Discussiones Mathematicae
General Algebra and Applications 22 (2002) 183–198

QUASI-IMPLICATION ALGEBRAS

Ivan Chajda and Kamil Dušek

Department of Algebra and Geometry
Palacký University of Olomouc
Tomkova 40, CZ–77900 Olomouc, Czech Republic
e-mail: chajda@risc.upol.cz

Abstract

A quasi-implication algebra is introduced as an algebraic counterpart of an implication reduct of propositional logic having non-involutory negation (e.g. intuitionistic logic). We show that every pseudocomplemented semilattice induces a quasi-implication algebra (but not conversely). On the other hand, a more general algebra, a so-called pseudocomplemented q-semilattice is introduced and a mutual correspondence between this algebra and a quasi-implication algebra is shown.

Keywords: implication, non-involutory negation, quasi-implication algebra, implitcation algebra, pseudocomplemented semilattice, q-semilattice.

2000 Mathematics Subject Classification: 03G25, 06A12, 06D15.

1. Introduction

Various motivations can be found in the historical background of intuitionistic calculi. Intuitionistic propositional logic does not accept the indirect proofs so frequently used in classical logic. From the intuitionistic standpoint, a contradiction shows that the negation of the statement is false and nothing more. Thus the *laws of double negation* and *of excluded middle* are rejected.

The algebraic semantics adequate for a treatment of intuitionistic propositional logic is usually provided by the class of relatively pseudocomplemented lattices. The lattice operations $\vee$ (join) and $\wedge$ (meet) are interpreted as the propositional connectives disjunction and conjunction, respectively, and the connective implication $x \Rightarrow y$ is interpreted by means of the relative pseudocomplement $x * y$.

Let us mention that this algebraic semantic has several paradoxical properties:

(a) from one point of view, it is a calculus which is too weak, because none of the connectives negation, conjunction, disjunction and implication can be derived by the remaining (e.g. in the classical logic, we have $x \Rightarrow y$ is equivalent to $\neg x \vee y$);

(b) on the other hand, it is too strong because every relative pseudocomplemented lattice is distributive (see, e.g., [2], [4]);

(c) on the contrary to (b), when the implication reduct of intuitionistic propositional logic is axiomatized, we obtain the so-called Hilbert algebra which can be algebraically characterized only as an ordered set with the greatest element.

To avoid such rather strange properties, we introduce another algebraic semantic for intuitionistic implication which is based on pseudocomplemented lattices. It enables us to characterize these so-called quasi-implication algebras by means of so-called pseudocomplemented q-semilattices (introduced by the first author in [3]).

2. The concept of a quasi-implication algebra

The concept of an implication algebra was introduced by J.C. Abbott [1] for studying the implication reduct of classical propositional calculus. Having a Boolean algebra $(A; \vee, \wedge, \neg, \mathbf{0}, \mathbf{1})$, the connective implication is introduced as $x \Rightarrow y := \neg x \vee y = \neg(x \wedge \neg y)$. Writing $x \cdot y$ instead of $x \Rightarrow y$ (for the sake of brevity), this connective can be characterized by three simple axioms (see [1]):

(I1) $(x \cdot y) \cdot x = x$,

(I2) $(x \cdot y) \cdot y = (y \cdot x) \cdot x$,

(I3) $x \cdot (y \cdot z) = y \cdot (x \cdot z)$.

Hence, by an *implication algebra* is understood as a groupoid $\mathcal{A} = (A; \cdot)$ satisfying the identities (I1), (I2), (I3). It was shown by J.C. Abbott that every implication algebra satisfies also the identity $x \cdot x = y \cdot y$ and hence there exists a constant (denoted by the symbol $\mathbf{1}$) such that $x \cdot x = \mathbf{1}$ for each $x \in A$. Moreover, one can introduce an *induced relation* $\leq$ by the setting

$$x \leq y \quad \text{if and only if} \quad x \cdot y = \mathbf{1} \tag{1}$$

which is an order relation on A and $\mathbf{1}$ is the greatest element. With respect to this order, $(A; \leq)$ is a $\vee$-semilattice, where

$$x \vee y := (x \cdot y) \cdot y.$$

Similarly, a logical connective implication in intuitionistic logic was described by A. Diego [5] as follows:

By a *Hilbert algebra* is meant an algebra $\mathcal{A} = (A; \cdot, \mathbf{1})$ of type $(2, 1)$ satisfying the axioms

(H1) $x \cdot (y \cdot x) = \mathbf{1}$,

(H2) $(x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = \mathbf{1}$,

(H3) $x \cdot y = \mathbf{1}$ and $y \cdot x = \mathbf{1}$ imply $x = y$.

