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FREE ABELIAN EXTENSIONS IN THE CONGRUENCE-PERMUTABLE VARIETIES

Pavel Zhdanovich

Volgograd State Pedagogical University, Russia
e-mail: comlab@vspu.ru

Abstract

We obtain the construction of free abelian extensions in a congurence-permutable variety $\mathcal{V}$ using the construction of a free abelian extension in a variety of algebras with one ternary Mal'cev operation and a monoid of unary operations. We also use this construction to obtain a free solvable $\mathcal{V}$-algebra.

Keywords: abelian extension, solvable algebra, congurence-permutable variety.

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1. Introduction

The theory of congruence commutators in congruence modular varieties develops an important tool for a generalization of several important concepts from the theory of groups and rings such as Abelian algebras, solvable algebras, a center of an algebra. The appearance of the commutator theory was prepared by a set of basic results. Historically one of the first of them was the well known Mal'cev theorem:

Theorem 1.1 (see, e.g., [5], p. 172, [6]). *The variety of algebras $\mathcal{V}$ is congruence-permutable if and only if there exists a ternary basic term $p(x, y, z)$ such that the following are the identities of $\mathcal{V}$:*

$$p(x, x, y) = p(y, x, x) = y. \tag{1}$$

The commutator theory is exposed in [4], [6], [7]. For all undefined notations and terminology the reader can consult [4]. Recall the most important facts about commutators and Abelian congruences. Throughout section we shall consider an arbitrary algebra G from a fixed congruence modular variety $\mathcal{M}$. The *commutator* is the largest binary operation $(\alpha, \beta) \mapsto f(\alpha, \beta)$ on the congruence lattice $\mathrm{Con}(G)$ such that

1. $f(\alpha, \beta) \leq \alpha \cap \beta$,
2. $f(\alpha, \beta \vee \gamma) = f(\alpha, \beta) \vee f(\alpha, \gamma)$,
3. $f(\alpha \vee \beta, \gamma) = f(\alpha, \gamma) \vee f(\beta, \gamma)$,
4. $\varphi^{-1}(f(\alpha, \beta)) = f(\varphi^{-1}(\alpha), \varphi^{-1}(\beta)) \vee Ker(\varphi)$ for any epimorphism

   $\varphi : B \to G$ from an algebra B.

Commutator of congruences α and β is denoted by $[\alpha, \beta]$. A congruence α is *Abelian* if $[\alpha, \alpha] = 0$. It is known from [4] (see Theorem 5.5, p. 47) that there exists a so-called *ternary difference term* d such that $d(x, x, y) = y$ is an identity of M. Furthermore, if we denote $(a_1, ..., a_n)$, $(b_1, ..., b_n)$ and $(c_1, ..., c_n)$ by $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, respectively, then a congruence $\alpha \in \mathrm{Con}(G)$ is Abelian if and only if

$$d(t(\mathbf{a}), t(\mathbf{b}), t(\mathbf{c})) = t(d(a_1, b_1, c_1), \ldots, d(a_n, b_n, c_n)),$$

for any basic operation $t(x_1, \ldots, x_n) = t(\boldsymbol{x})$ and for all elements a_i, b_i, c_i with $a_i \, \alpha \, b_i \, \alpha \, c_i$ $(1 \leq i \leq n)$. In this case the following properties hold:

1. For any fixed element $\overline{g}$ the congruence class $[\overline{g}]_\alpha$ is an Abelian group with respect to the addition

$$x + y = d(x, \overline{g}, y) \tag{2}$$

with zero element $\overline{g}$. Moreover, $d(x, y, z) = x - y + z$ for all $x, y, z \in [\overline{g}]_\alpha$. The set $[\overline{g}]_\alpha$ with this operation $d(x, y, z)$ is called *ternary* group and (2) is called *ternary* addition.

2. Each term n-ary operation t and each ordered set $(g_1, \ldots, g_n) \in G^n$ define a system of group homomorphisms $h_i : [g_i]_\alpha \mapsto [t(g_1, \ldots, g_n)]_\alpha$ such that
$$t(x_1, \ldots, x_n) = \sum_{i=1}^{n} h_i(x_i) + t(\overline{g_1}, \ldots, \overline{g_n}),$$
where $x_i \in [g_i]_\alpha$. As it is mentioned in [1],

(3) $$h_i(x) = t(\overline{g_1}, \ldots, \overline{g_{i-1}}, x, \overline{g_{i+1}}, \ldots, \overline{g_n}) - t(\overline{g_1}, \ldots, \overline{g_n}),$$

and these homomorphisms are compatible with compositions of operations.

In particular, the operation p with Mal'cev identities (1) can be taken as the difference term d in any congruence-permutable variety.

Remark 1.1. It is known from [6] that, for any two elements e and e' from the same congruence class of an Abelian congruence, the mapping $f(x) = d(e', e, x)$ is an isomorphism between the ternary groups defined on the given congruence class with the help of two zero elements e and e', respectively.

