# FREE ABELIAN EXTENSIONS IN THE CONGRUENCE-PERMUTABLE VARIETIES 

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#### Abstract

We obtain the construction of free abelian extensions in a congurence-permutable variety $\mathcal{V}$ using the construction of a free abelian extension in a variety of algebras with one ternary Mal'cev operation and a monoid of unary operations. We also use this construction to obtain a free solvable $\mathcal{V}$-algebra.


Keywords: abelian extension, solvable algebra, congurence-permutable variety.

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## 1. Introduction

The theory of congruence commutators in congruence modular varieties develops an important tool for a generalization of several important concepts from the theory of groups and rings such as Abelian algebras, solvable algebras, a center of an algebra. The appearance of the commutator theory was prepared by a set of basic results. Historically one of the first of them was the well known Mal'cev theorem:

Theorem 1.1 (see, e.g., [5], p. 172, [6]). The variety of algebras $\mathcal{V}$ is congruence-permutable if and only if there exists a ternary basic term $p(x, y, z)$ such that the following are the identities of $\mathcal{V}$ :

$$
\begin{equation*}
p(x, x, y)=p(y, x, x)=y \tag{1}
\end{equation*}
$$

The commutator theory is exposed in [4], [6], [7]. For all undefined notations and terminology the reader can consult [4]. Recall the most important facts about commutators and Abelian congruences. Throughout section we shall consider an arbitrary algebra $G$ from a fixed congruence modular variety $\mathcal{M}$. The commutator is the largest binary operation $(\alpha, \beta) \mapsto f(\alpha, \beta)$ on the congruence lattice $\operatorname{Con}(G)$ such that

1. $f(\alpha, \beta) \leq \alpha \cap \beta$,
2. $f(\alpha, \beta \vee \gamma)=f(\alpha, \beta) \vee f(\alpha, \gamma)$,
3. $f(\alpha \vee \beta, \gamma)=f(\alpha, \gamma) \vee f(\beta, \gamma)$,
4. $\varphi^{-1}(f(\alpha, \beta))=f\left(\varphi^{-1}(\alpha), \varphi^{-1}(\beta)\right) \vee \operatorname{Ker}(\varphi)$ for any epimorphism $\varphi: B \rightarrow G$ from an algebra $B$.

Commutator of congruences $\alpha$ and $\beta$ is denoted by $[\alpha, \beta]$. A congruence $\alpha$ is Abelian if $[\alpha, \alpha]=0$. It is known from [4] (see Theorem 5.5, p. 47) that there exists a so-called ternary difference term $d$ such that $d(x, x, y)=y$ is an identity of $M$. Furthermore, if we denote $\left(a_{1}, \ldots, a_{n}\right),\left(b_{1}, \ldots, b_{n}\right)$ and $\left(c_{1}, \ldots, c_{n}\right)$ by a, $\mathbf{b}$ and $\mathbf{c}$, respectively, then a congruence $\alpha \in \operatorname{Con}(G)$ is Abelian if and only if

$$
d(t(\mathbf{a}), t(\mathbf{b}), t(\mathbf{c}))=t\left(d\left(a_{1}, b_{1}, c_{1}\right), \ldots, d\left(a_{n}, b_{n}, c_{n}\right)\right),
$$

for any basic operation $t\left(x_{1}, \ldots, x_{n}\right)=t(\boldsymbol{x})$ and for all elements $a_{i}, b_{i}, c_{i}$ with $a_{i} \alpha b_{i} \alpha c_{i}(1 \leq i \leq n)$. In this case the following properties hold:

1. For any fixed element $\bar{g}$ the congruence class $[\bar{g}]_{\alpha}$ is an Abelian group with respect to the addition

$$
\begin{equation*}
x+y=d(x, \bar{g}, y) \tag{2}
\end{equation*}
$$

with zero element $\bar{g}$. Moreover, $d(x, y, z)=x-y+z$ for all $x, y, z \in[\bar{g}]_{\alpha}$. The set $[\bar{g}]_{\alpha}$ with this operation $d(x, y, z)$ is called ternary group and (2) is called ternary addition.
2. Each term $n$-ary operation $t$ and each ordered set $\left(g_{1}, \ldots, g_{n}\right) \in G^{n}$ define a system of group homomorphisms $h_{i}:\left[g_{i}\right]_{\alpha} \mapsto\left[t\left(g_{1}, \ldots, g_{n}\right)\right]_{\alpha}$ such that

$$
t\left(x_{1}, \ldots, x_{n}\right)=\sum_{i=1}^{n} h_{i}\left(x_{i}\right)+t\left(\overline{g_{1}}, \ldots, \overline{g_{n}}\right),
$$

where $x_{i} \in\left[g_{i}\right]_{\alpha}$. As it is mentioned in [1],

$$
\begin{equation*}
h_{i}(x)=t\left(\overline{g_{1}}, \ldots, \overline{g_{i-1}}, x, \overline{g_{i+1}}, \ldots, \overline{g_{n}}\right)-t\left(\overline{g_{1}}, \ldots, \overline{g_{n}}\right), \tag{3}
\end{equation*}
$$

and these homomorphisms are compatible with compositions of operations.
In particular, the operation $p$ with Mal'cev identities (1) can be taken as the difference term $d$ in any congruence-permutable variety.

Remark 1.1. It is known from [6] that, for any two elements $e$ and $e^{\prime}$ from the same congruence class of an Abelian congruence, the mapping $f(x)=d\left(e^{\prime}, e, x\right)$ is an isomorphism between the ternary groups defined on the given congruence class with the help of two zero elements $e$ and $e^{\prime}$, respectively.

