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A SCHEME FOR CONGRUENCE SEMIDISTRIBUTIVITY

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Abstract

A diagrammatic statement is developed for the generalized semidistributive law in case of single algebras assuming that their congruences are permutable. Without permutable congruences, a diagrammatic statement is developed for the $\wedge$-semidistributive law.

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Some attempts show that instead of identities in congruence lattices, certain diagrammatic statements are reasonable to consider, see [2] and [5]. The aim of the present paper is to show that this phenomenon can be extended to lattice Horn sentences as well.

Definition 1. A lattice L is $\wedge$-*semidistributive* if it satisfies the following implication for all $\alpha, \beta, \gamma \in L$:

$$\alpha \wedge \beta = \alpha \wedge \gamma \quad \Rightarrow \quad \alpha \wedge (\beta \vee \gamma) = \alpha \wedge \beta.$$

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The $\wedge$-semidistributive law above is often denoted by $SD_\wedge$. More general (in fact, weaker) Horn sentences have been investigated in Geyer [4] and Czédli [3]. For $n \geq 2$ put $\mathbf{n} = \{0, 1, \dots, n-1\}$ and let $P_2(\mathbf{n})$ denote the set $\{S : S \subseteq \mathbf{n} \text{ and } |S| \geq 2\}$.

Definition 2. For $\emptyset \neq H \subseteq P_2(\mathbf{n})$ we define the *generalized meet semidistributive law* $SD_\wedge(n, H)$ for lattices as follows: for all $\alpha, \beta_0, \dots, \beta_{n-1}$

$$\alpha \wedge \beta_0 = \alpha \wedge \beta_1 = \cdots = \alpha \wedge \beta_{n-1} \quad \Rightarrow \quad \alpha \wedge \beta_0 = \alpha \wedge \bigwedge_{I \in H} \bigvee_{i \in I} \beta_i.$$

As a particular case, when $H = \{S : S \subseteq \mathbf{n} \text{ and } |S| = 2\}$, $SD_\wedge(n, H)$ is denoted by $SD_\wedge(n, 2)$.

Notice that $SD_\wedge(n, 2)$ is the following lattice Horn sentence:

$$\alpha \wedge \beta_0 = \alpha \wedge \beta_1 = \cdots = \alpha \wedge \beta_{n-1} \quad \Rightarrow \quad \alpha \wedge \bigwedge_{0 \leq i < j < n} (\beta_i \vee \beta_j) = \alpha \wedge \beta_0,$$

which was originally studied by Geyer [4], and $SD_\wedge(2, 2)$ is the $\wedge$-semidistributivity law defined in Definition 1. Czédli [3] has noticed that $SD_\wedge(n, 2)$ is strictly weakening in n, i. e. $SD_\wedge(n, 2)$ implies $SD_\wedge(n+1, 2)$ but not conversely.

Our goal is to study $SD_\wedge(n, H)$ in congruence lattices of *single* algebras. Although it is usual to consider lattice identities and Horn sentences in congruence lattices of all algebras of a variety, this is not our case. The reason is that, for an arbitrary variety $\mathcal{V}$, if $SD_\wedge(n, H)$ holds in $\{\mathrm{Con}\,(A) \,:\, A \in \mathcal{V}\}$ then so does $SD_\wedge$. (This was proved by Czédli [3] and an anonymous referee of [3] who pointed out that both Kearnes and Szendrei [6] and Lipparini [7] contain implicitly the statement that if a lattice Horn sentence λ can be characterized by a weak Mal'cev condition and, for each nontrivial module variety $\mathcal{M}$, λ fails in $\mathrm{Con}\, M$ for some $M \in \mathcal{M}$, then for an arbitrary variety $\mathcal{V}$, if λ holds in $\{\mathrm{Con}\,(A) \,:\, A \in \mathcal{V}\}$, then so does $SD_\wedge$, cf. the last paragraph in [3].) In particular, for any variety $\mathcal{V}$ and any $n \geq 2$, $SD_\wedge(n, 2)$ and $SD_\wedge$ are equivalent for the class $\{\mathrm{Con}\,(A) \,:\, A \in \mathcal{V}\}$. Hence $SD_\wedge(n, 2)$ does not deserve a separate study for *varieties*.

First, we consider congruence permutable algebras.

Theorem 1. *Let A be a congruence permutable algebra. Then* $\mathrm{Con}\,(A)$ *satisfies* $SD_\wedge(n, 2)$ *if and only if A satisfies the scheme depicted in Figure* 1 *for* $\alpha, \beta_0, \dots, \beta_{n-1} \in \mathrm{Con}\,(A)$ *and* $x_0, \dots, x_k, y, z \in A$*, where* $k = \frac{n(n-1)}{2} - 1$ *and δ stands for* $\beta_0 \cap \beta_1 \cap \cdots \cap \beta_{n-1}$.