A homomorphism of $\mathcal{M}$-algebras is *Abelian* if its kernel is an Abelian congruence. We use the following notations from [2]:
$$I_G^0 = 1_G, \quad I_G^1 = [1_G, 1_G], \ldots, I_G^k = [I_G^{k-1}, I_G^{k-1}].$$
An algebra G is *solvable of degree at most k* if $I_G^k = 0_G$.

Let $G \in M$ be generated by a subset X. $\mathcal{M}$-algebra A is an *Abelian* extension of G if A is generated by the same set X and there exists an Abelian epimorphism $\psi : A \to G$ which is identical on X. An Abelian extension $AE(G)$ of G, with an Abelian epimorphism $\varphi : AE(G) \to G$, is said to be *free* if for any Abelian extension B of G, with an Abelian epimorphism $\psi : B \to G$ being identical on X, there exists a homomorphism $\tau : AE(G) \to B$ such that $\varphi = \psi\tau$. The free Abelian extension can be obtained as follows. Let F be the free $\mathcal{M}$-algebra generated by X and

$\gamma \in \mathrm{Con}(F)$ such that $G = F/\gamma$. Then $AE(G) = F/[\gamma, \gamma]$. The idea of Abelian extension is used intensively in commutator theory. For example, each free solvable algebra of degree k is obtained as a free Abelian extension of free solvable algebra of degree $(k-1)$. The construction of free solvable algebra with one ternary Mal'cev operation p is given in [3]. These results were generalized in [2] for a general congruence modular variety. The paper [1] contains the general approach to the construction of free Abelian extensions in any given congruence modular variety.

Now let Ω be a system of operations. We use the ideas from [1] and apply the results obtained in [8] and [9] for $\langle p, S\rangle$–algebras to the construction of free Abelian extensions in any congruence-permutable variety. The main result of the present paper is the Theorem 2.14. We also apply this construction to the structure of free solvable algebras.

2. Construction of the free Abelian extension

Consider a congruence-permutable variety $\mathcal{V}$ of Ω-algebras, and let p be a term operation from Theorem 1.1. We denote the clone of $\mathcal{V}$ by $T = \{T_n \mid n \in \mathbb{N}\}$, where T_n is the set of all n-ary term operations that are distinct in $\mathcal{V}$. Recall that T is a system of operations which is closed under all compositions and contains all projections i.e. the operations p_{jn} such that $p_{jn}(x_1, \dots x_n) = x_j, \quad j = 1, \dots, n$. Let A be an arbitrary algebra from $\mathcal{V}$ with a fixed Abelian congruence α and a set of generators X. Consider a set E of representatives of α-cosets such that:

1. if $x \in X$ and $e \in E \cap [x]_\alpha$, then $e \in X$;
2. if $e \in E$, then

$$e = t_e(x_1, \dots, x_n) \tag{4}$$

for some n-ary term t_e, where $x_1, \dots, x_n \in X \cap E$.

Any class $[e]_\alpha$, $e \in E$, will be considered as a ternary group with the zero element e. Following [1] denote by S_Ω the set of all symbols

$$\frac{\partial t}{\partial i}(\mathbf{e}),\ \frac{\partial p_{jn}}{\partial i}(\mathbf{e})$$

for each positive integer n, for each n-ary term t and for all $\mathbf{e} = (e_1, \dots, e_n) \in E^n$. Let $h(x_1, \dots, x_n), g_i(x_1, \dots, x_{m_i})$ be arbitrary term operations on A, $i = 1, \dots, n$. Put

$$d_i = \begin{cases} 0, & i = 0, \\ m_1 + \cdots + m_i, & i = 1, \ldots, n. \end{cases}$$

Consider now the following term operations on A:

$$h = t(g_1(x_1, \ldots, x_{d_1}), \ldots, g_n(x_{d_{m-1}+1}, \ldots, x_{d_m})).$$

It is shown in [1] (see Proposition 2.1) that if $\mathbf{e}_j \in E^{m_j},\ 1 \le j \le n,\ \mathbf{e} = (e_1, \ldots, e_n)$, then

$$\frac{\partial h}{\partial i}(\mathbf{e}) = \left(\frac{\partial t}{\partial j}(g_1(\mathbf{e}_1), \ldots, g_n(\mathbf{e}_n))\right)\left(\frac{\partial g_j}{\partial(i - d_{j-1})}(\mathbf{e}_j)\right), \tag{5}$$

where $d_{j-1} \le i \le d_j$. It means that S_Ω is closed under multiplication (5).