Thus $x \cdot y$ satisfying (H1), (H2), (H3) is the intuitionistic implication. It was shown in [5] that:

(a) every implication algebra is a Hilbert algebra;

(b) the axiom (H2) can be replaced by two more simple axioms:

$$x \cdot (y \cdot z) = y \cdot (x \cdot z),$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z);$$

(c) the axiom (H3) can be replaced by identities and hence the class of all Hilbert algebras forms a variety;

(d) the induced relation $\leq$ on A by (1) is also an order on A with the greatest element $\mathbf{1}$ but $(A; \leq)$ need not be a semilattice.

In fact, if $(A;\leq)$ is an arbitrary ordered set with the greatest element $\mathbf{1}$, then for "$\cdot$" introduced by the rule

$$x\cdot y=\mathbf{1}\quad\text{if}\quad x\leq y \text{ and } x\cdot y=y \text{ otherwise}$$

we obtain a Hilbert algebra.

Our aim is to set up another description of the logical connective implication which is very close to that of a Hilbert algebra but based on another algebraic structure. This implication is more similar to that of classical logic. As accepted in intuitionistic logic, we will have $\neg\neg\neg x=\neg x$ (but not necessarily $\neg\neg x=x$).

Definition 1. By a *quasi-implication algebra* is meant an algebra $\mathcal{A}=(A;\cdot,\mathbf{0})$ of type $(2,0)$ satisfying the following identities:

(q1) $(x\cdot y)\cdot x=(x\cdot\mathbf{0})\cdot\mathbf{0}$,

(q2) $(x\cdot y)\cdot y=(y\cdot x)\cdot x$,

(q3) $x\cdot(y\cdot z)=y\cdot(x\cdot z)$,

(q4) $((x\cdot y)\cdot\mathbf{0})\cdot\mathbf{0}=((x\cdot\mathbf{0})\cdot\mathbf{0})\cdot((y\cdot\mathbf{0})\cdot\mathbf{0})=x\cdot y$.

Theorem 1. *Let $\mathcal{A}=(A;\cdot,\mathbf{0})$ be a quasi-implication algebra. The relation Θ_g introduced on A by the setting*

$$\langle x,y\rangle\in\Theta_g \text{ if and only if } x\cdot\mathbf{0}=y\cdot\mathbf{0} \tag{2}$$

is a congruence on $\mathcal{A}$ and the factor algebra $(A/\Theta_g;\cdot)$ is an implication algebra.

Proof. It is clear that Θ_g is an equivalence on A. We prove the substitution property with respect to the binary operation: let $\langle x,y\rangle\in\Theta_g$ and $\langle u,v\rangle\in\Theta_g$. Then $x\cdot\mathbf{0}=y\cdot\mathbf{0}$ and $u\cdot\mathbf{0}=v\cdot\mathbf{0}$. Applying axiom (q4), we have $(x\cdot u)\cdot\mathbf{0}=(((x\cdot u)\cdot\mathbf{0})\cdot\mathbf{0})\cdot\mathbf{0}=(((x\cdot\mathbf{0})\cdot\mathbf{0})\cdot((u\cdot\mathbf{0})\cdot\mathbf{0}))\cdot\mathbf{0}=(((y\cdot\mathbf{0})\cdot\mathbf{0})\cdot((v\cdot\mathbf{0})\cdot\mathbf{0}))\cdot\mathbf{0}=(((y\cdot v)\cdot\mathbf{0})\cdot\mathbf{0})\cdot\mathbf{0}=(y\cdot v)\cdot\mathbf{0}$; thus $\langle x\cdot u,y\cdot v\rangle\in\Theta_g$.

