Discussiones Mathematicae
General Algebra and Applications 23 (2003) 19–30

ON ABSOLUTE RETRACTS AND ABSOLUTE CONVEX RETRACTS IN SOME CLASSES OF ℓ-GROUPS

Ján Jakubík

Matematický ústav SAV,
Grešákova 6, 040 01 Košice, Slovakia
email: kstefan@saske.sk

Abstract

By dealing with absolute retracts of ℓ-groups we use a definition analogous to that applied by Halmos for the case of Boolean algebras. The main results of the present paper concern absolute convex retracts in the class of all archimedean ℓ-groups and in the class of all complete ℓ-groups.

Keywords: ℓ-group, absolute retract, absolute convex retract, archimedean ℓ-group, complete ℓ-group, orthogonal completeness.

2000 Mathematics Subject Classification: 06F20, 06F15.

1. Introduction

Retracts of abelian ℓ-groups and of abelian cyclically ordered groups were investigated in [6], [7], [8].

Suppose that $\mathcal{C}$ is a class of algebras. An algebra $A \in \mathcal{C}$ is called an absolute retract in $\mathcal{C}$ if, whenever $B \in \mathcal{C}$ and A is a subalgebra of B, then A is a retract of B (i.e., there is a homomorphism h of B onto A such that $h(a) = a$ for each $a \in A$). Cf., e.g., Halmos [3].

Further, let $\mathcal{C}$ be a class of ℓ-groups. An element $A \in \mathcal{C}$ will be called an absolute convex retract in $\mathcal{C}$ if, whenever $B \in \mathcal{C}$ and A is a convex ℓ-subgroup of B, then A is a retract of B.

Let $\mathcal{G}$ and Arch be the class of all ℓ-groups, or the class of all archimedean ℓ-groups, respectively.

It is easy to verify (cf. Section 2 below) that for $A \in \mathcal{G}$ the following conditions are equivalent:

(i) A is an absolute retract in $\mathcal{G}$;

(ii) A is an absolute convex retract in $\mathcal{G}$;

(iii) $A = \{0\}$.

In this note we prove

(α) Let A be an absolute retract in the class Arch. Then the ℓ-group A is divisible, complete and orthogonally complete.

By applying a result of [5] we obtain

(β) Let $A \in$ Arch and suppose that the ℓ-group A is complete and orthogonally complete. Then A is an absolute convex retract in the class Arch.

The question whether the implication in (α) (or in (β), respectively) can be reversed remains open.

Let us denote by
Compl - the class of all complete ℓ-groups;
Compl* - the class of all ℓ-groups which are complete and orthogonally complete.

(γ) Let $A \in$ Compl. Then the following conditions are equivalent:

(i) A is orthogonally complete.

(ii) A is an absolute convex retract in the class Compl.

As a corollary we obtain that each ℓ-group belonging to Compl* is an absolute convex retract in the class Compl*.

We prove that if the class $\mathcal{C} \subseteq \mathcal{G}$ is closed with respect to direct products and if A_i ($i \in I$) are asbolute (convex) retracts in $\mathcal{C}$, then their direct product $\prod_{i \in I} A_i$ is also an absolute (convex) retract in $\mathcal{C}$.

2. Preliminaries

For ℓ-groups we apply the notation as in Conrad [1]. Hence, in particular, the group operation in an ℓ-group is written additively.

We recall some relevant notions. Let G be an ℓ-group. G is *divisible* if for each $a \in G$ and each positive integer n there is $x \in G$ with $nx = a$. A system $\emptyset \neq \{x_i\}_{i\in I} \subseteq G^+$ is called *orthogonal* (or *disjoint*) if $x_{i(1)} \wedge x_{i(2)} = 0$ whenever $i(1)$ and $i(2)$ are distinct elements of I. If each orthogonal subset of G possesses the supremum in G then G is said to be *orthogonally complete*. G is *complete* if each nonempty bounded subset of G has the supremum and the infimum in G.

G is *archimedean* if, whenever $0 < x \in G$ and $y \in G$, then there is a positive integer n such that $nx \nleqq y$. For each archimedean ℓ-group G there exists a complete ℓ-group $D(G)$ (the Dedekind completion of G) such that

(i) G is a closed ℓ-subgroup of $D(G)$;

(ii) for each $x \in D(G)$ there are subsets $\{y_i\}_{i\in I}$ and $\{z_j\}_{j\in J}$ of G such that the relations

$$\sup\{x_i\}_{i\in I} = x = \inf\{z_j\}_{j\in J}$$

are valid in $D(G)$.