A homomorphism of $\mathcal{M}$-algebras is Abelian if its kernel is an Abelian congruence. We use the following notations from [2]:

$$
I_{G}^{0}=1_{G}, \quad I_{G}^{1}=\left[1_{G}, 1_{G}\right], \ldots, I_{G}^{k}=\left[I_{G}^{k-1}, I_{G}^{k-1}\right] .
$$

An algebra $G$ is solvable of degree at most $k$ if $I_{G}^{k}=0_{G}$.
Let $G \in M$ be generated by a subset $X$. $\mathcal{M}$-algebra $A$ is an Abelian extension of $G$ if $A$ is generated by the same set $X$ and there exists an Abelian epimorphism $\psi: A \rightarrow G$ which is identical on $X$. An Abelian extension $A E(G)$ of $G$, with an Abelian epimorphism $\varphi: A E(G) \rightarrow G$, is said to be free if for any Abelian extension $B$ of $G$, with an Abelian epimorphism $\psi: B \rightarrow G$ being identical on $X$, there exists a homomorphism $\tau: A E(G) \rightarrow B$ such that $\varphi=\psi \tau$. The free Abelian extension can be obtained as follows. Let $F$ be the free $\mathcal{M}$-algebra generated by $X$ and
$\gamma \in \operatorname{Con}(F)$ such that $G=F / \gamma$. Then $A E(G)=F /[\gamma, \gamma]$. The idea of Abelian extension is used intensively in commutator theory. For example, each free solvable algebra of degree $k$ is obtained as a free Abelian extension of free solvable algebra of degree $(k-1)$. The construction of free solvable algebra with one ternary Mal'cev operation $p$ is given in [3]. These results were generalized in [2] for a general congruence modular variety. The paper [1] contains the general approach to the construction of free Abelian extensions in any given congruence modular variety.

Now let $\Omega$ be a system of operations. We use the ideas from [1] and apply the results obtained in [8] and [9] for $\langle p, S\rangle$-algebras to the construction of free Abelian extensions in any congruence-permutable variety. The main result of the present paper is the Theorem 2.14. We also apply this construction to the structure of free solvable algebras.

## 2. Construction of the free Abelian extension

Consider a congruence-permutable variety $\mathcal{V}$ of $\Omega$-algebras, and let $p$ be a term operation from Theorem 1.1. We denote the clone of $\mathcal{V}$ by $T=\left\{T_{n} \mid n \in \mathbb{N}\right\}$, where $T_{n}$ is the set of all $n$-ary term operations that are distinct in $\mathcal{V}$. Recall that $T$ is a system of operations which is closed under all compositions and contains all projections i.e. the operations $p_{j n}$ such that $p_{j n}\left(x_{1}, \ldots x_{n}\right)=x_{j}, \quad j=1, \ldots, n$. Let $A$ be an arbitrary algebra from $\mathcal{V}$ with a fixed Abelian congruence $\alpha$ and a set of generators $X$. Consider a set $E$ of representatives of $\alpha$-cosets such that:

1. if $x \in X$ and $e \in E \cap[x]_{\alpha}$, then $e \in X$;
2. if $e \in E$, then

$$
\begin{equation*}
e=t_{e}\left(x_{1}, \ldots, x_{n}\right) \tag{4}
\end{equation*}
$$

for some $n$-ary term $t_{e}$, where $x_{1}, \ldots, x_{n} \in X \cap E$.
Any class $[e]_{\alpha}, e \in E$, will be considered as a ternary group with the zero element $e$. Following [1] denote by $S_{\Omega}$ the set of all symbols

$$
\frac{\partial t}{\partial i}(\mathbf{e}), \frac{\partial p_{j n}}{\partial i}(\mathbf{e})
$$

for each positive integer $n$, for each $n$-ary term $t$ and for all $\mathbf{e}=\left(e_{1}, \ldots, e_{n}\right) \in$ $E^{n}$. Let $h\left(x_{1}, \ldots, x_{n}\right), g_{i}\left(x_{1}, \ldots, x_{m_{i}}\right)$ be arbitrary term operations on $A$, $i=1, \ldots, n$. Put

$$
d_{i}= \begin{cases}0, & i=0 \\ m_{1}+\cdots+m_{i}, & i=1, \ldots, n\end{cases}
$$

Consider now the following term operations on $A$ :

$$
h=t\left(g_{1}\left(x_{1}, \ldots, x_{d_{1}}\right), \ldots, g_{n}\left(x_{d_{m-1}+1}, \ldots, x_{d_{m}}\right)\right) .
$$

It is shown in [1] (see Proposition 2.1) that if $\mathbf{e}_{j} \in E^{m_{j}}, 1 \leq j \leq n, \mathbf{e}=$ $\left(e_{1}, \ldots, e_{n}\right)$, then

$$
\begin{equation*}
\frac{\partial h}{\partial i}(\mathbf{e})=\left(\frac{\partial t}{\partial j}\left(g_{1}\left(\mathbf{e}_{1}\right), \ldots, g_{n}\left(\mathbf{e}_{n}\right)\right)\right)\left(\frac{\partial g_{j}}{\partial\left(i-d_{j-1}\right)}\left(\mathbf{e}_{j}\right)\right) \tag{5}
\end{equation*}
$$

where $d_{j-1} \leq i \leq d_{j}$. It means that $S_{\Omega}$ is closed under multiplication (5).
Proposition 2.1. Multiplication (5) is associative.

Proof. Let
(6) $\delta=\frac{\partial t}{\partial i}\left(a_{1}, \ldots, a_{s}\right), \quad \beta=\frac{\partial u}{\partial j}\left(b_{1}, \ldots, b_{q}\right), \quad \gamma=\frac{\partial v}{\partial k}\left(c_{1}, \ldots, c_{r}\right)$,
where $a_{1}, \ldots, a_{s}, b_{1}, \ldots, b_{q}, c_{1}, \ldots, c_{r} \in E$. Then

$$
\beta \gamma=\frac{\partial h}{\partial(j+k-1)}\left(b_{1}, \ldots, b_{j-1}, c_{1}, \ldots, c_{r}, b_{j+1}, \ldots, b_{q}\right),
$$

where $h=u\left(x_{1}, \ldots, x_{j-1}, v\left(x_{j}, \ldots, x_{j+r-1}\right), x_{j+r}, \ldots, x_{q+r-1}\right)$, and therefore
(7) $\delta(\beta \gamma)=$

$$
=\frac{\partial g}{\partial(i+j+k-2)}\left(a_{1}, \ldots, a_{i-1}, b_{1}, \ldots, b_{j-1}, c_{1}, \ldots, c_{r}, b_{j+1}, \ldots, b_{q}, a_{i+1}, \ldots, a_{s}\right),
$$

where

$$
\begin{aligned}
g=t\left(x_{1}, \ldots, x_{i-1}, u( \right. & x_{i}, \ldots, x_{i+j-2}, v\left(x_{i+j-1}, \ldots, x_{i+j+r-2}\right) \\
& \left.\left.x_{i+j+r-1}, \ldots, x_{i+q+r-2}\right), x_{i+q+r-1}, \ldots, x_{s+q+r-2}\right) .
\end{aligned}
$$