Proposition 2.1. *Multiplication* (5) *is associative.*

Proof. Let

$$\delta = \frac{\partial t}{\partial i}(a_1, \ldots, a_s), \quad \beta = \frac{\partial u}{\partial j}(b_1, \ldots, b_q), \quad \gamma = \frac{\partial v}{\partial k}(c_1, \ldots, c_r), \tag{6}$$

where $a_1, \ldots, a_s, b_1, \ldots, b_q, c_1, \ldots, c_r \in E$. Then

$$\beta\gamma = \frac{\partial h}{\partial(j + k - 1)}(b_1, \ldots, b_{j-1}, c_1, \ldots, c_r, b_{j+1}, \ldots, b_q),$$

where $h = u(x_1, \ldots, x_{j-1}, v(x_j, \ldots, x_{j+r-1}), x_{j+r}, \ldots, x_{q+r-1})$, and therefore

(7) $\delta(\beta\gamma) =$

$$= \frac{\partial g}{\partial(i+j+k-2)}(a_1, \ldots, a_{i-1}, b_1, \ldots, b_{j-1}, c_1, \ldots, c_r, b_{j+1}, \ldots, b_q, a_{i+1}, \ldots, a_s),$$

where

$$g = t\left(x_1, \dots, x_{i-1}, u\left(x_i, \dots, x_{i+j-2}, v(x_{i+j-1}, \dots, x_{i+j+r-2}),\right.\right.$$
$$\left.\left. x_{i+j+r-1}, \dots, x_{i+q+r-2}\right), x_{i+q+r-1}, \dots, x_{s+q+r-2}\right).$$

On the other hand

$$\delta\beta = \frac{\partial w}{\partial(i+j-1)}(a_1, \dots, a_{i-1}, b_1, \dots, b_q, a_{i+1}, \dots, a_s),$$

where

$$(8) \qquad w = t(x_1, \dots, x_{i-1}, u(x_i, \dots, x_{i+q-1}), x_{i+q}, \dots, x_{s+q-1}).$$

At the final step we calculate $(\delta\beta)\gamma$ and show that it is equal to (7). ■

Assign to each element $\frac{\partial t}{\partial i}(e_1, \dots, e_n) \in S_\Omega$ the unary operation on A as follows:

$$(9) \qquad f_{\frac{\partial t}{\partial i}(e_1,\dots,e_n)}(x) = t(e_1, \dots, e_{i-1}, x, e_{i+1} \dots, e_n).$$

Proposition 2.2. *The equality* (9) *defines an action on* A *of the monoid* S_Ω *with the multiplication* (5).

Proof. Suppose that δ, β are from (6), and $x \in A$. Then

$$f_\beta(x) = u(b_1, \dots, b_{j-1}, x, a_{j+1}, \dots, b_q);$$

$$f_\delta(f_\beta(x)) = t(a_1, \dots, a_{i-1}, u(b_1, \dots, b_{j-1}, x, b_{j+1}, \dots, b_q), a_{i+1}, \dots, a_s) =$$

$$= f_{\frac{\partial w}{\partial(i+j-1)}(a_1,\dots,a_{i-1},b_1,\dots,b_q,a_{i+1},\dots,a_s)}(x) = f_{\delta\beta}(x),$$

where w is from (8). ■

Here are some properties of the action (9):

1. if p_{in} is a projection, then
$$f_{\frac{\partial p_{in}}{\partial i}(\mathbf{e})}(x) = x;$$
2. if $\mathbf{e} \in E^n$ and t is an arbitrary n-ary term operation, then

$$f_{\frac{\partial t}{\partial i}(\mathbf{e})}(e_i) = t(\mathbf{e}), \quad i = 1, \ldots, n. \tag{10}$$

Let $\left[\frac{\partial t}{\partial i}(\mathbf{e})\right]$ stand for the homomorphism h_i from (3). For all $a_1 \in [e_1]_\alpha, \ldots, a_n \in [e_n]_\alpha$ we have

$$t(a_1, \ldots, a_n) = \sum_{i=1}^{n} \left[\frac{\partial t}{\partial i}(\mathbf{e})\right](a_i) + t(e_1, \ldots, e_n). \tag{11}$$

Denote by θ the congruence on S_Ω generated by all pairs of the form:

$$\left(\frac{\partial p}{\partial 2}(e, e, e),\ 1\right),$$

$$\left(\frac{\partial f}{\partial i}(\mathbf{e}),\ \frac{\partial g}{\partial i}(\mathbf{e})\right), \tag{12}$$

$$\left(\frac{\partial p_{in}}{\partial i}(\mathbf{e}),\ 1\right), \tag{13}$$

$$\left(\frac{\partial p_{in}}{\partial j}(e_1, \ldots, e_n),\ \frac{\partial p_{km}}{\partial l}(e'_1, \ldots, e'_m)\right), \quad \text{for} \quad i \neq j, \quad k \neq l, \quad e_i = e'_k, \tag{14}$$

$$\left(\frac{\partial t}{\partial i}(e_1, \ldots, e_{i-1}, c, e_{i+1}, \ldots, e_n), \frac{\partial t}{\partial i}(e_1, \ldots, e_{i-1}, d, e_{i+1}, \ldots, e_n)\right), \tag{15}$$

for $c, d \in E$,

where $\mathbf{e} \in E^n$, and $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ is a defining identity of $\mathcal{V}$ [1]; (see Theorem 2.4). The monoid S_Ω/θ will be denoted by $S(E)$.