Let G_1 be a linearly ordered group and let G_2 be an ℓ-group. The symbol $G_1 \circ G_2$ denotes the lexicographic product of G_1 and G_2. The elements of $G_1 \circ G_2$ are pairs (g_1, g_2) with $g_1 \in G_1$ and $g_2 \in G_2$. For each $g_2 \in G_2$, the pair $(0, g_2)$ will be identified with the element g_2 of G_2. Then G_2 is a convex ℓ-subgroup of $G_1 \circ G_2$.

Lemma 2.1. *Let A be an ℓ-group, $A \neq \{0\}$, and let G_1 be a linearly ordered group, $G_1 \neq \{0\}$. Put $B = G_1 \circ A$. Then A fails to be a retract of B.*

Proof. By way of contradiction, suppose that A is a retract of B. Let h be the corresponding retract homomorphism of B onto A; i.e., $h(a) = a$ for each $a \in A$. There exists $g_1 \in G_1$ with $g_1 > 0$. Denote $(g_1, 0) = b$, $h(b) = a$. Further, there exists $a_1 \in A$ with $a_1 > a$. We have $a_1 < b$, whence $h(a_1) \leqq h(b)$, thus $a_1 \leqq a$, which is a contradiction. ∎

Let us denote by $\mathcal{A}$ the class of all abelian lattice ordered groups. If A, G_1 and B are as in Lemma 2.1 and $A, G_1 \in \mathcal{A}$, then also B belongs to $\mathcal{A}$. Thus Lemma 2.1 yields

Proposition 2.2. *Let $\mathcal{C} \in \{\mathcal{G}, \mathcal{A}\}$ and let A be an absolute retract (or an absolute convex retract, respectively) in the class $\mathcal{C}$. Then $A = \{0\}$.* ■

It is obvious that $\{0\}$ is an absolute (convex) retract in both the classes $\mathcal{G}$ and $\mathcal{A}$.

Let us remark that if G_1, $B \in \mathcal{G}$ and if G_1 is a retract of B, then G_1 need not be a convex ℓ-subgroup of B. This is verified by the following example:

Let G_1 be a linearly ordered group, $G_1 \neq \{0\}$. Further, let $G_2 \in \mathcal{G}$, $G_2 \neq \{0\}$. Put $B = G_1 \circ G_2$. If $g_1 \in G_1$, then the element $(g_1, 0)$ of B will be identified with the element g_1 of G_1. Thus G_1 turns out to be an ℓ-subgroup of B which is not a convex subset of B. For each $(g_1, g_2) \in B$ we put $h((g_1, g_2)) = g_1$. Then h is a homomorphism of B onto G_1 such that $h(g_1) = g_1$ for each $g_1 \in G_1$. Hence G_1 is a retract of B.

3. Proofs of (α), (β) and (γ)

In this section we assume that A is an archimedean ℓ-group. Hence A is abelian.

It is well-known that there exists the divisible hull A^d of A. Thus

(i) A^d is a divisible ℓ-group;

(ii) A is an ℓ-subgroup of A^d;

(iii) if $g \in A^d$, then there are $a \in A$, a positive integer n and an integer m such that $ng = ma$.

Lemma 3.1. *Assume that A is an absolute retract in the class* Arch*. Then the ℓ-group A is divisible.*

Proof. By way of contradiction, suppose that A fails to be divisible. Thus there are $a_1 \in A$ and $n \in N$ such that there is no x in A with $nx = a_1$.

Put $B = A^d$. In view of the assumption, A is a retract of B; let h be the corresponding retract homomorphism.

There exists $b \in B$ with $nb = a_1$. Then $b \notin A$. Denote $h(b) = a$. We have

$$a_1 = h(a_1) = h(nb) = nh(b) = na,$$

which is a contradiction. ■

Lemma 3.2. *Assume that A is an absolute retract in the class* Arch. *Then A is a complete ℓ-group.*

Proof. By way of contradiction, suppose that A fails to be complete. Put $B = D(A)$. Then A is an ℓ-subgroup of B and $A \neq B$. Thus there is $b \in B$ such that b does not belong to A.

In view of the assumption, A is a retract of B; let h be the corresponding retract homomorphism. Put $h(b) = a$.

There exists a subset $\{a_i\}_{i \in I}$ of A such that the relation

$$b = \bigvee_{i \in I} a_i$$

is valid in B. Hence $a_i \leqq b$ for each $i \in I$. This yields

$$a_i = h(a_i) \leqq h(b) = a$$

for each $i \in I$. Thus $b \leqq a$.