On the other hand

$$
\delta \beta=\frac{\partial w}{\partial(i+j-1)}\left(a_{1}, \ldots, a_{i-1}, b_{1}, \ldots, b_{q}, a_{i+1}, \ldots, a_{s}\right)
$$

where

$$
\begin{equation*}
w=t\left(x_{1}, \ldots, x_{i-1}, u\left(x_{i}, \ldots, x_{i+q-1}\right), x_{i+q}, \ldots, x_{s+q-1}\right) \tag{8}
\end{equation*}
$$

At the final step we calculate $(\delta \beta) \gamma$ and show that it is equal to (7).
Assign to each element $\frac{\partial t}{\partial i}\left(e_{1}, \ldots, e_{n}\right) \in S_{\Omega}$ the unary operation on $A$ as follows:

$$
\begin{equation*}
f_{\frac{\partial t}{\partial i}\left(e_{1}, \ldots, e_{n}\right)}(x)=t\left(e_{1}, \ldots, e_{i-1}, x, e_{i+1} \ldots, e_{n}\right) \tag{9}
\end{equation*}
$$

Proposition 2.2. The equality (9) defines an action on $A$ of the monoid $S_{\Omega}$ with the multiplication (5).

Proof. Suppose that $\delta, \beta$ are from (6), and $x \in A$. Then

$$
\begin{aligned}
& f_{\beta}(x)=u\left(b_{1}, \ldots, b_{j-1}, x, a_{j+1}, \ldots, b_{q}\right) \\
& f_{\delta}\left(f_{\beta}(x)\right)=t\left(a_{1}, \ldots, a_{i-1}, u\left(b_{1}, \ldots, b_{j-1}, x, b_{j+1}, \ldots, b_{q}\right), a_{i+1}, \ldots, a_{s}\right)= \\
& =f_{\frac{\partial w}{\partial(i+j-1)}\left(a_{1}, \ldots, a_{i-1}, b_{1}, \ldots, b_{q}, a_{i+1}, \ldots, a_{s}\right)}(x)=f_{\delta \beta}(x)
\end{aligned}
$$

where $w$ is from (8).

Here are some properties of the action (9):

1. if $p_{i n}$ is a projection, then

$$
f_{\frac{\partial p_{i n}}{\partial i}(\mathbf{e})}(x)=x ;
$$

2. if $\mathbf{e} \in E^{n}$ and $t$ is an arbitrary $n$-ary term operation, then

$$
\begin{equation*}
f_{\frac{\partial t}{\partial i}(\mathbf{e})}\left(e_{i}\right)=t(\mathbf{e}), \quad i=1, \ldots, n \tag{10}
\end{equation*}
$$

Let $\left[\frac{\partial t}{\partial i}(\mathbf{e})\right]$ stand for the homomorphism $h_{i}$ from (3). For all $a_{1} \in\left[e_{1}\right]_{\alpha}, \ldots, a_{n} \in$ $\left[e_{n}\right]_{\alpha}$ we have

$$
\begin{equation*}
t\left(a_{1}, \ldots, a_{n}\right)=\sum_{i=1}^{n}\left[\frac{\partial t}{\partial i}(\mathbf{e})\right]\left(a_{i}\right)+t\left(e_{1}, \ldots, e_{n}\right) \tag{11}
\end{equation*}
$$

Denote by $\theta$ the congruence on $S_{\Omega}$ generated by all pairs of the form:

$$
\left(\frac{\partial p}{\partial 2}(e, e, e), 1\right)
$$

$$
\begin{equation*}
\left(\frac{\partial f}{\partial i}(\mathbf{e}), \frac{\partial g}{\partial i}(\mathbf{e})\right) \tag{12}
\end{equation*}
$$

$$
\begin{equation*}
\left(\frac{\partial p_{i n}}{\partial i}(\mathbf{e}), 1\right) \tag{13}
\end{equation*}
$$

(14) $\left(\frac{\partial p_{i n}}{\partial j}\left(e_{1}, \ldots, e_{n}\right), \frac{\partial p_{k m}}{\partial l}\left(e_{1}^{\prime}, \ldots, e_{m}^{\prime}\right)\right)$, for $i \neq j, k \neq l, e_{i}=e_{k}^{\prime}$,
(15) $\left(\frac{\partial t}{\partial i}\left(e_{1}, \ldots, e_{i-1}, c, e_{i+1}, \ldots, e_{n}\right), \frac{\partial t}{\partial i}\left(e_{1}, \ldots, e_{i-1}, d, e_{i+1}, \ldots, e_{n}\right)\right)$,
for $c, d \in E$,
where $\mathbf{e} \in E^{n}$, and $f\left(x_{1}, \ldots, x_{n}\right)=g\left(x_{1}, \ldots, x_{n}\right)$ is a defining identity of $\mathcal{V}$ [1]; (see Theorem 2.4). The monoid $S_{\Omega} / \theta$ will be denoted by $S(E)$.

Proposition 2.3. Each $\theta$-class generated by $\frac{\partial p_{i n}}{\partial j}\left(e_{1}, \ldots, e_{n}\right)$ for $i \neq j$ is a left zero of $S(E)$.