Proposition 2.3. *Each θ-class generated by $\frac{\partial p_{in}}{\partial j}(e_1, \ldots, e_n)$ for $i \neq j$ is a left zero of $S(E)$.*

Proof. Without loss of generality, we put $j < i$. If $t(x_1, \ldots, x_n)$ is a term operation, then we set

$$h(x_1, \ldots, x_{m+n-1}) =$$

$$= p_{(m+i-1),(m+n-1)}(x_1, \ldots, x_{j-1}, t(x_j, \ldots, x_{m+j-1}), x_{m+j}, \ldots, x_{m+n-1}).$$

Observe that the identity

$$h(x_1, \ldots, x_{m+n-1}) = p_{(m+i-1),(m+n-1)}(x_1, \ldots, x_{m+n-1})$$

holds on $\mathcal{V}$. By (12), (14), we have

$$\frac{\partial p_{in}}{\partial j}(e_1, \ldots, e_n)\frac{\partial t}{\partial k}(e'_1, \ldots, e'_m) =$$

$$= \frac{\partial h}{\partial(j+k-1)}(e_1, \ldots, e_{j-1}, e'_1, \ldots, e'_m, e_{j+1}, \ldots, e_n) =$$

$$= \frac{\partial p_{(m+i-1)(m+n-1)}}{\partial(j+k-1)}(e_1, \ldots, e_{j-1}, e'_1, \ldots, e'_m, e_{j+1}, \ldots, e_n)\,\theta\,\frac{\partial p_{in}}{\partial j}(e_1, \ldots, e_n).$$

■

It has proved that A is a polygon over the monoid $S(E)$. A ternary operation p satisfying Mal'cev identities (1) is also defined on A. Hence A is a $\langle p, S(E)\rangle$-*algebra* in the sense of [9]. Moreover, we will prove the following fact.

Proposition 2.4. *X generates the $\langle p, S(E)\rangle$-algebra A. If any subset Y generates the $\langle p, S(E)\rangle$-algebra A, then $Y \cup E$ generates Ω-algebra.*

Proof. Recall that for any $x \in X$ the zero of the ternary Abelian group $[x]_\alpha$ belongs to X. Let $u \in A$. Then for some operation t and $x_1, \ldots, x_n \in X$ we have

$$u = t(x_1, \dots, x_n) =$$

$$= \sum_{i=1}^{n} \left(f_{\frac{\partial t}{\partial 1}(e_1,\dots,e_n)}(x_i) - f_{\frac{\partial t}{\partial i}(e_1,\dots,e_n)}(e_i) \right) + f_{\frac{\partial t}{\partial 1}(e_1,\dots,e_n)}(e_1). \tag{16}$$

Note that $e_1 \in X$ and if $e \in E \cap [u]_\alpha$, then $e = t_e(x'_1, \dots, x'_n)$ for some $x'_1, \dots, x'_n \in X$. Now let a set Y generate A as a $\langle p, S(E)\rangle$-algebra. Each operation from $S(E)$ is a polynomial of the Ω-algebra A. Since each element from X is the value of some term for the elements of $Y \cup E$, then it is also true for an arbitrary element u from (16). ■

Proposition 2.5. *Congruence α is Abelian on the $\langle p, S(E)\rangle$-algebra A.*

Proof. Assume that $(u, v) \in \alpha$. From (15) we get

$$\alpha \ni (t(e_1, \dots, e_{i-1}, u, e_{i+1}, \dots, e_n), t(e_1, \dots, e_{i-1}, v, e_{i+1}, \dots, e_n)) =$$

$$= \left(f_{\frac{\partial t}{\partial i}(\mathbf{e})}(u), f_{\frac{\partial t}{\partial i}(\mathbf{e})}(v) \right).$$

Consequently α is a congruence with respect to the new operations. We know from [7] that the commutator $[\alpha, \alpha]$ on $\langle p, S(E)\rangle$-algebra A is generated by all pairs of the form

$$\begin{aligned} &\Big(p(p(u_1, u_2, u_3), p(v_1, v_2, v_3), p(w_1, w_2, w_3)), \\ &p(p(u_1, v_1, w_1), p(u_2, v_2, w_2), p(u_3, v_3, w_3)) \Big), \end{aligned} \tag{17}$$

$$\left(p(f_{\frac{\partial t}{\partial i}(\mathbf{e})}(u), f_{\frac{\partial t}{\partial i}(\mathbf{e})}(v), f_{\frac{\partial t}{\partial i}(\mathbf{e})}(w)),\ f_{\frac{\partial t}{\partial i}(\mathbf{e})}(p(u, v, w)) \right), \tag{18}$$

where u_i, v_i, w_i, u, v, w are congruent modulo α, $\quad i = 1, 2, 3$. In terms of Ω-algebra the congruence $[\alpha, \alpha]$ is generated by pairs (17), and also by the pairs:

$$(19)\qquad \Big(p(t(u_1,\dots,u_n),t(v_1,\dots,v_n),\quad t(w_1,\dots,w_n)),\\ t(p(u_1,v_1,w_1),\dots,p(u_n,v_n,w_n))\Big),$$

where u_i, v_i, w_i are congruent modulo α, $\quad i=1,\dots,n$. But, as one can see, the pairs from (18) belong to the set of those from (19) and that all the pairs from (17), (18) generate the smallest congruence on A. Hence α is Abelian. ■