At the same time, there exists a subset $\{a'_j\}_{j \in J}$ of A such that the relation

$$b = \bigwedge_{j \in J} a'_j$$

holds in B. Hence $b \leqq a'_j$ for each $j \in J$, thus by applying the homomorphism h we obtain that $a \leqq a'_j$ for each $j \in J$. Therefore $a \leqq b$. Summarizing, $a = b$ and we arrived at a contradiction. ∎

Lemma 3.3. *Suppose that H is a complete ℓ-group. Then there exists an ℓ-group K such that*

(i) *H is a convex ℓ-subgroup of K;*

(ii) *K is complete and orthogonally complete;*

(iii) *for each $0 < k \in K$ there exists a disjoint subset $\{x_i\}_{i \in I}$ of H such that the relation*

$$k = \bigvee_{i \in I} x_i$$

is valid in K.

Proof. This is a consequence of results of [5]. ■

Lemma 3.4. *Assume that A is an absolute retract in the class* Arch. *Then the ℓ-group A is orthogonally complete.*

Proof. In view of Lemma 3.2, A is complete. Put $A = H$ and let K be as in Lemma 3.3. According to the assumption, A is a retract of K. Let h be the corresponding retract homomorphism.

Let $0 < k \in K$ and let $\{x_i\}_{i\in I}$ be as in Lemma 3.3. Put $h(k) = a$. Then $a \geqq h(x_i) = x_i$ for each $i \in I$, whence $k \leqq a$. Thus the condition (i) of Lemma 3.3 yields that $k \in A$. Hence $K^+ \subseteq A$ and then $K \subseteq A$. Therefore $K = A$ and so A is orthogonally complete. ■

From Lemmas 3.1, 3.2 and 3.4 we conclude that (α) is valid.

Let $G_1, G_2 \in \mathcal{G}$; their direct product is denoted by $G_1 \times G_2$. If $g_1 \in G_1$, then the element $(g_1, 0)$ of $G_1 \times G_2$ will be identified with g_1. Similarly, for $g_2 \in G_2$, the element $(0, g_2)$ of $G_1 \times G_2$ will be identified with g_2. Under this identification, both G_1 and G_2 are convex ℓ-subgroups of $G_1 \times G_2$.

Definition 3.5. (Cf. [2].) Let $G_1 \in$ Arch. We say that G_1 *has the splitting property* if, whenever $H \in$ Arch and G_1 is a convex ℓ-subgroup of H, then G_1 is a direct factor of H.

Proposition 3.6. (Cf. [4].) Let $G_1 \in$ Arch. Then the following conditions are equivalent:

(i) G_1 has the splitting property.

(ii) The ℓ-group G_1 is complete and orthogonally complete.

Lemma 3.7. *Let $H \in \mathcal{G}$ and let G_1 be a direct factor of H. Then G_1 is a retract of H.*

Proof. There exists $G_2 \in \mathcal{G}$ such that $H = G_1 \times G_2$. For $(g_1, g_2) \in H$ we put $h((g_1, g_2)) = g_1$. Then h is a retract homomorphism of H onto G_1. ■

Proof of (β). Let $A, B \in$ Arch and suppose that A is a convex ℓ-subgroup of B. Further, suppose that A is complete and orthogonally complete. In view of Proposition 3.6, A is a direct factor of B. Hence according to Lemma 3.7, A is a retract of B. Therefore A is an absolute convex retract in the class Arch. ■

Lemma 3.8. *Let* $A \in$ Compl. *Suppose that* A *is an absolute convex retract in the class* Compl. *Then* A *is orthogonally complete.*

Proof. Put $H = A$ and let K be as in Lemma 3.3. In view of Lemma 3.3 (i), A is a convex ℓ-subgroup of K. Hence according to the assumption, A is a retract of K. Now it suffices to apply the same method as in the proof of Lemma 3.4. ■

Lemma 3.9. *Let* $A \in$ Compl. *Suppose that* A *is orthogonally complete. Then* A *is an absolute convex retract in the class* Compl.

Proof. In view of (β), A is an absolute convex retract in the class Arch. It is well-known that the class Compl is a subclass of Arch. Hence A is an absolute convex retract in the class Compl. ■

From Lemmas 3.8 and 3.9 we conclude that (γ) holds.