Proof. Without loss of generality, we put $j<i$. If $t\left(x_{1}, \ldots, x_{n}\right)$ is a term operation, then we set

$$
\begin{aligned}
& h\left(x_{1}, \ldots, x_{m+n-1}\right)= \\
& =p_{(m+i-1),(m+n-1)}\left(x_{1}, \ldots, x_{j-1}, t\left(x_{j}, \ldots, x_{m+j-1}\right), x_{m+j}, \ldots, x_{m+n-1}\right)
\end{aligned}
$$

Observe that the identity

$$
h\left(x_{1}, \ldots, x_{m+n-1}\right)=p_{(m+i-1),(m+n-1)}\left(x_{1}, \ldots, x_{m+n-1}\right)
$$

holds on $\mathcal{V}$. By (12), (14), we have

$$
\begin{aligned}
& \frac{\partial p_{i n}}{\partial j}\left(e_{1}, \ldots, e_{n}\right) \frac{\partial t}{\partial k}\left(e_{1}^{\prime}, \ldots, e_{m}^{\prime}\right)= \\
& =\frac{\partial h}{\partial(j+k-1)}\left(e_{1}, \ldots, e_{j-1}, e_{1}^{\prime}, \ldots, e_{m}^{\prime}, e_{j+1}, \ldots, e_{n}\right)= \\
& =\frac{\partial p_{(m+i-1)(m+n-1)}}{\partial(j+k-1)}\left(e_{1}, \ldots, e_{j-1}, e_{1}^{\prime}, \ldots, e_{m}^{\prime}, e_{j+1}, \ldots, e_{n}\right) \theta \frac{\partial p_{i n}}{\partial j}\left(e_{1}, \ldots, e_{n}\right)
\end{aligned}
$$

It has proved that $A$ is a polygon over the monoid $S(E)$. A ternary operation $p$ satisfying Mal'cev identities (1) is also defined on $A$. Hence $A$ is a $\langle p, S(E)\rangle$-algebra in the sense of [9]. Moreover, we will prove the following fact.

Proposition 2.4. $X$ generates the $\langle p, S(E)\rangle$-algebra $A$. If any subset $Y$ generates the $\langle p, S(E)\rangle$-algebra $A$, then $Y \cup E$ generates $\Omega$-algebra.

Proof. Recall that for any $x \in X$ the zero of the ternary Abelian group $[x]_{\alpha}$ belongs to $X$. Let $u \in A$. Then for some operation $t$ and $x_{1}, \ldots, x_{n} \in X$ we have

$$
\begin{aligned}
& u=t\left(x_{1}, \ldots, x_{n}\right)= \\
& =\sum_{i=1}^{n}\left(f_{\frac{\partial t}{\partial 1}\left(e_{1}, \ldots, e_{n}\right)}\left(x_{i}\right)-f_{\frac{\partial t}{\partial i}\left(e_{1}, \ldots, e_{n}\right)}\left(e_{i}\right)\right)+f_{\frac{\partial t}{\partial 1}\left(e_{1}, \ldots, e_{n}\right)}\left(e_{1}\right) .
\end{aligned}
$$

Note that $e_{1} \in X$ and if $e \in E \cap[u]_{\alpha}$, then $e=t_{e}\left(x_{1}^{\prime}, \ldots, x_{n}^{\prime}\right)$ for some $x_{1}^{\prime}, \ldots, x_{n}^{\prime} \in X$. Now let a set $Y$ generate $A$ as a $\langle p, S(E)\rangle$-algebra. Each operation from $S(E)$ is a polynomial of the $\Omega$-algebra $A$. Since each element from $X$ is the value of some term for the elements of $Y \cup E$, then it is also true for an arbitrary element $u$ from (16).

Proposition 2.5. Congruence $\alpha$ is Abelian on the $\langle p, S(E)\rangle$-algebra $A$.

Proof. Assume that $(u, v) \in \alpha$. From (15) we get

$$
\begin{aligned}
& \alpha \ni\left(t\left(e_{1}, \ldots, e_{i-1}, u, e_{i+1}, \ldots, e_{n}\right), t\left(e_{1}, \ldots, e_{i-1}, v, e_{i+1}, \ldots, e_{n}\right)\right)= \\
& =\left(f_{\frac{\partial t}{\partial i}(\mathbf{e})}(u), f_{\frac{\partial t}{\partial i}(\mathbf{e})}(v)\right) .
\end{aligned}
$$

Consequently $\alpha$ is a congruence with respect to the new operations. We know from [7] that the commutator [ $\alpha, \alpha$ ] on $\langle p, S(E)\rangle$-algebra $A$ is generated by all pairs of the form

$$
\begin{array}{r}
\left(p\left(p\left(u_{1}, u_{2}, u_{3}\right), p\left(v_{1}, v_{2}, v_{3}\right), p\left(w_{1}, w_{2}, w_{3}\right)\right),\right. \\
\left.p\left(p\left(u_{1}, v_{1}, w_{1}\right), p\left(u_{2}, v_{2}, w_{2}\right), p\left(u_{3}, v_{3}, w_{3}\right)\right)\right), \\
\left(p\left(f_{\frac{\partial t}{\partial i}(\mathbf{e})}(u), f_{\frac{\partial t}{\partial i}(\mathbf{e})}(v), f_{\frac{\partial t}{\partial i}(\mathbf{e})}(w)\right), f_{\frac{\partial t}{\partial i}(\mathbf{e})}(p(u, v, w))\right), \tag{18}
\end{array}
$$

where $u_{i}, v_{i}, w_{i}, u, v, w$ are congruent modulo $\alpha, \quad i=1,2,3$. In terms of $\Omega$-algebra the congruence $[\alpha, \alpha]$ is generated by pairs (17), and also by the pairs:

$$
\begin{align*}
& \left(p\left(t\left(u_{1}, \ldots, u_{n}\right), t\left(v_{1}, \ldots, v_{n}\right), \quad t\left(w_{1}, \ldots, w_{n}\right)\right)\right.  \tag{19}\\
& \left.t\left(p\left(u_{1}, v_{1}, w_{1}\right), \ldots, p\left(u_{n}, v_{n}, w_{n}\right)\right)\right)
\end{align*}
$$

where $u_{i}, v_{i}, w_{i}$ are congruent modulo $\alpha, \quad i=1, \ldots, n$. But, as one can see, the pairs from (18) belong to the set of those from (19) and that all the pairs from (17), (18) generate the smallest congruence on $A$. Hence $\alpha$ is Abelian.

Remark 2.1. Let $B$ be a $\mathcal{V}$-algebra such that there exists a homomorphism $\lambda$ from $A$ onto $B$ and $\operatorname{Ker}(\lambda) \subseteq \alpha$. Then, by the basic properties of commutators, $\lambda(\alpha)$ is an Abelian congruence. For each $\mathbf{e} \in E^{n}, t \in T_{n}, b \in B$ we put

$$
f_{\frac{\partial t}{\partial i}(\mathbf{e})}(b)=t\left(\lambda\left(e_{1}\right), \ldots, \lambda\left(e_{i-1}\right), b, \lambda\left(e_{i+1}\right), \ldots, \lambda\left(e_{n}\right)\right)
$$

Thus $B$ becomes a $\langle p, S(E)\rangle$-algebra. Moreover, $\lambda$ is an Abelian homomorphism between the two $\langle p, S(E)\rangle$-algebras.