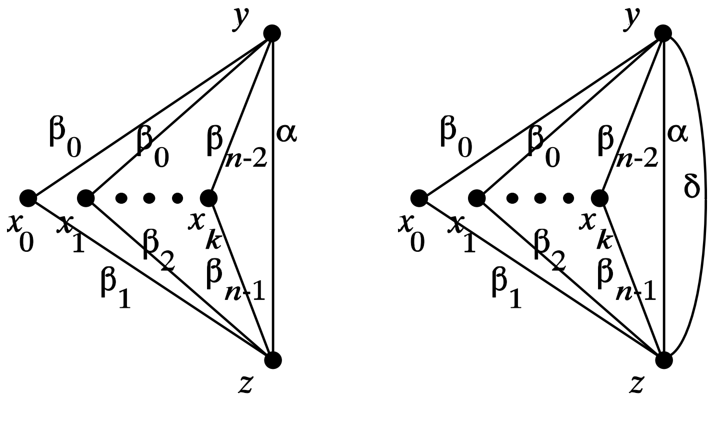

Figure 1

Proof. Suppose $SD_\wedge(n,2)$ holds. Using the premise of $SD_\wedge(n,2)$ we obtain

$$\alpha\cap\beta_0=(\alpha\cap\beta_0)\cap\cdots\cap(\alpha\cap\beta_{n-1})=\alpha\cap(\beta_0\cap\cdots\cap\beta_{n-1})\subseteq\delta,$$

whence $\mathrm{Con}\,(A)$ satisfies the Horn sentence

$$\alpha\cap\beta_0=\cdots=\alpha\cap\beta_{n-1}\quad\Rightarrow\quad\alpha\cap\bigcap_{0\le i<j<n}(\beta_i\vee\beta_j)\le\delta.$$

This implies the scheme, for the situation on the left hand side in Figure 1 then gives

$$(y,z)\in\alpha\cap\bigcap_{0\le i<j<n}(\beta_i\circ\beta_j)\subseteq\alpha\cap\bigcap_{0\le i<j<n}(\beta_i\vee\beta_j)\subseteq\delta.$$

To show the converse, suppose that the scheme given by Figure 1 holds, $\alpha,\beta_0,\dots,\beta_{n-1}\in\mathrm{Con}\,(A)$ with $\alpha\cap\beta_0=\cdots=\alpha\cap\beta_{n-1}$, and suppose that $(y,z)\in\alpha\cap\bigcap_{0\le i<j<n}(\beta_i\vee\beta_j)$. Since $\beta_i\vee\beta_j=\beta_i\circ\beta_j$ by congruence permutability, there exist $x_0,x_1,\dots,x_k$ of A such that for each j $(1\le j\le k)$ there exist u, v such that $(z,x_j)\in\beta_u$ and $(x_j,y)\in\beta_v$ (according to the left hand side of Figure 1). Then the scheme applies and we conclude $(y,z)\in\delta$. Since $\delta\subseteq\beta_0$, $(y,z)\in\beta_0$. Hence $(y,z)\in\alpha\cap\beta_0$. This proves the "$\le$" part of $SD_\wedge(n,2)$. The reverse part is simpler and does not need the scheme: $\alpha\supseteq\alpha\cap\beta_0$ and $\beta_i\vee\beta_j\supseteq\beta_i\supseteq\alpha\cap\beta_i=\alpha\cap\beta_0$ clearly give

$$\alpha \cap \bigcap_{0\le i<j<n} (\beta_i \vee \beta_j) \supseteq \alpha \cap \beta_0,$$

proving the theorem.

In the particular case when $n = 2$, we trivially conclude the following assertion:

Theorem 2. *Let A be a congruence permutable algebra. Then* Con (A) *is $\wedge$-semidistributive if and only if A satisfies the so-called triangular scheme in Figure* 2 *for any $\alpha, \beta, \gamma \in$* Con (A) *and $x, y, z \in A$.*