Corollary 3.10. *Let* $A \in \text{Compl}^*$. *Then* A *is an absolute convex retract in the class* Compl^*. ■

4. DIRECT PRODUCTS

Let A_i $(i \in I)$ be ℓ-groups; consider their direct product

(1) $$A = \prod_{i \in I} A_i.$$

Without loss of generality we can suppose that $A_{i(1)} \cap A_{i(2)} = \{0\}$ whenever $i(1)$ and $i(2)$ are distinct elements of I. For $a \in A$ and $i \in I$, we denote by a_i or by $a(A_i)$ the component of a in the direct factor A_i.
Let $i \in I$. Put

$$A'_i = \{a \in A : a_i = 0\}.$$

Then we have

(2) $$A = A_i \times A'_i,$$

$$A'_i = \prod_{j \in I \setminus \{i\}} A_j.$$

Let $i(0) \in I$ and $a^{i(0)} \in A_{i(1)}$. There exists $a \in A$ such that

$$a_i = \begin{cases} a^{i(0)} & \text{if } i = i(0), \\ 0 & \text{otherwise.} \end{cases}$$

Then the element a of A will be identified with the element $a^{i(0)}$ of $A_{i(0)}$. Under this identification, each A_i turns out to be a convex ℓ-subgroup of A.

Lemma 4.1. *Let B be an ℓ-group and let A be an ℓ-subgroup of B. Suppose that* (1) *is valid. Let i be a fixed element of I and assume that A_i is a retract of B; the corresponding retract homomorphism will be denoted by h_i. Then for each $a \in A$ the relation*

$$h_i(a) = a_i$$

is valid.

Proof. a) At first let $0 \leqq a' \in A'_i$ and $0 \leqq a^i \in A_i$. Then $a' \wedge a^i = 0$, thus

$$0 = h_i(a') \wedge h_i(a^i) = h_i(a') \wedge a^i.$$

Since this is valid for each $a^i \in A_i$ and $h_i(a') \in A_i$ we conclude that $h_i(a') = 0$. Then $h_i(-a') = 0$ as well and this yields that $h_i(a'') = 0$ for each $a'' \in A'_i$.

b) Let $a \in A$. In view of (2) we have

$$a = a_i + a(A'_i).$$

Thus

$$h_i(a) = h_i(a_i) + h_i(a(A'_i)).$$

According to a), $h_i(a(A'_i)) = 0$. Thus $h_i(a) = a_i$. ■

Lemma 4.2. *Let B be an ℓ-group and let A be an ℓ-subgroup of B. Suppose that* (1) *is valid and that for each $i \in I$, A_i is a retract of B; the corresponding retract homomorphism will be denoted by h_i. For $b \in B$ we put*

$$h(b) = b^1 \in A,$$

where $b^1_i = h_i(b)$ for each $i \in I$. Then

(i) *h is a homomorphism of B into A;*

(ii) *$h(a) = a$ for each $a \in A$.*

Proof. The definition of h and the relation (1) immediately yield that (i) is valid. Let $a \in A$ and $i \in I$. Put $h(a) = a^1$. We have

$$a = a_i + a(A_i'),$$

thus by applying (i),

$$h(a) = h(a_i) + h(a(A_i'),$$

$$a_i^1 = h_i(a_i) + h_i(a(A_i')).$$

Since $h_i(a_i) = a_i$ and because $(a(A_i'))_i = 0$, according to Lemma 4.1, we obtain

$$a_i^1 = a_i \quad \text{for each } i \in I,$$

thus $a^1 = a$. ■

Corollary 4.3. *Let the assumptions of Lemma* 4.2 *be valid. Then A is a re-tract*
of B. ■

From Corollary 4.3 we immediately conclude

Proposition 4.4. *Assume that $\mathcal{C}$ is a class of ℓ-groups which is closed with respect to direct products. Let A_i $(i \in I)$ be absolute retracts in $\mathcal{C}$ and let* (1) *be valid. Then A is an absolute retract in $\mathcal{C}$.* ■

Proposition 4.5. *Assume that $\mathcal{C}$ is a class of ℓ-groups which is closed with respect to direct products. Let A_i $(i \in I)$ be absolute convex retracts in $\mathcal{C}$ and let* (1) *be valid. Then A is an absolute convex retract in $\mathcal{C}$.*

Proof. Let $B \in \mathcal{C}$ and suppose that A is a convex ℓ-subgroup of B. Then all A_i are convex ℓ-subgroups of B. Hence in view of the assumption, all A_i are retracts of B. Thus according to Corollary 4.3, A is a retract of B. Therefore A is an absolute convex retract in the class $\mathcal{C}$. ■

5. An example

The assertions of the following two lemmas are easy to verify; the proofs will be omitted.