Remark 2.1. Let B be a $\mathcal{V}$-algebra such that there exists a homomorphism λ from A onto B and $Ker(\lambda)\subseteq\alpha$. Then, by the basic properties of commutators, $\lambda(\alpha)$ is an Abelian congruence. For each $\mathbf{e}\in E^n$, $t\in T_n$, $b\in B$ we put

$$f_{\frac{\partial t}{\partial i}(\mathbf{e})}(b)=t(\lambda(e_1),\dots,\lambda(e_{i-1}),b,\lambda(e_{i+1}),\dots,\lambda(e_n)).$$

Thus B becomes a $\langle p,S(E)\rangle$-algebra. Moreover, λ is an Abelian homomorphism between the two $\langle p,S(E)\rangle$-algebras.

Rremark 2.2. We can generalize the preceding remark. In fact, E can be viewed as an Ω-algebra isomorphic to A/α. By Remark 2.1, each Abelian extension of E (including E itself) is a $\langle p,S(E)\rangle$-algebra where the elements from E are fixed by (4). The obtained algebra is an Abelian extension of E.

Let D be a $\langle p,S(E)\rangle$-algebra generated by X and there exists an epimorphism $\xi: D\to A$ such that $\xi(X)=X$ and the congruence $\beta=\xi^{-1}(\alpha)$ is Abelian. Since $Ker(\xi)\subseteq\beta$ then D is an Abelian extension of A. Consider a set E' of all elements $f_{\frac{\partial t_e}{\partial 1}(x_1\dots,x_n)}(x_1)$. Certainly, $\xi(E')=E$. Since there is only one element of E' in each β-class, then we can treat each element from E' as the zero element of the corresponding ternary group.

Proposition 2.6. *The restriction of ξ to each β-class is a group epimorphism.*

Proof. First we note that ξ preserves the operation p. As mentioned above, $\xi(E')=E$ and hence ξ preserves the addition on each β-class as ternary group. ■

Proposition 2.7.

$$\xi\left(\left[\frac{\partial t}{\partial i}(\mathbf{e})\right](u)\right) = \left[\frac{\partial t}{\partial i}(\mathbf{e})\right](\xi(u))$$

for each term operation $t(x_1, \ldots, x_n)$ *and for all* $\mathbf{e} \in E^n$.

Proof. For $e_i' \in \xi^{-1}(e_i) \cap E'$ we get

$$\begin{aligned}\xi\left(\left[\frac{\partial t}{\partial i}(\mathbf{e})\right](u)\right) = {} & \xi\left(f_{\frac{\partial t}{\partial i}(\mathbf{e})}(u) - f_{\frac{\partial t}{\partial i}(\mathbf{e})}(e_i')\right) = \\ & = f_{\frac{\partial t}{\partial i}(\mathbf{e})}(\xi(u)) - f_{\frac{\partial t}{\partial i}(\mathbf{e})}(e_i) = \left[\frac{\partial t}{\partial i}(\mathbf{e})\right](\xi(u)).\end{aligned}$$

■

Let ω be a congruence on D generated by pairs of the form

$$\begin{gathered}\Big(f_{\frac{\partial t(g_1,\ldots,g_n)}{\partial i}(\mathbf{e})}(u), \\ \sum_{j=1}^{n} f_{\frac{\partial h_j}{\partial(i+j-1)}(\overline{g_1},\ldots,\overline{g_{j-1}},\mathbf{e},\overline{g_{j+1}}\ldots,\overline{g_n})}(u) - (n-1) f_{\frac{\partial t}{\partial 1}(\overline{g_1},\ldots,\overline{g_n})}(g_1')\Big);\end{gathered} \tag{20}$$

$$\left(f_{\frac{\partial g}{\partial 1}(\mathbf{e})}(e_1'),\ f_{\frac{\partial g}{\partial j}(\mathbf{e})}(e_j')\right), \quad j = 2, \ldots, m; \tag{21}$$

$$\left(f_{\frac{\partial p_{jm}}{\partial i}(\mathbf{e})}(u),\ e_j'\right), \quad j \neq i. \tag{22}$$

Here g is an m-ary term operation, $\mathbf{e} = (e_1, \ldots, e_m) \in E^m$, $e_i' \in E'$, $\xi(e_i') = e_i$, $u \in [e_i']_\beta$, $\quad i = 1, \ldots, m$,

$$h_j = t(x_1, \ldots, x_{j-1}, g_j(x_j, \ldots, x_{j+m-1}), x_{j+m}, \ldots, x_{n+m-1}),$$

$\overline{g_i} = E \cap [g_i(\mathbf{e})]_\alpha$, and $g_i' \in E'$ such that $\xi(g_i') = \overline{g_i}$. The sum in (20) is denoting the addition in the group $[t(g_1, \ldots, g_n)(e_1', \ldots, e_m')]_\beta$.