Since (I2) coincides with (q2) and (I3) with (q3), we need only to prove (I1). Let $a,b\in A/\Theta_g$. By (q4), we have $((z\cdot\mathbf{0})\cdot\mathbf{0})\cdot\mathbf{0}=z\cdot\mathbf{0}$; thus

$\langle z, (z \cdot \mathbf{0}) \cdot \mathbf{0}\rangle \in \Theta_g$ for each $z \in A$. It means that there exists an $x \in A$ such that $a = [x \cdot 0]_{\Theta_g}$. Applying (q1) and (q4), we obtain $(a \cdot b) \cdot a = (a \cdot [\mathbf{0}]_{\Theta_g}) \cdot [\mathbf{0}]_{\Theta_g} = ([x \cdot \mathbf{0}]_{\Theta_g} \cdot [\mathbf{0}]_{\Theta_g}) \cdot [\mathbf{0}]_{\Theta_g} = [((x \cdot \mathbf{0}) \cdot \mathbf{0}) \cdot \mathbf{0}]_{\Theta_g} = [x \cdot \mathbf{0}]_{\Theta_g} = a$. ■

Let $\mathcal{A} = (A; \cdot, \mathbf{0})$ be a quasi-implication algebra. Denote by $A_0 := \{x\mathbf{0} : x \in A\}$ the so-called *skeleton of* $\mathcal{A}$. The following result follows directly by Theorem 1:

Corollary 1. *Let* $\mathcal{A} = (A; \cdot, 0)$ *be a quasi-implication algebra. Then*

(i) $\mathcal{A}_0 = (A_0; \cdot)$ *is an implication algebra*

(ii) $\mathcal{A}_0 \simeq (A; \cdot)/\Theta_g$.

We can prove the following ■

Lemma 1. *Let* $\mathcal{A} = (A; \cdot, \mathbf{0})$ *be a quasi-implication algebra. Then, for each* $x, y \in A$, *we have*

(i) $x \cdot y \in A_0$,

(ii) $x \cdot x = y \cdot y$.

Proof. Accoroding to (q4), $xy = ((xy)\,\mathbf{0})\,\mathbf{0} \in A_0$ for all $x, y \in A$; thus we obtain (i). Taking into account (q4) and (i) of Corollary 1, we have

$$xx = ((xx)\,\mathbf{0})\,\mathbf{0} = ((x\,\mathbf{0})\,\mathbf{0})((x\,\mathbf{0})\,\mathbf{0}) = ((y\,\mathbf{0})\,\mathbf{0})((y\,\mathbf{0})\,\mathbf{0}) = ((yy)\mathbf{0})\,\mathbf{0} = yy$$

for all $x, y \in A$ proving (ii). ■

Remark. By (ii) of Lemma 1, we know that every quasi-implication algebra $\mathcal{A} = (A; \cdot, \mathbf{0})$ has a constant $\mathbf{1}$ such that $x \cdot x = \mathbf{1}$. Hence, also $\mathbf{0} \cdot \mathbf{0} = \mathbf{1}$, i.e. $\mathbf{1}$ is an *algebraic constant* (i.e. a nullary term operation). In the interpretation of propositional logic, $\mathbf{0}$ corresponds to the value "FALSE" and $\mathbf{1}$ to the value "TRUE".

3. Algebraic properties of quasi-implication algebras

By Theorem 1 and Lemma 1, for any quasi-implication algebra $\mathcal{A} = (A; \cdot, \mathbf{0})$ and every $x, y \in A$, we have $x \cdot y \in A_0$, where $(A_0; \cdot)$, the skeleton of $\mathcal{A}$, is an implication algebra. Hence, applying the results of J.C. Abbott, we can prove:

Theorem 2. *Every quasi-implication algebra satisfies the following identities:*

(a) $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$,

(b) $x \cdot (x \cdot y) = x \cdot y$,

(c) $x \cdot ((y \cdot \mathbf{0}) \cdot \mathbf{0}) = x \cdot y$,

(d) $x \cdot \mathbf{1} = \mathbf{1},\ x \cdot x = \mathbf{1},\ \mathbf{1} \cdot (x \cdot y) = x \cdot y,\ x \cdot (\mathbf{1} \cdot y) = x \cdot y$,

(e) $x \cdot y = (y \cdot \mathbf{0}) \cdot (x \cdot \mathbf{0})$,

(f) $x \cdot (y \cdot x) = \mathbf{1}$.