Rremark 2.2. We can generalize the preceding remark. In fact, $E$ can be viewed as an $\Omega$-algebra isomorphic to $A / \alpha$. By Remark 2.1, each Abelian extension of $E$ (including $E$ itself) is a $\langle p, S(E)\rangle$-algebra where the elements from $E$ are fixed by (4). The obtained algebra is an Abelian extension of $E$.

Let $D$ be a $\langle p, S(E)\rangle$-algebra generated by $X$ and there exists an epimorphism $\xi: D \rightarrow A$ such that $\xi(X)=X$ and the congruence $\beta=\xi^{-1}(\alpha)$ is Abelian. Since $\operatorname{Ker}(\xi) \subseteq \beta$ then $D$ is an Abelian extension of $A$. Consider a set $E^{\prime}$ of all elements $f_{\frac{\partial t_{e}}{\partial 1}\left(x_{1} \ldots, x_{n}\right)}\left(x_{1}\right)$. Certainly, $\xi\left(E^{\prime}\right)=E$. Since there is only one element of $E^{\prime}$ in each $\beta$-class, then we can treat each element from $E^{\prime}$ as the zero element of the corresponding ternary group.

Proposition 2.6. The restriction of $\xi$ to each $\beta$-class is a group epimorphism.

Proof. First we note that $\xi$ preserves the operation $p$. As mentioned above, $\xi\left(E^{\prime}\right)=E$ and hence $\xi$ preserves the addition on each $\beta$-class as ternary group.

## Proposition 2.7.

$$
\xi\left(\left[\frac{\partial t}{\partial i}(\mathbf{e})\right](u)\right)=\left[\frac{\partial t}{\partial i}(\mathbf{e})\right](\xi(u))
$$

for each term operation $t\left(x_{1}, \ldots, x_{n}\right)$ and for all $\mathbf{e} \in E^{n}$.

Proof. For $e_{i}^{\prime} \in \xi^{-1}\left(e_{i}\right) \cap E^{\prime}$ we get

$$
\begin{aligned}
\xi\left(\left[\frac{\partial t}{\partial i}(\mathbf{e})\right](u)\right)= & \xi\left(f_{\frac{\partial t}{\partial i}(\mathbf{e})}(u)-f_{\frac{\partial t}{\partial i}(\mathbf{e})}\left(e_{i}^{\prime}\right)\right)= \\
& =f_{\frac{\partial t}{\partial i}(\mathbf{e})}(\xi(u))-f_{\frac{\partial t}{\partial i}(\mathbf{e})}\left(e_{i}\right)=\left[\frac{\partial t}{\partial i}(\mathbf{e})\right](\xi(u))
\end{aligned}
$$

Let $\omega$ be a congruence on $D$ generated by pairs of the form

$$
\left(f_{\frac{\partial t\left(g_{1}, \ldots, g_{n}\right)}{\partial i}(\mathbf{e})}(u),\right.
$$

$$
\begin{equation*}
\left.\sum_{j=1}^{n} f_{\frac{\partial h_{j}}{\partial(i+j-1)}\left(\overline{g_{1}}, \ldots, \overline{g_{j-1}}, \mathbf{e}, \overline{g_{j+1}} \ldots, \overline{g_{n}}\right)}(u)-(n-1) f_{\frac{\partial t}{\partial 1}\left(\overline{g_{1}}, \ldots, \overline{g_{n}}\right)}\left(g_{1}^{\prime}\right)\right) \tag{20}
\end{equation*}
$$

$$
\begin{equation*}
\left(f_{\frac{\partial g}{\partial 1}(\mathbf{e})}\left(e_{1}^{\prime}\right), f_{\frac{\partial g}{\partial j}(\mathbf{e})}\left(e_{j}^{\prime}\right)\right), \quad j=2, \ldots, m \tag{21}
\end{equation*}
$$

$$
\begin{equation*}
\left(f_{\frac{\partial p_{j m}}{\partial i}(\mathbf{e})}(u), e_{j}^{\prime}\right), \quad j \neq i \tag{22}
\end{equation*}
$$

Here $g$ is an $m$-ary term operation, $\mathbf{e}=\left(e_{1}, \ldots, e_{m}\right) \in E^{m}, e_{i}^{\prime} \in E^{\prime}$, $\xi\left(e_{i}^{\prime}\right)=e_{i}, u \in\left[e_{i}^{\prime}\right]_{\beta}, \quad i=1, \ldots, m$,

$$
h_{j}=t\left(x_{1}, \ldots, x_{j-1}, g_{j}\left(x_{j}, \ldots, x_{j+m-1}\right), x_{j+m}, \ldots, x_{n+m-1}\right)
$$

$\overline{g_{i}}=E \cap\left[g_{i}(\mathbf{e})\right]_{\alpha}$, and $g_{i}^{\prime} \in E^{\prime}$ such that $\xi\left(g_{i}^{\prime}\right)=\overline{g_{i}}$. The sum in (20) is denoting the addition in the group $\left[t\left(g_{1}, \ldots, g_{n}\right)\left(e_{1}^{\prime}, \ldots, e_{m}^{\prime}\right)\right]_{\beta}$.

Proposition 2.8. $\omega \leq \operatorname{Ker}(\xi)$.

Proof. A direct calculation shows that pairs (21), (22) are in $\operatorname{Ker}(\xi)$. If $e_{1}, \ldots, e_{m} \in E$ then we have

$$
\begin{equation*}
t\left(g_{1}, \ldots, g_{n}\right)\left(e_{1}, \ldots, e_{m}\right)=t\left(g_{1}\left(e_{1}, \ldots, e_{m}\right), \ldots, g_{n}\left(e_{1}, \ldots, e_{m}\right)\right) \tag{23}
\end{equation*}
$$

Therefore each pair of the form (20) belongs to $\operatorname{Ker}(\xi)$.
It follows immediately from Proposition 2.8 that the congruence $\omega$ is Abelian and $D / \omega$ is an Abelian extension of $A$. Let $\rho$ be the fractional congruence $\beta / \omega$. For each $a \in A$, we denote by $\rho(a)$ the $\rho$-class corresponding to $a$.

