the associative and commutative law. Therefore $V \subseteq V_n$.

Conversely, assume that $V \subseteq V_n$. We show that V is H_n^{op}-solid.

At first we derive some more identities in V.

(I) $x^{n+1}y \approx z^{n+1}w \in IdV$. Indeed, from $x^{n+1}y \approx xy^{n+1}$ and $x^{n+1} \approx y^{n+1}$ there follows $z^{n+1}y \approx xt^{n+1}$ and $x^{n+1}y \approx z^{n+1}w$.

(II) $x^{n+2} \approx x^{n+1} \in IdV$.

From $x^{n+1} \approx y^{n+1}$ there follows $x^{n+1} \approx x^{2(n+1)}$. If we replace in (I) y and z by x and w by x^{n+1}, we get $x^{n+2} \approx x^{2(n+1)}$. This gives $x^{n+2} \approx x^{n+1}$. Using the relation $\sim_V$ defined on $Hyp(\tau)$ by $\sigma_1 \sim_V \sigma_2$ if and only if $\sigma_1(f) \approx \sigma_2(f) \in IdV$, because of

$$\hat{\sigma}_1[s] \approx \hat{\sigma}_2[t] \in IdV \text{ and } \sigma_1 \sim_V \sigma_2 \Rightarrow \hat{\sigma}_2[s] \approx \sigma_2[t] \in IdV,$$

we can restrict ourselves to one representative from each equivalence class with respect to the relation $\sim_V$.

Using the identities in V_n which are also satisfied in $V \subseteq V_n$ we consider the following list of representatives:
$\sigma_{x^{n+1}}, \sigma_{x^a y^b}$, for $a, b \in \{1, \ldots, n\}, a + b = n + 1$.

Let $u \approx v \in IdV$. Then $\hat{\sigma}_{x^{n+1}}[u] \approx leftmost(u)^s$ for some $s \geq n + 1$ (where $leftmost(u)$ is the first variable occurring in u) and $\hat{\sigma}_{x^{n+1}}[v] \approx leftmost(v)^r$ for some $r \geq n + 1$. Using the identity (I) and $x^{n+1} \approx y^{n+1} \in IdV$ we have $\hat{\sigma}_{x^{n+1}}[u] \approx \hat{\sigma}_{x^{n+1}}[v] \in IdV$.

Now we consider hypersubstitutions of the form $\sigma_{x^a y^b}$ with $a, b \in \{1, \ldots, n\}, a + b = n + 1$. Let $u \approx v \in IdV$, and $u = u_1 \ldots u_m, v = v_1 \ldots v_p$ for variables $u_1, \ldots, u_m, v_1, \ldots, v_p \in X$.

We consider the following cases for m and p:

(1) $m = p = 1$,

(2) $m = 1, p \geq 2$,

(3) $m \geq 2, p = 1$,

(4) $m = p = 2$,

(5) $m = 2, p \geq 3$,

(6) $m \geq 3, p = 2$,

(7) $m \geq 3, p \geq 3$.

ad (1): In this case we have $u = u_1 \approx v_1 = v \in IdV$ and thus $\hat{\sigma}_{x^a y^b}[u] = \hat{\sigma}_{x^a y^b}[u_1] = u_1 \approx v_1 = \hat{\sigma}_{x^a y^b}[v_1] = \hat{\sigma}_{x^a y^b}[v] \in IdV$.

ad (2): In this case we have $u = u_1 \approx v \in IdV$. If we substitute for all variables occurring in u_1 and v the variable x, we get $x \approx x^p \in IdV$ and together with $x^{n+2} \approx x^{n+1} \in IdV$ we have $x^{np} \approx x^2 \in IdV$ and since $np \geq n+2$ we have $x^{np} \approx x^{n(p-1)} \approx x$ and then $x \approx x^2 \in IdV$. But then $\sigma_{x^a y^b} \sim_V \sigma_{xy}$ and we can apply σ_{xy} to $u \approx v$ which gives $\hat{\sigma}_{xy}[u] = u \approx v = \hat{\sigma}_{xy}[v] \in IdV$.

ad (3): $m \geq 2, p = 1$ goes in the same way as Case (2).

ad (4): For $m = p = 2$ we consider the following three cases:

(4.1) $u_1 = v_1$ and $u_2 = v_2$,

(4.2) $u_1 = v_2$ and $u_2 = v_1$,

(4.3) $u_1 \neq v_1$ or $u_2 \neq v_2$ and $u_1 \neq v_2$ or $u_2 \neq v_1$.

In the first case we have $u = v$ and therefore $\hat{\sigma}_{x^a y^b}[u] = \hat{\sigma}_{x^a y^b}[v] \in IdV$.

In the second case, i.e. if $u_1 v_1 \approx v_1 u_1$ we obtain $\hat{\sigma}_{x^a y^b}[u_1 v_1] = u_1^a v_1^b \approx v_1^a u_1^b = \hat{\sigma}_{x^a y^b}[v_1 u_1] \in IdV$ because of the identities in the basis of V_n which are also satisfied in $V \subseteq V_n$.

In the third case we have $u_1 \neq v_1, v_2$ or $u_2 \neq v_1, v_2$ or $v_2 \neq u_1, u_2$ or $v_1 \neq u_1, u_2$. Without restriction of the generality we assume that $u_1 \neq v_1, v_2$. If we replace u_1 by x^2 and $w \in W(X) \setminus \{u_1\}$ by x in $u \approx v$, we obtain $x^3 \approx x^2$ (If $u_1 = u_2$, then we get at first $x^4 \approx x^2$ which gives also $x^3 \approx x^2$ because of $x^{n+2} \approx x^{n+1}$).