Proof. As mentioned above, we have immediately the conditions (a), (b), (d) and (f). For (e), we apply (a), (f), (d) and (q3), (q1) to show $(y \cdot \mathbf{0}) \cdot (x \cdot \mathbf{0}) = x \cdot ((y \cdot \mathbf{0}) \cdot \mathbf{0}) = x \cdot ((y \cdot x) \cdot y) = (x \cdot (y \cdot x)) \cdot (x \cdot y) = \mathbf{1} \cdot (x \cdot y) = x \cdot y$. Now, $x \cdot ((y \cdot \mathbf{0}) \cdot \mathbf{0}) = (y \cdot \mathbf{0}) \cdot (x \cdot \mathbf{0}) = x \cdot y$ by (q3) and (e); thus also (c) is proved. ■

Remark. By (f) of Theorem 2, every quasi-implication algebra satisfies the axiom (H1) and, by (a) and (d), also (H2). However, a quasi-implication algebra is not a Hilbert algebra in general as one can check by the following:

Example. Let $A = \{\mathbf{0}, y, \mathbf{1}\}$ and the binary operation be given by the table

$\cdot$	$\mathbf{0}$	y	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
y	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$

Then clearly $(A; \cdot, \mathbf{0})$ is a quasi-implication algebra but it is not a Hilbert algebra since axiom (H3) does not hold; namely, $y \cdot \mathbf{1} = \mathbf{1} = \mathbf{1} \cdot y$ but $y \neq \mathbf{1}$.

Since quasi-implication algebras do not satisfy axiom (H3), we cannot expect that the induced relation will be an order. However, we are able to prove:

Lemma 2. *Let $\mathcal{A} = (A; \cdot, \mathbf{0})$ be a quasi-implication algebra and Q be a binary relation on A defined by the setting*

$$\langle x, y\rangle \in Q \text{ if and only if } x \cdot y = \mathbf{1}.$$

Then Q is a quasi-order (i.e. a reflexive and transitive relation) on A and $\langle \mathbf{0}, x\rangle \in Q$, $\langle x, \mathbf{1}\rangle \in Q$ for each $x \in A$.

Proof. Reflexivity of Q follows directly by $x \cdot x = \mathbf{1}$ and $x \cdot \mathbf{1} = \mathbf{1}$ yields $\langle x, \mathbf{1}\rangle \in Q$ immediately. Prove transitivity of Q: Let $\langle x, y\rangle \in Q$ and $\langle y, z\rangle \in Q$. Then $x\cdot y = \mathbf{1} = y\cdot z$. Applying (d) of Theorem 2 and axioms (q2), (q3), we obtain $x\cdot z = x\cdot(\mathbf{1}\cdot z) = x\cdot((y\cdot z)\cdot z) = x\cdot((z\cdot y)\cdot y) = (z\cdot y)\cdot(x\cdot y) = (z\cdot y)\cdot \mathbf{1} = \mathbf{1}$; thus also $\langle x, z\rangle \in Q$. It remains to show $\langle \mathbf{0}, x\rangle \in Q$. Applying (e) and (d) of Theorem 2, we derive $\mathbf{0} \cdot x = (x \cdot \mathbf{0}) \cdot (\mathbf{0} \cdot \mathbf{0}) = (x \cdot \mathbf{0}) \cdot \mathbf{1} = \mathbf{1}$ proving $\langle \mathbf{0}, x\rangle \in Q$ for each $x \in A$. ∎

4. QUASI-IMPLICATION ALGEBRAS INDUCED BY PSEUDOCOMPLEMENTED SEMILATTICES

By a *pseudocomplemented semilattice* is meant an algebra $\mathcal{S} = (S; \wedge, ^*, \mathbf{0})$ of type (2,1,0) such that $(S; \wedge)$ is a meet-semilattice with the least element $\mathbf{0}$ and for each $a \in S$, a^* is its *pseudocomplement*, i.e. the greatest element of S satysfying $a \wedge a^* = \mathbf{0}$; in other words,

$$a \wedge b = \mathbf{0} \text{ if and only if } b \leq a^*. \tag{3}$$

Pseudocomplemented semilattices were treated, e.g., by R. Balbes ([2]) and O. Frink ([6]).

Let $\mathcal{S} = (S; \wedge, ^*, \mathbf{0})$ be a pseudocomplemented semilattice. Denote by $\leq$ its induced order, i.e. $x \leq y$ if and only if $x \wedge y = x$. At first we list some useful properties of these algebras:

Lemma 3. *Let $\mathcal{S} = (S; \wedge, ^*, \mathbf{0})$ be a pseudocomplemented semilattice. Then $\mathcal{S}$ has the greatest element $\mathbf{1}$ with respect to the induced order and for every $a, b \in S$ the following identities are satisfied:*

(i) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$,

(ii) $(a \wedge b^*)^* \wedge b^* = (b \wedge a^*)^* \wedge a^* = a^* \wedge b^*$.