Proposition 2.8. $\omega \leq Ker(\xi)$.

Proof. A direct calculation shows that pairs (21), (22) are in $Ker(\xi)$. If $e_1, \ldots, e_m \in E$ then we have

$$t(g_1, \ldots, g_n)(e_1, \ldots, e_m) = t(g_1(e_1, \ldots, e_m), \ldots, g_n(e_1, \ldots, e_m)). \tag{23}$$

Therefore each pair of the form (20) belongs to $Ker(\xi)$. ■

It follows immediately from Proposition 2.8 that the congruence ω is Abelian and D/ω is an Abelian extension of A. Let ρ be the fractional congruence β/ω. For each $a \in A$, we denote by $\rho(a)$ the ρ-class corresponding to a.

Proposition 2.9. *ρ is an Abelian congruence.*

Proof. Let $\varepsilon : D \to D/\omega$ be the natural homomorphism. Then,

$$\varepsilon^{-1}([\rho, \rho]) = [\varepsilon^{-1}(\rho), \varepsilon^{-1}(\rho)] \vee \omega = [\beta, \beta] \vee \omega = \omega.$$

■

Since all pairs (21) belong to ω then we can write $\widetilde{t}(e'_1, \ldots, e'_n)$ for $f_{\frac{\partial t}{\partial j}(\mathbf{e})}(e'_j)$, $j = 1, \ldots, n$ where $\mathbf{e}, e'_j$ are such as in (21). Let t be an arbitrary term n-ary operation from Ω. Put

$$t(b_1, \ldots, b_n) = \sum_{i=1}^{n} \left[\frac{\partial t}{\partial i}(e_1, \ldots, e_n)\right](b_i) + \widetilde{t}(e'_1, \ldots, e'_n) \tag{24}$$

for all $b_1, \ldots, b_n$ from $\rho(e_1), \ldots, \rho(e_n)$ respectively.

Proposition 2.10. *D/ω is a $\mathcal{V}$-algebra with respect to the operations* (24).

Proof. We have to check that (24) defines a homomorphism from the clone T of all term operations on $\mathcal{V}$ to the clone $\mathcal{O}(D/\omega)$ of operations on D/ω. From (22), (13), we see that

$$p_{im}(u_1, \ldots, u_m) = \sum_{j=1}^{m} \left[\frac{\partial p_{im}}{\partial j}(\mathbf{e})\right](u_i) + \widetilde{p_{im}}(\mathbf{e}') = u_i + me'_i = u_i$$

for $u \in \rho(e_i)$. By (20), (5), for $t, h_i, \mathbf{e}, \mathbf{e}'$ and u such as in (20), we get

$$\left[\frac{\partial t(g_1,\ldots,g_n)}{\partial i}(\mathbf{e})\right](u) = f_{\frac{\partial t(g_1,\ldots,g_n)}{\partial i}(\mathbf{e})}(u) - f_{\frac{\partial t(g_1,\ldots,g_n)}{\partial i}(\mathbf{e})}(e_i') =$$

$$= \sum_{j=1}^{n} f_{\frac{\partial h_j}{\partial(i+j-1)}(\overline{g_1},\ldots,\overline{g_{j-1}},\mathbf{e},\overline{g_{j+1}}\ldots,\overline{g_n})}(u) - (n-1) f_{\frac{\partial t}{\partial 1}(\overline{g_1},\ldots,\overline{g_n})}(g_1') -$$

$$- \sum_{j=1}^{n} f_{\frac{\partial h_j}{\partial(i+j-1)}(\overline{g_1},\ldots,\overline{g_{j-1}},\mathbf{e},\overline{g_{j+1}}\ldots,\overline{g_n})}(e_i') + (n-1) f_{\frac{\partial t}{\partial 1}(\overline{g_1},\ldots,\overline{g_n})}(g_1') =$$

$$= \sum_{j=1}^{n} \left[\frac{\partial h_j}{\partial(i+j-1)}(\overline{g_1},\ldots,\overline{g_{j-1}},\mathbf{e},\overline{g_{j+1}}\ldots,\overline{g_n})\right](u) =$$

$$= \sum_{j=1}^{n} \left[\frac{\partial t}{\partial j}(\overline{g_1},\ldots,\overline{g_n})\right]\left[\frac{\partial g_j}{\partial i}(\mathbf{e})\right](u).$$