Proposition 2.9. $\rho$ is an Abelian congruence.

Proof. Let $\varepsilon: D \rightarrow D / \omega$ be the natural homomorphism. Then,

$$
\varepsilon^{-1}([\rho, \rho])=\left[\varepsilon^{-1}(\rho), \varepsilon^{-1}(\rho)\right] \vee \omega=[\beta, \beta] \vee \omega=\omega
$$

Since all pairs (21) belong to $\omega$ then we can write $\widetilde{t}\left(e_{1}^{\prime}, \ldots, e_{n}^{\prime}\right)$ for $f_{\frac{\partial t}{\partial j}(\mathbf{e})}\left(e_{j}^{\prime}\right)$, $j=1, \ldots, n$ where $\mathbf{e}, e_{j}^{\prime}$ are such as in (21). Let $t$ be an arbitrary term $n$-ary operation from $\Omega$. Put

$$
\begin{equation*}
t\left(b_{1}, \ldots, b_{n}\right)=\sum_{i=1}^{n}\left[\frac{\partial t}{\partial i}\left(e_{1}, \ldots, e_{n}\right)\right]\left(b_{i}\right)+\widetilde{t}\left(e_{1}^{\prime}, \ldots, e_{n}^{\prime}\right) \tag{24}
\end{equation*}
$$

for all $b_{1}, \ldots, b_{n}$ from $\rho\left(e_{1}\right), \ldots, \rho\left(e_{n}\right)$ respectively.
Proposition 2.10. $D / \omega$ is a $\mathcal{V}$-algebra with respect to the operations (24).

Proof. We have to check that (24) defines a homomorphism from the clone $T$ of all term operations on $\mathcal{V}$ to the clone $\mathcal{O}(D / \omega)$ of operations on $D / \omega$. From (22), (13), we see that

$$
p_{i m}\left(u_{1}, \ldots, u_{m}\right)=\sum_{j=1}^{m}\left[\frac{\partial p_{i m}}{\partial j}(\mathbf{e})\right]\left(u_{i}\right)+\widetilde{p_{i m}}\left(\mathbf{e}^{\prime}\right)=u_{i}+m e_{i}^{\prime}=u_{i}
$$

for $u \in \rho\left(e_{i}\right) . \operatorname{By}(20),(5)$, for $t, h_{i}, \mathbf{e}, \mathbf{e}^{\prime}$ and $u$ such as in (20), we get

$$
\begin{aligned}
& {\left[\frac{\partial t\left(g_{1}, \ldots, g_{n}\right)}{\partial i}(\mathbf{e})\right](u)=f_{\frac{\partial t\left(g_{1}, \ldots, g_{n}\right)}{\partial i}(\mathbf{e})}(u)-f_{\frac{\partial t\left(g_{1}, \ldots, g_{n}\right)}{\partial i}(\mathbf{e})}\left(e_{i}^{\prime}\right)=} \\
& =\sum_{j=1}^{n} f_{\frac{\partial h_{j}}{\partial(i+j-1)}\left(\overline{g_{1}}, \ldots, \overline{g_{j-1}}, \mathbf{e}, \overline{g_{j+1}} \ldots, \overline{g_{n}}\right)}(u)-(n-1) f_{\frac{\partial t}{\partial 1}\left(\overline{g_{1}}, \ldots, \overline{g_{n}}\right)}\left(g_{1}^{\prime}\right)- \\
& -\sum_{j=1}^{n} f_{\frac{\partial h_{j}}{\partial(i+j-1)}\left(\overline{g_{1}}, \ldots, \overline{g_{j-1}}, \mathbf{e}, \overline{g_{j+1}} \ldots, \overline{\left.g_{n}\right)}\right.}\left(e_{i}^{\prime}\right)+(n-1) f_{\frac{\partial t}{\partial 1}\left(\overline{g_{1}}, \ldots, \overline{\left.g_{n}\right)}\right.}\left(g_{1}^{\prime}\right)= \\
& =\sum_{j=1}^{n}\left[\frac{\partial h_{j}}{\partial(i+j-1)}\left(\overline{g_{1}}, \ldots, \overline{g_{j-1}}, \mathbf{e}, \overline{g_{j+1}} \ldots, \overline{g_{n}}\right)\right](u)= \\
& =\sum_{j=1}^{n}\left[\frac{\partial t}{\partial j}\left(\overline{g_{1}}, \ldots, \overline{g_{n}}\right)\right]\left[\frac{\partial g_{j}}{\partial i}(\mathbf{e})\right](u) .
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
& t\left(g_{1}, \ldots, g_{n}\right)(\mathbf{e})=f_{\frac{\partial t\left(g_{1}, \ldots, g_{n}\right)}{\partial 1}(\mathbf{e})}\left(e_{1}^{\prime}\right)= \\
& =\sum_{j=1}^{n} f_{\frac{\partial h_{j}}{\partial j}\left(\overline{g_{1}}, \ldots, \overline{g_{j-1}}, \mathbf{e}, \overline{g_{j+1}} \ldots, \overline{g_{n}}\right)}\left(e_{1}^{\prime}\right)-(n-1) f_{\frac{\partial t}{\partial 1}\left(\overline{g_{1}}, \ldots, \overline{g_{n}}\right)}\left(g_{1}^{\prime}\right)= \\
& =\sum_{j=1}^{n} f_{\frac{\partial t}{\partial j}\left(\overline{g_{1}}, \ldots, \overline{g_{n}}\right)}\left(f_{\frac{\partial g_{j}}{\partial 1}(\mathbf{e})}\left(e_{1}^{\prime}\right)\right)-(n-1) f_{\frac{\partial t}{\partial 1}\left(\overline{g_{1}}, \ldots, \overline{g_{n}}\right)}\left(g_{1}^{\prime}\right)= \\
& =\sum_{j=1}^{n}\left[\frac{\partial t}{\partial j}\left(\overline{g_{1}}, \ldots, \overline{g_{n}}\right)\right]\left(f_{\frac{\partial g_{j}}{\partial 1}(\mathbf{e})}\left(e_{1}^{\prime}\right)\right)+f_{\frac{\partial t}{\partial 1}\left(\overline{g_{1}}, \ldots, \overline{g_{n}}\right)}\left(g_{1}^{\prime}\right)= \\
& =\sum_{j=1}^{n}\left[\frac{\partial t}{\partial j}\left(\overline{g_{1}}, \ldots, \overline{g_{n}}\right)\right]\left(\widetilde{g_{j}}(\mathbf{e})\right)+f_{\frac{\partial t}{\partial 1}\left(\overline{g_{1}}, \ldots, \overline{g_{n}}\right)}\left(g_{1}^{\prime}\right) .
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
& t\left(g_{1}, \ldots, g_{n}\right)\left(u_{1}, \ldots, u_{m}\right)= \\
& =\sum_{i=1}^{m}\left[\frac{\partial t\left(g_{1}, \ldots, g_{n}\right)}{\partial i}(\mathbf{e})\right]\left(u_{i}\right)+t\left(g_{1}, \ldots, g_{n}\right)(a)= \\
& =\sum_{i=1}^{m} \sum_{j=1}^{n}\left[\frac{\partial t}{\partial j}\left(\overline{g_{1}}, \ldots, \overline{g_{n}}\right)\right]\left[\frac{\partial g_{j}}{\partial i}(\mathbf{e})\right]\left(u_{i}\right)+\sum_{j=1}^{n}\left[\frac{\partial t}{\partial j}\left(\overline{g_{1}}, \ldots, \overline{g_{n}}\right)\right]\left(\widetilde{g_{j}}(\mathbf{e})\right)+ \\
& +f_{\frac{\partial t}{\partial 1}\left(\overline{g_{1}}, \ldots, \overline{g_{n}}\right)}\left(g_{1}^{\prime}\right)=\sum_{j=1}^{n}\left[\frac{\partial t}{\partial j}\left(\overline{g_{1}}, \ldots, \overline{g_{n}}\right)\right]\left(\sum_{i=1}^{m}\left[\frac{\partial g_{j}}{\partial i}(\mathbf{e})\right]\left(u_{i}\right)+\widetilde{g_{j}}(\mathbf{e})\right)= \\
& +f_{\frac{\partial t}{\partial 1}\left(\overline{\left.g_{1}, \ldots, \overline{g_{n}}\right)}\right.}\left(g_{1}^{\prime}\right)=t\left(g_{1}\left(u_{1}, \ldots, u_{m}\right), \ldots, g_{n}\left(u_{1}, \ldots, u_{m}\right)\right) .
\end{aligned}
$$