Proof. Immediately by (3), $\mathbf{0}^*$ is the greatest element of $\mathcal{S}$, i.e. $\mathbf{0}^* = \mathbf{1}$.

Prove (i): Since $a \wedge b \leq a$, $a \wedge b \leq b$, we have $(a \wedge b)^{**} \leq a^{**}$ and $(a \wedge b)^{**} \leq b^{**}$, whence $(a \wedge b)^{**} \leq a^{**} \wedge b^{**}$. Further, $a \wedge b \wedge (a \wedge b)^* = \mathbf{0}$; thus, by (2), $a \wedge (a \wedge b)^* \leq b^* = b^{***}$ whence $a \wedge b^{**} \wedge (a \wedge b)^* = \mathbf{0}$. Hence, $b^{**} \wedge (a \wedge b)^* \leq a^* = a^{***}$, i.e. $a^{**} \wedge b^{**} \wedge (a \wedge b)^* = \mathbf{0}$ which yields $a^{**} \wedge b^{**} \leq (a \wedge b)^{**}$ proving the converse inequality.

Prove (ii): Since $a \wedge b^* \leq a$, also $(a \wedge b^*)^* \geq a^*$ and thus $(a \wedge b^*)^* \wedge b^* \geq a^* \wedge b^*$.

Conversely, we have $(a \wedge b^*)^* \wedge (a \wedge b^*) = \mathbf{0}$, i.e. $(a \wedge b^*)^* \wedge b^* \leq a^*$ and hence $(a \wedge b^*)^* \wedge b^* \leq a^* \wedge b^*$.

We have shown $(a \wedge b^*)^* \wedge b^* = a^* \wedge b^*$. The second equality follows by symmetry. ■

Theorem 3. *Let $\mathcal{S} = (S; \wedge, ^*, \mathbf{0})$ be a pseudocomplemented semilattice. Introduce a term operation*

$$x \cdot y := (x \wedge y^*)^*.$$

Then $\mathcal{A} = (S; \cdot, \mathbf{0})$ is a quasi-implication algebra.

Proof. It is clear by the definition that, for each $a \in S$, we have $a \cdot \mathbf{0} = (a \wedge \mathbf{0}^*)^* = (a \wedge \mathbf{1})^* = a^*$. Prove the axioms of quasi-implication algebra:

(q1): Clearly $(x \wedge y^*)^* \geq x^*$; thus $(x \cdot y) \cdot x = ((x \wedge y^*)^* \wedge x^*)^* = x^{**} = (x \cdot \mathbf{0}) \cdot \mathbf{0}$.

(q2): $(x \cdot y) \cdot y = ((x \wedge y^*)^* \wedge y^*)^* = ((y \wedge x^*)^* \wedge x^*)^* = (y \cdot x) \cdot x$ by (ii) of Lemma 3.

(q3): $x \cdot (y \cdot z) = (x \wedge (y \wedge z^*)^{**})^* = (x^{**} \wedge (y^{**} \wedge z^{***}))^* = (y^{**} \wedge (x^{**} \wedge z^{***}))^* = (y \wedge (x \wedge z^*)^{**})^* = y \cdot (x \cdot z)$ by (i) of Lemma 3.

(q4): $((x \cdot y) \cdot \mathbf{0}) \cdot \mathbf{0} = (x \wedge y^*)^{***} = (x \wedge y^*)^* = x \cdot y = (x^{**} \wedge y^{***})^* = ((x \cdot \mathbf{0}) \cdot \mathbf{0}) \cdot ((y \cdot \mathbf{0}) \cdot \mathbf{0})$ by (i) of Lemma 3 again. ■

Remark. The quasi-implication algebra $\mathcal{A} = (S; \cdot, \mathbf{0})$ obtained from $\mathcal{S} = (S; \wedge, ^*, \mathbf{0})$ as given in Theorem 3 will be called *induced by the pseudocomplemented semilattice* $\mathcal{S}$.

Theorem 3 shows that every pseudocomplemented semilattice induces a quasi-implication algebra, but the converse is not true. Namely, we have seen in Lemma 2 that a quasi-implication algebra induces only a quasiorder but a pseudocomplemented semilattice has an induced order.