Furthermore,

$$t(\widetilde{g_1,\ldots},g_n)(\mathbf{e}) = f_{\frac{\partial t(g_1,\ldots,g_n)}{\partial 1}(\mathbf{e})}(e_1') =$$

$$= \sum_{j=1}^{n} f_{\frac{\partial h_j}{\partial j}(\overline{g_1},\ldots,\overline{g_{j-1}},\mathbf{e},\overline{g_{j+1}}\ldots,\overline{g_n})}(e_1') - (n-1) f_{\frac{\partial t}{\partial 1}(\overline{g_1},\ldots,\overline{g_n})}(g_1') =$$

$$= \sum_{j=1}^{n} f_{\frac{\partial t}{\partial j}(\overline{g_1},\ldots,\overline{g_n})}\left(f_{\frac{\partial g_j}{\partial 1}(\mathbf{e})}(e_1')\right) - (n-1) f_{\frac{\partial t}{\partial 1}(\overline{g_1},\ldots,\overline{g_n})}(g_1') =$$

$$= \sum_{j=1}^{n} \left[\frac{\partial t}{\partial j}(\overline{g_1},\ldots,\overline{g_n})\right]\left(f_{\frac{\partial g_j}{\partial 1}(\mathbf{e})}(e_1')\right) + f_{\frac{\partial t}{\partial 1}(\overline{g_1},\ldots,\overline{g_n})}(g_1') =$$

$$= \sum_{j=1}^{n} \left[\frac{\partial t}{\partial j}(\overline{g_1},\ldots,\overline{g_n})\right](\widetilde{g_j}(\mathbf{e})) + f_{\frac{\partial t}{\partial 1}(\overline{g_1},\ldots,\overline{g_n})}(g_1').$$

Consequently,

$$
\begin{aligned}
&t(g_1,\ldots,g_n)(u_1,\ldots,u_m) = \\
&= \sum_{i=1}^{m} \left[\frac{\partial t(g_1,\ldots,g_n)}{\partial i}(\mathbf{e})\right](u_i) + t(\widetilde{g_1,\ldots,g_n})(a) = \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} \left[\frac{\partial t}{\partial j}(\overline{g_1},\ldots,\overline{g_n})\right]\left[\frac{\partial g_j}{\partial i}(\mathbf{e})\right](u_i) + \sum_{j=1}^{n}\left[\frac{\partial t}{\partial j}(\overline{g_1},\ldots,\overline{g_n})\right](\widetilde{g_j}(\mathbf{e})) + \\
&+ f_{\frac{\partial t}{\partial 1}(\overline{g_1},\ldots,\overline{g_n})}(g_1') = \sum_{j=1}^{n}\left[\frac{\partial t}{\partial j}(\overline{g_1},\ldots,\overline{g_n})\right]\left(\sum_{i=1}^{m}\left[\frac{\partial g_j}{\partial i}(\mathbf{e})\right](u_i) + \widetilde{g_j}(\mathbf{e})\right) = \\
&+ f_{\frac{\partial t}{\partial 1}(\overline{g_1},\ldots,\overline{g_n})}(g_1') = t(g_1(u_1,\ldots,u_m),\ldots,g_n(u_1,\ldots,u_m)).
\end{aligned}
$$

Hence, (23) holds on D/ω. Finally, if the equality $f_1(x_1,\ldots,x_k) = f_2(x_1,\ldots,x_k)$ holds in T, then, by (12), it also holds on D/ω. ■

In particular, we observe the following important fact.

Proposition 2.11. *The operation*

$$
\begin{aligned}
&p'(u_1,u_2,u_3) = \\
&= \left[\frac{\partial p}{\partial 1}(e_1,e_2,e_3)\right](u_1) + \left[\frac{\partial p}{\partial 2}(e_1,e_2,e_3)\right](u_2) + \\
&+ \left[\frac{\partial p}{\partial 3}(e_1,e_2,e_3)\right](u_3) + \widetilde{p}(e_1',e_2',e_3'),
\end{aligned}
$$

where $u_i \in [e_i']_\beta$ and $e_i = \xi(e_i') \in E$, satisfies the Mal'cev identities (1).

Proof. Let $f = p(p_{12}, p_{12}, p_{22})$. Then $f(x, y) = p_{22}(x, y)$ is an identity of $\mathcal{V}$. By (12), (14) and (20), we get

$$
\begin{aligned}
&p'(a,a,b) = \\
&= \left[\frac{\partial p}{\partial 1}(e_1,e_1,e_2)\right](a) + \left[\frac{\partial p}{\partial 2}(e_1,e_1,e_2)\right](a) + \\
&+ \left[\frac{\partial p}{\partial 3}(e_1,e_1,e_2)\right](b) + \widetilde{p}(e_1',e_1',e_2') = \\
&= \left[\frac{\partial p}{\partial 1}(p_{12}(e_1,e_2),p_{12}(e_1,e_2),p_{22}(e_1,e_2))\right]\left[\frac{\partial p_{12}}{\partial 1}(e_1,e_2)\right](a) + \\
&+ \left[\frac{\partial p}{\partial 2}(p_{12}(e_1,e_2),p_{12}(e_1,e_2),p_{22}(e_1,e_2))\right]\left[\frac{\partial p_{12}}{\partial 1}(e_1,e_2)\right](a) + \\
&+ \left[\frac{\partial p}{\partial 3}(p_{12}(e_1,e_2),p_{12}(e_1,e_2),p_{22}(e_1,e_2))\right]\left[\frac{\partial p_{22}}{\partial 1}(e_1,e_2)\right](a) + \\
&+ \left[\frac{\partial p}{\partial 1}(p_{12}(e_1,e_2),p_{12}(e_1,e_2),p_{22}(e_1,e_2))\right]\left[\frac{\partial p_{12}}{\partial 2}(e_1,e_2)\right](b) + \\
&+ \left[\frac{\partial p}{\partial 2}(p_{12}(e_1,e_2),p_{12}(e_1,e_2),p_{22}(e_1,e_2))\right]\left[\frac{\partial p_{12}}{\partial 2}(e_1,e_2)\right](b) + \\
&+ \left[\frac{\partial p}{\partial 3}(p_{12}(e_1,e_2),p_{12}(e_1,e_2),p_{22}(e_1,e_2))\right]\left[\frac{\partial p_{22}}{\partial 2}(e_1,e_2)\right](b) + e_2' = \\
&= \left[\frac{\partial f}{\partial 1}(e_1,e_2)\right](a) + \left[\frac{\partial f}{\partial 2}(e_1,e_2)\right](b) + \widetilde{f}(e_1',e_2') = e_2' + 1(b) + e_2' = b.
\end{aligned}
$$