Lemma 5.1. *Let A be an ℓ-group which is complete and divisible. Then*

(i) *we can define (in a unique way) a multiplication of elements of A with reals such that A turns out to be a vector lattice;*

(ii) *if $r > 0$ is a real, $0 < a \in A$, $X = \{q_1 \in Q : 0 < q_1 \leqq r\}$, $Y = \{q_2 \in R : r \leqq q_2\}$, then the relations*

$$sup(q_1 a) = ra = \inf(q_2 a)$$

are valid in A;

(iii) *if A_1 is an ℓ-subgroup of A such that A_1 is complete and divisible, and $a_1 \in A$, then for each real r the multiplication ra_1 in A_1 gives the same result as the multiplication ra_1 in A.* ■

Lemma 5.2. *Let A be as in Lemma 5.1 and suppose that $A = \prod_{i\in I} A_i$. Then all A_i are complete and divisible; moreover, for each real r, each $a \in A$ and each $i \in I$ we have*

$$(ra)_i = ra_i.$$

■

Let R be the additive group of all reals with the natural linear order. We denote by $\mathcal{C}_{\mathcal{R}}$ the class of all lattice ordered groups which can be expressed as direct products of ℓ-groups isomorphic to R.

We remark that if $B \in \mathcal{C}_{\mathcal{R}}$ and if A is an ℓ-subgroup of B which is isomorphic to R, then A need not be a convex ℓ-subgroup of B. In fact, suppose that

$$B = \prod_{i\in I} B_i,$$

where each B_i is isomorphic to R; let φ_i be and isomorphism of R onto B_i. For each $r \in R$ put

$$\varphi(r) = (\dots, \varphi_i(r), \dots)_{i\in I},$$

$$A = \varphi(R).$$

A is an ℓ-subgroup of B; if I has more than one element, then A fails to be convex in B.

Let B be as above; suppose that A is an ℓ-group isomorphic to R and that A is an ℓ-subgroup of B. Let $0 < a \in A$. Then $a_i = a(B_i) \geqq 0$ for each $i \in I$ and there exists $i(0) \in I$ with $a_{i(0)} > 0$. Thus, in view of Lemma 5.1, we have $(ra)_{i(0)} > 0$ for each $r \in R$ with $r \neq 0$. Further, for each $a_1 \in A$ there exists a uniquely determined element $r \in R$ with $a_1 = ra$. This yields that the mapping

$$\varphi_{i(0)} : a_1 \mapsto (a_1)_{i(0)}$$

is an isomorphism of A into $B_{i(0)}$.

Let $b \in B_{i(0)}$. There exists a unique $r \in R$ such that

$$b = ra_{i(0)}.$$

Then, in view of Lemma 5.2, $b = (ra)_{i(0)}$ and hence the mapping $\varphi_{i(0)}$ is an isomorphism of A onto $B_{i(0)}$.

For each $b \in B$ we put

$$h(b) = \varphi^{-1}_{i(0)}\,(b_{i(0)}).$$

Then h is a homomorphism of B into A. For $a_1 \in A$ the definition of $\varphi_{i(0)}$ yields

$$h(a_1) = a_1.$$

Thus we obtain

Lemma 5.3. *Let $B \in \mathcal{C}_{\mathcal{R}}$ and let A be an ℓ-subgroup of B such that A is isomorphic to R. Then A is a retract of B.* ∎

Corollary 5.4. *Let A be an ℓ-group isomorphic to R. Then A is an absolute retract in the class $\mathcal{C}_{\mathcal{R}}$.* ∎

From Lemma 5.4 and Corollary 4.5 we conclude

Proposition 5.5. *Each element of $\mathcal{C}_{\mathcal{R}}$ is an absolute retract in the class $\mathcal{C}_{\mathcal{R}}$.* ∎

References

[1] P. Conrad, Lattice Ordered Groups, Tulane University, New Orleans, LA, 1970.

[2] P. Conrad, *The essential closure of an archimedean lattice-ordered group*, Duke Math. J. **38** (1971), 151–160.

[3] P. Halmos, *Injective and projective Boolean algebras*, p. 114–122 in: "*Lattice Theory*" (R.P. Dilworth, editor), Amer. Math. Soc., Providence, RI,1961.

[4] J. Jakubík, *Splitting property of lattice ordered groups*, Czechoslovak Math. J. **24** (1974), 257–269.

[5] J. Jakubík, *Orthogonal hull of a strongly projectable lattice ordered group*, Czechoslovak Math. J. **28** (1978), 484–504.

[6] J. Jakubík, *Retracts of abelian cyclically ordered groups*, Archivum Math. (Brno) **25** (1989), 13–18.

[7] J. Jakubík, *Retracts of abelian lattice ordered groups*, Czechoslovak Math. J. **39** (1989), 477–489.

[8] J. Jakubík, *Retract varieties of lattice ordered groups*, Czechoslovak Math. J. **40** (1990), 104–112.

Received 21 October 2002