The relation Θ on a pseudocomplemented semilattice $\mathcal{S} = (S; \wedge, ^*, \mathbf{0})$, defined by

$$\langle x, y\rangle \in \Theta \text{ if and only if } x^* = y^*, \tag{4}$$

is the so-called *Glivenko congruence;* and it is well-known that $\mathcal{S}/\Theta$ is a Boolean algebra with respect to the induced order. Consider the quasi-implication algebra $\mathcal{A} = (S; \cdot, \mathbf{0})$ induced by $\mathcal{S}$ and the relation Θ_g on S defined by (2) of Theorem 1. Then $(S/\Theta_g, \cdot)$ is an implication algebra which is clearly induced by $\mathcal{S}/\Theta$, i.e. $\Theta = \Theta_g$ on the set S. Hence, our relation Θ_g from Theorem 1 is in fact the Glivenko congruence whenever $\mathcal{A} = (S; \cdot, \mathbf{0})$ is induced by a pseudocomplemented semilattice $\mathcal{S} = (S; \wedge, ^*, \mathbf{0})$. Unfortunately, not every quasi-implication algebra is induced by a pseudocomplemented semilattice. We will improve this in the following section.

5. Representation of quasi-implication algebras

The concept of q-semilattice was introduced by the first author in [3]:
By a *q-semilattice* is meant an algebra $\mathcal{S} = (S; \wedge)$ of type (2) satisfying the axioms

(P1) $x \wedge y = y \wedge x$,

(P2) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,

(P3) $x \wedge y = (x \wedge x) \wedge y$.

For our reasons, we need a q-semilattice equipped with a unary operation having similar properties as the pseudocomplementation in semilattices:

Definition 2. A *pseudocomplemented q-semilattice* is an algebra $\mathcal{P} = (A; \wedge, ^*, \mathbf{0})$ of type (2,1,0) such that $(A; \wedge)$ is a q-semilattice and the following axioms are satisfied:

(K1) $x^* \wedge x^* = x^*$,

(K2) $x^{***} = x^*$,

(K3) $x \wedge y = \mathbf{0}$ if and only if $y \wedge x^* = y \wedge y$.

Having a pseudocomplemented q-semilattice, one can introduce the induced relation $\leq$ as follows:

$$x \leq y \text{ if and only if } x \wedge y = x \wedge x. \tag{5}$$

Of course, $\leq$ is reflexive. Prove transitivity of $\leq$: Let $x \leq y$ and $y \leq z$. Then $x \wedge y = x \wedge x$ and $y \wedge z = y \wedge y$. Hence,

$$x \wedge z = (x \wedge x) \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z) =$$

$$= x \wedge (y \wedge y) = (y \wedge y) \wedge x = y \wedge x = x \wedge y = x \wedge x;$$

thus also $x \leq z$.

The quasiorder $\leq$ given by (5) will be called an *induced quasiorder* of the pseudocomplemented q-semilattice.

Further, for each $x, y \in A$, we have

$$(x \wedge y) \wedge x = (x \wedge y) \wedge (x \wedge y), \text{ i.e. } x \wedge y \leq x,$$

and

$$(x \wedge y) \wedge y = (x \wedge y) \wedge (x \wedge y), \text{ i.e. } x \wedge y \leq y;$$

thus $x \wedge y$ is a lower bound of x, y with respect to $\leq$. Moreover, if $c \in A$ and $c \leq x$, $c \leq y$, then $x \wedge c = c \wedge c$, $y \wedge c = c \wedge c$ and

$$(x \wedge y) \wedge c = (x \wedge y) \wedge (c \wedge c) = (x \wedge c) \wedge (y \wedge c) = (c \wedge c) \wedge (c \wedge c) = c \wedge c,$$

showing $c \leq x \wedge y$. Thus $x \wedge y$ is a greatest lower bound, i.e. $x \wedge y = inf_{\leq}(x, y)$.

What concerns the unary operation * in a pseudocomplemented q-semilattice $\mathcal{P} = (A; \wedge, ^*, \mathbf{0})$, we can show that it has similar properties as the pseudocomplementation in a semilattice (and hence it will be called a *pseudocomplementation* of $\mathcal{P}$):