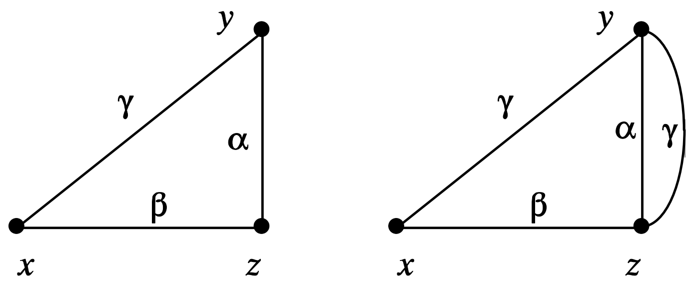

Figure 2

Proof. If Con (A) is $\wedge$-semidistributive, then the premise of the Triangular Scheme gives $(y, z) \in \beta \cap \gamma \subseteq \gamma$ by Theorem 1. Conversely, if the Triangular Scheme holds for A then its premise, after interchanging the role of β and γ, implies $(y, z) \in \beta \cap \gamma$, so $SD_\wedge(2, 2)$, which is the usual $\wedge$-semidistributivity, holds in Con (A) by Theorem 1.

One may observe that this scheme in Theorem 2 is the same as that in [2] characterising congruence distributivity in the congruence permutable case. This implies that: *in presence of congruence permutability, congruence $\wedge$-semidistributivity is equivalent to congruence distributivity.*

This follows also from another direction. Let A be congruence permutable and satisfying $SD_\wedge$. In this case A is congruence distributive, since otherwise its congruence lattice, being modular due to congruence permutability, contains M_3 but with the choice α, β, γ on Figure 3 we see that $SD_\wedge$ fails.

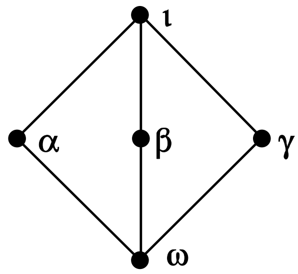

Figure 3

Remark. For $SD_\wedge(n, H)$, a similar scheme can be derived as in Theorem 1.

Without congruence permutability, for the case $SD_\wedge(2,2) = SD_\wedge$, the following theorem can be stated:

Theorem 3. *Let A be an algebra. The congruence lattice* $\mathrm{Con}\,(A)$ *is $\wedge$-semidistributive if and only if for each n, A satisfies the scheme in Figure 4 for $\alpha, \beta, \gamma \in \mathrm{Con}\,(A)$ and $x, y, z \in A$, where $\Lambda_0 = \beta$ and $\Lambda_{m+1} = \Lambda_m \circ \gamma \circ \beta$.*

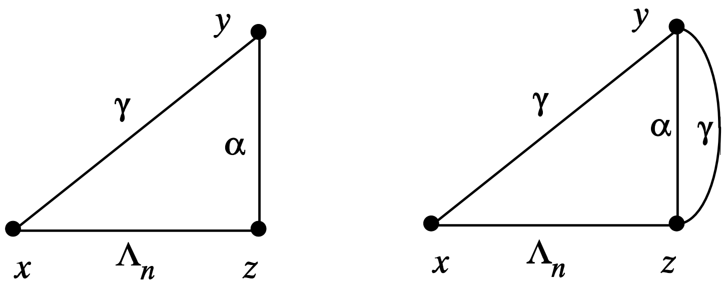

Figure 4

Proof. Suppose that $\mathrm{Con}\,(A)$ is $\wedge$- semidistributive and $\alpha, \beta, \gamma \in \mathrm{Con}\,(A)$ with $\alpha \cap \beta = \alpha \cap \gamma$. Let $x, y, z \in A$ and let $(x, y) \in \gamma$, $(y, z) \in \alpha$ and $(x, z) \in \Lambda_n$. Then

$$(y, z) \in \alpha \cap (\Lambda_n \circ \gamma) \subseteq \alpha \cap (\beta \vee \gamma) = \alpha \cap \beta = \alpha \cap \gamma$$

due to the $\wedge$-semidistributivity. Thus $(y, z) \in \gamma$, proving the validity of the scheme.

Conversely, let A satisfy the scheme for each $n \in \mathbf{N}_0$, let $\alpha, \beta, \gamma \in \mathrm{Con}\,(A)$ with $\alpha \cap \beta = \alpha \cap \gamma$. Suppose $(z, y) \in \alpha \cap (\beta \vee \gamma)$. Then there exists $n \in \mathbf{N}_0$ such that $(z, y) \in \alpha \cap (\Lambda_n \circ \gamma)$ and hence $(x, y) \in \gamma$ and $(y, z) \in \alpha$ and $(x, z) \in \Lambda_n$ for some $x \in A$. Due to the scheme, we conclude $(x, y) \in \alpha \cap \gamma$, i.e. $\alpha \cap (\beta \vee \gamma) \subseteq \alpha \cap \gamma \subseteq \alpha \cap \beta$. The converse inclusion is trivial, thus $\mathrm{Con}\,(A)$ is $\wedge$-semidistibutive.

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