■

Proposition 2.12. *p' coincides with p on each ρ-class.*

Proof. Let $a, b, c \in [e']_\rho$, $e' \in E'$, and $e^* = \xi(e')$. Since p commutes with p' on $[e']_\rho$, then by (11), (21)

$$\begin{aligned} p'(a,b,c) = \ & p'(p(a,e',e'),p(e',b,e'),p(e',e',c)) = \\ & = p(p'(a,e',e'),p'(e',b,e'),p'(e',e',c)) = \\ & = a + \left[\frac{\partial p}{\partial 2}(e^*,e^*,e^*)\right](b) + c = a + b + c. \end{aligned}$$

■

Let F be the subalgebra in D/ω generated by X with respect to the operations (24).

Theorem 2.13. *ξ induces an Abelian epimorphism of Ω-algebras $F \to A$.*

Proof. Since $\omega \subseteq Ker(\xi)$ then there is an epimorphism of $\langle p, S(E)\rangle$-algebras $\varphi : D/\omega \to A, \quad [u]_\omega \mapsto \xi(u)$. Observe that

$$\varphi(\widetilde{t}(e'_1,\ldots,e'_n) = \varphi(f_{\frac{\partial t}{\partial 1}(e_1,\ldots,e_n)}(e'_1)) = t(e_1,\ldots,e_n),$$

by (10) and thus, by (11), (24), the mapping φ commutes with each operation from Ω. Moreover,

$$Ker(\varphi) = Ker(\xi)/\omega \subseteq \beta/\omega = \rho. \tag{25}$$

Thus $Ker(\varphi)$ is Abelian. ■

Now let G be an Ω-algebra with a set of generators X. According to Remark 2.2, we define the structure of a $\langle p, S(G)\rangle$-algebra on both G and it's free Abelian extension A generated by X. By Proposition 2.4, the $\langle p, S(G)\rangle$-algebra G has a free Abelian extension D generated by X. As the $\langle p, S(G)\rangle$-algebra A is an Abelian extension of G, there exists an Abelian epimorphism ξ of $\langle p, S(G)\rangle$-algebras from D onto A which identically maps X onto itself. By α we mean the kernel of the Abelian homomorphism from A onto G. Let F be obtained from D as described above. In terms of Theorem 2.13, β is the Abelian kernel of the epimorphism form D onto G. Now the following main result follows immediately from Theorem 2.13:

Corollary 2.14. $F \cong A$. ■

Note that the construction of S and F depends only on G. Hence we obtain the construction of $AE(G)$ in terms of G.

3. Free solvable algebra

Finally we combine the results from [8] and the technique used in the previous section to obtain a construction of the free solvable $\mathcal{V}$-algebra. Let F_k be a free solvable Ω-algebra of degree k over a given set X. Let $\alpha = I_{F_k}^{k-1}$. We construct the set E for α and consider the free solvable algebra D_k of degree k generated by X. We begin with the construction of the free solvable Abelian algebra F_1. In this case $E = \{e\}$ for a fixed element e from F_1 and S_Ω consists of all elements of the form $\frac{\partial t}{\partial i}(e, \ldots, e)$ for each operation t from T. By Proposition 2.5, the $\langle p, S\{e\}\rangle$-algebra F_1 is Abelian. Let ω be a congruence of D_1 defined by (20)–(22). Then, as it was shown in the previous section, $F_1' = D_1/\omega$ becomes a $\mathcal{V}$-algebra with respect to the operations (24).

Theorem 3.1. $F_1' \cong F_1$.

Proof. At first we note that F_1' is generated by X. Then we observe that $\alpha = 1_{F_1}$, $\beta = \xi^{-1}(\alpha) = 1_{D_1}$ and, from (25), we see that $F_1' \times F_1' \subseteq \rho$; thus F_1' is an Abelian Ω-algebra generated by X. Now the desired conclusion follows from Theorem 2.13. ∎

Now the construction of the free solvable Ω-algebra F_k can be obtained by induction on k as the free Abelian extension of F_{k-1}.

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