If $a = b = 1$, then clearly $\hat{\sigma}_{x^a y^b}[u] \approx \hat{\sigma}_{x^a y^b}[v] \in IdV$.

Without restriction of the generality we assume now that $a \geq 2$. Then

$$\begin{aligned}
\hat{\sigma}_{x^a y^b}[u] \quad \approx \quad & u_1^a u_2^b \\
& \approx u_1^{n+1} u_1^a u_2^b \text{ because of } x^3 \approx x^2 \\
& \approx v_1^{n+1}(\hat{\sigma}_{x^a y^b}[v]) \text{ because of (I)} \\
& \approx \hat{\sigma}_{x^a y^b}[v] \text{ since the term } \hat{\sigma}[v] \text{ starts with } v_1^2 \text{ and then we} \\
& \quad \text{can use } x^3 \approx x^2.
\end{aligned}$$

ad (5): We replace all variables occurring in $u \approx v$ by x and get $x^2 \approx x^p$.

Together with (II), we get $x^3 \approx x^2$. Then similar to (4.3) we get $\hat{\sigma}_{x^a y^b}[u] = \hat{\sigma}_{x^a y^b}[v] \in IdV$.

ad (6): This case is similar to (5).

ad (7): Without loss of generality we may assume that the brackets in u and in v are canonical. Then there are terms $\bar{u}$ and $\bar{v}$ such that $\hat{\sigma}_{x^a y^b}[u] \approx (\bar{u}^a u_{m-1}^b)^a u_m^b \approx \bar{u}^{aa} u_{m-1}^{ba} u_m^b$ and $\hat{\sigma}_{x^a y^b}[v] \approx (\bar{v}^a v_{p-1}^b)^a v_p^b \approx \bar{v}^{aa} u_{p-1}^{ba} u_p^b$.

It is easy to see that $aa \geq n+1$ or $bb \geq n+1$. In the first case we get $\bar{u}^{aa} u_{m-1}^{ba} u_m^b \approx x^{n+1} z \approx x^{n+1}$ and $\bar{v}^{aa} v_{p-1}^{ba} u_p^b \approx x^{n+1} z$ by (I), i.e. $\hat{\sigma}_{x^a y^b}[u] \approx \bar{u}^{aa} u_{m-1}^{ba} u_m^b \approx x^{n+1} z \approx \bar{v}^{aa} v_{p-1}^{ba} u_p^b \approx \hat{\sigma}_{x^a y^b}[v]$.

If $bb \geq n+1$, then we use $x^a y^b \approx x^b y^a \in IdV$ and get $\bar{u}^{aa} u_{m-1}^{ba} u_m^b \approx \bar{u}^{aa} u_{m-1}^{bb} u_m^a$ and $\bar{v}^{aa} v_{p-1}^{ba} u_p^b \approx \bar{v}^{aa} v_{p-1}^{bb} u_p^a$. By (I) we have $\bar{u}^{aa} u_{m-1}^{bb} u_m^a \approx x^{n+1} z$ and $\bar{v}^{aa} v_{p-1}^{bb} u_p^a \approx x^{n+1} z$ and thus $\hat{\sigma}_{x^a y^b}[u] \approx \hat{\sigma}_{x^a y^b}[v] \in IdV$. ■

Now we apply our results to a particular case. We noticed already that $H_1^{op} = Pre(\tau)$ is the set $Hyp \setminus \{\sigma_x, \sigma_y\}$ of all pre-hypersubstitutions and then $V_{PC} = V_1$ is the greatest pre-solid variety of commutative semigroups, i.e. $V_{PC} = Mod\{x(yz) \approx (xy)z, xy \approx yx, x^2 \approx y^2, x^2 y \approx xy^2\}$. Every subvariety of V_{PC} is pre-solid. In [2] the lattice of subvarieties of V_{PC} was determined.

If $p_m = x_0 x_1 \ldots x_m \approx y_0 y_1 \ldots y_m$, $I_m = \{x(yz) \approx (xy)z, xy \approx yx, x^2 \approx y^2, x^2 y \approx xy^2, p_m\}$, $P_m = ModI_m$ for every natural number m, then the varieties $P_m, m \in \mathbb{N}$ are exactly all subvarieties of V_{PC}.

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References

[1] Th. Changphas and K. Denecke, *Green's relations on the seminearring of full hypersubstitutions of type* (n), preprint 2002.

[2] K. Denecke and J. Koppitz, *Pre-solid varieties of semigroups*, Tatra Mt. Math. Publ. **5** (1995), 35–41.

[3] K. Denecke, J. Koppitz and N. Pabhapote, *The greatest regular-solid variety of semigroups*, preprint 2002.

[4] K. Denecke, J. Koppitz and S. L. Wismath, *Solid varieties of arbitrary type*, Algebra Universalis **48** (2002), 357–378.

[5] K. Denecke and S. L. Wismath, *Hyperidentities and Clones*, Gordon and Breach Science Publishers, Amsterdam 2000.

[6] K. Denecke and S. L. Wismath, *Universal Algebra and Applications in Theoretical Computer Science*, Chapman & Hall/CRC Publishers, Boca Raton, FL, 2002.

[7] K. Denecke and S. L. Wismath, *Valuations of terms*, preprint 2002.

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