Hence, (23) holds on $D / \omega$. Finally, if the equality $f_{1}\left(x_{1}, \ldots, x_{k}\right)=$ $f_{2}\left(x_{1}, \ldots, x_{k}\right)$ holds in $T$, then, by (12), it also holds on $D / \omega$.

In particular, we observe the following important fact.

Proposition 2.11. The operation

$$
\begin{aligned}
& p^{\prime}\left(u_{1}, u_{2}, u_{3}\right)= \\
& =\left[\frac{\partial p}{\partial 1}\left(e_{1}, e_{2}, e_{3}\right)\right]\left(u_{1}\right)+\left[\frac{\partial p}{\partial 2}\left(e_{1}, e_{2}, e_{3}\right)\right]\left(u_{2}\right)+ \\
& +\left[\frac{\partial p}{\partial 3}\left(e_{1}, e_{2}, e_{3}\right)\right]\left(u_{3}\right)+\widetilde{p}\left(e_{1}^{\prime}, e_{2}^{\prime}, e_{3}^{\prime}\right)
\end{aligned}
$$

where $u_{i} \in\left[e_{i}^{\prime}\right]_{\beta}$ and $e_{i}=\xi\left(e_{i}^{\prime}\right) \in E$, satisfies the Mal'cev identities (1).

Proof. Let $f=p\left(p_{12}, p_{12}, p_{22}\right)$. Then $f(x, y)=p_{22}(x, y)$ is an identity of $\mathcal{V}$. By (12), (14) and (20), we get

$$
\begin{aligned}
& p^{\prime}(a, a, b)= \\
& =\left[\frac{\partial p}{\partial 1}\left(e_{1}, e_{1}, e_{2}\right)\right](a)+\left[\frac{\partial p}{\partial 2}\left(e_{1}, e_{1}, e_{2}\right)\right](a)+ \\
& +\left[\frac{\partial p}{\partial 3}\left(e_{1}, e_{1}, e_{2}\right)\right](b)+\widetilde{p}\left(e_{1}^{\prime}, e_{1}^{\prime}, e_{2}^{\prime}\right)= \\
& =\left[\frac{\partial p}{\partial 1}\left(p_{12}\left(e_{1}, e_{2}\right), p_{12}\left(e_{1}, e_{2}\right), p_{22}\left(e_{1}, e_{2}\right)\right)\right]\left[\frac{\partial p_{12}}{\partial 1}\left(e_{1}, e_{2}\right)\right](a)+ \\
& +\left[\frac{\partial p}{\partial 2}\left(p_{12}\left(e_{1}, e_{2}\right), p_{12}\left(e_{1}, e_{2}\right), p_{22}\left(e_{1}, e_{2}\right)\right)\right]\left[\frac{\partial p_{12}}{\partial 1}\left(e_{1}, e_{2}\right)\right](a)+ \\
& +\left[\frac{\partial p}{\partial 3}\left(p_{12}\left(e_{1}, e_{2}\right), p_{12}\left(e_{1}, e_{2}\right), p_{22}\left(e_{1}, e_{2}\right)\right)\right]\left[\frac{\partial p_{22}}{\partial 1}\left(e_{1}, e_{2}\right)\right](a)+ \\
& +\left[\frac{\partial p}{\partial 1}\left(p_{12}\left(e_{1}, e_{2}\right), p_{12}\left(e_{1}, e_{2}\right), p_{22}\left(e_{1}, e_{2}\right)\right)\right]\left[\frac{\partial p_{12}}{\partial 2}\left(e_{1}, e_{2}\right)\right](b)+ \\
& +\left[\frac{\partial p}{\partial 2}\left(p_{12}\left(e_{1}, e_{2}\right), p_{12}\left(e_{1}, e_{2}\right), p_{22}\left(e_{1}, e_{2}\right)\right)\right]\left[\frac{\partial p_{12}}{\partial 2}\left(e_{1}, e_{2}\right)\right](b)+ \\
& +\left[\frac{\partial p}{\partial 3}\left(p_{12}\left(e_{1}, e_{2}\right), p_{12}\left(e_{1}, e_{2}\right), p_{22}\left(e_{1}, e_{2}\right)\right)\right]\left[\frac{\partial p_{22}}{\partial 2}\left(e_{1}, e_{2}\right)\right](b)+e_{2}^{\prime}= \\
& =\left[\frac{\partial f}{\partial 1}\left(e_{1}, e_{2}\right)\right](a)+\left[\frac{\partial f}{\partial 2}\left(e_{1}, e_{2}\right)\right](b)+\widetilde{f}\left(e_{1}^{\prime}, e_{2}^{\prime}\right)=e_{2}^{\prime}+1(b)+e_{2}^{\prime}=b .
\end{aligned}
$$

Proposition 2.12. $p^{\prime}$ coincides with $p$ on each $\rho$-class.

Proof. Let $a, b, c \in\left[e^{\prime}\right]_{\rho}, e^{\prime} \in E^{\prime}$, and $e^{*}=\xi\left(e^{\prime}\right)$. Since $p$ commutes with $p^{\prime}$ on $\left[e^{\prime}\right]_{\rho}$, then by (11), (21)

$$
\begin{aligned}
p^{\prime}(a, b, c)= & p^{\prime}\left(p\left(a, e^{\prime}, e^{\prime}\right), p\left(e^{\prime}, b, e^{\prime}\right), p\left(e^{\prime}, e^{\prime}, c\right)\right)= \\
& =p\left(p^{\prime}\left(a, e^{\prime}, e^{\prime}\right), p^{\prime}\left(e^{\prime}, b, e^{\prime}\right), p^{\prime}\left(e^{\prime}, e^{\prime}, c\right)\right)= \\
& =a+\left[\frac{\partial p}{\partial 2}\left(e^{*}, e^{*}, e^{*}\right)\right](b)+c=a+b+c
\end{aligned}
$$

Let $F$ be the subalgebra in $D / \omega$ generated by $X$ with respect to the operations (24).

Theorem 2.13. $\xi$ induces an Abelian epimorphism of $\Omega$-algebras $F \rightarrow A$.

Proof. Since $\omega \subseteq \operatorname{Ker}(\xi)$ then there is an epimorphism of $\langle p, S(E)\rangle$ algebras $\varphi: D / \omega \rightarrow A, \quad[u]_{\omega} \mapsto \xi(u)$. Observe that

$$
\varphi \widetilde{t}\left(e_{1}^{\prime}, \ldots, e_{n}^{\prime}\right)=\varphi\left(f_{\frac{\partial t}{\partial 1}\left(e_{1}, \ldots, e_{n}\right)}\left(e_{1}^{\prime}\right)\right)=t\left(e_{1}, \ldots, e_{n}\right)
$$

by (10) and thus, by (11), (24), the mapping $\varphi$ commutes with each operation from $\Omega$. Moreover,

$$
\begin{equation*}
\operatorname{Ker}(\varphi)=\operatorname{Ker}(\xi) / \omega \subseteq \beta / \omega=\rho \tag{25}
\end{equation*}
$$

Thus $\operatorname{Ker}(\varphi)$ is Abelian.
Now let $G$ be an $\Omega$-algebra with a set of generators $X$. According to Remark 2.2 , we define the structure of a $\langle p, S(G)\rangle$-algebra on both $G$ and it's free Abelian extension $A$ generated by $X$. By Proposition 2.4, the $\langle p, S(G)\rangle$ algebra $G$ has a free Abelian extension $D$ generated by $X$. As the $\langle p, S(G)\rangle$ algebra $A$ is an Abelian extension of $G$, there exists an Abelian epimorphism $\xi$ of $\langle p, S(G)\rangle$-algebras from $D$ onto $A$ which identically maps $X$ onto itself. By $\alpha$ we mean the kernel of the Abelian homomorphism from $A$ onto $G$. Let $F$ be obtained from $D$ as described above. In terms of Theorem 2.13, $\beta$ is the Abelian kernel of the epimorphism form $D$ onto $G$. Now the following main result follows immediately from Theorem 2.13:

Corollary 2.14. $F \cong A$.
Note that the construction of $S$ and $F$ depends only on $G$. Hence we obtain the construction of $A E(G)$ in terms of $G$.

## 3. Free solvable algebra

Finally we combine the results from [8] and the technique used in the previous section to obtain a construction of the free solvable $\mathcal{V}$-algebra. Let $F_{k}$ be a free solvable $\Omega$-algebra of degree $k$ over a given set $X$. Let $\alpha=I_{F_{k}}^{k-1}$. We construct the set $E$ for $\alpha$ and consider the free solvable algebra $D_{k}$ of degree $k$ generated by $X$. We begin with the construction of the free solvable Abelian algebra $F_{1}$. In this case $E=\{e\}$ for a fixed element $e$ from $F_{1}$ and $S_{\Omega}$ consists of all elements of the form $\frac{\partial t}{\partial i}(e, \ldots, e)$ for each operation $t$ from $T$. By Proposition 2.5, the $\langle p, S\{e\}\rangle$-algebra $F_{1}$ is Abelian. Let $\omega$ be a congruence of $D_{1}$ defined by (20)-(22). Then, as it was shown in the previous section, $F_{1}^{\prime}=D_{1} / \omega$ becomes a $\mathcal{V}$-algebra with respect to the operations (24).

Theorem 3.1. $F_{1}^{\prime} \cong F_{1}$.
Proof. At first we note that $F_{1}^{\prime}$ is generated by $X$. Then we observe that $\alpha=1_{F_{1}}, \quad \beta=\xi^{-1}(\alpha)=1_{D_{1}}$ and, from (25), we see that $F_{1}^{\prime} \times F_{1}^{\prime} \subseteq \rho$; thus $F_{1}^{\prime}$ is an Abelian $\Omega$-algebra generated by $X$. Now the desired conclusion follows from Theorem 2.13.

Now the construction of the free solvable $\Omega$-algebra $F_{k}$ can be obtained by induction on $k$ as the free Abelian extension of $F_{k-1}$.

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