Lemma 4. *Let $x, y \in A$. Then*

(a) $x \wedge x^* = \mathbf{0}$,

(b) $x \leq x^{**}$,

(c) *if $x \leq y$ then $y^* \leq x^*$.*

Proof. Since $x^* \wedge x^* = x^* \wedge x^*$, (K3) yields (a) immediately. However, if $x \wedge x^* = \mathbf{0}$, then (K3) yields also $x \wedge x^{**} = x \wedge x$, whence $x \leq x^{**}$. To prove (c), suppose $x \leq y$. Then $x \wedge y = x \wedge x$. Further $y \leq y^{**}$; thus, applying transitivity of $\leq$, $x \leq y^{**}$ which gives $x \wedge y^{**} = x \wedge x$. By (K3), we have $x \wedge y^* = \mathbf{0}$ and $y^* \wedge x^* = y^* \wedge y^*$, whence $y^* \leq x^*$. ∎

Lemma 5. *Let $\mathcal{P} = (A; \wedge, ^*, \mathbf{0})$ be a pseudocomplemented q-semilattice and $a, b \in A$. Then*

$$(a \wedge b)^{**} \leq a^{**} \wedge b^{**} \quad \textit{and} \quad a^{**} \wedge b^{**} \leq (a \wedge b)^{**}.$$

Proof. Since $a \wedge b \leq a$ and $a \wedge b \leq b$, we have $(a \wedge b)^{**} \leq a^{**}$ and $(a \wedge b)^{**} \leq b^{**}$ by (c) of Lemma 4; thus $(a \wedge b)^{**} \leq a^{**} \wedge b^{**}$. Conversely, by (a) of Lemma 4, we have $b \wedge (a \wedge (a \wedge b)^*) = (a \wedge b) \wedge (a \wedge b)^* = \mathbf{0}$; thus, by (K3), $b^* \wedge a \wedge (a \wedge b)^* = (a \wedge (a \wedge b)^*) \wedge (a \wedge (a \wedge b)^*)$, i.e. $a \wedge (a \wedge b)^* \leq b^* = b^{***}$. Applying (K3) once more, $a \wedge b^{**} \wedge (a \wedge b)^* = \mathbf{0}$; thus $b^{**} \wedge (a \wedge b)^* \leq a^* = a^{***}$. We obtain $a^{**} \wedge b^{**} \wedge (a \wedge b)^* = \mathbf{0}$ which yields $a^{**} \wedge b^{**} \leq (a \wedge b)^{**}$.

∎

Let $\mathcal{P} = (A; \wedge, ^*, \mathbf{0})$ be a pseudocomplemented q-semilattice. Denote by $A^* := \{a^* : a \in A\}$ the so-called *skeleton of* $\mathcal{P}$.

Theorem 4. *Let $\mathcal{P} = (A; \wedge, ^*, \mathbf{0})$ be a pseudocomplemented q-semilattice and A^* its skeleton. Then the restriction of the induced quasiorder $\leq$ to A^* is an order and $\mathcal{P}^* = (A^*; \wedge, ^*, \mathbf{0})$ is a pseudocomplemented semilattice (which is a subalgebra of $\mathcal{P}$).*

Proof. To prove that $\leq$ is an order on A^*, we need only to show the antisymmetry. Suppose $x^*, y^* \in A^*$ with $x^* \leq y^*$ and $y^* \leq x^*$. Then, by (5), (K1), and (P1), $x^* = x^* \wedge x^* = x^* \wedge y^* = y^* \wedge x^* = y^*$.

The fact that $(A^*; \wedge)$ is a semilattice follows immediately by (P1), (P2) and (K1). By (K3), x^{**} is a pseudocomplement of x^* for each $x \in A$, because of (K) and (5). By Lemma 5, A^* is closed with respect to $\wedge$, since

$$a^* \wedge b^* = a^{***} \wedge b^{***} = (a^* \wedge b^*)^{**},$$

due to the fact that $\leq$ is an order on A^*. By Lemma 4, $x^* \wedge x^{**} = \mathbf{0}$; thus also $\mathbf{0} \in A^*$, i.e. $\mathcal{P}^* = (A^*; \wedge, ^*, \mathbf{0})$ is a pseudocomplemented semilattice which is a subalgebra of $\mathcal{P}$. ■

Lemma 6. *Let $\mathcal{P} = (A; \wedge, ^*, \mathbf{0})$ be a pseudocomplemented q-semilattice. Denote $\mathbf{0}^*$ by $\mathbf{1}$. Then*

$$\mathbf{0} \leq x \quad \textit{and} \quad x \leq \mathbf{1}$$

for each $x \in A$.

Proof. By Theorem 4, $\mathbf{0} \in A^*$ and, by (K3), $\mathbf{0} \wedge \mathbf{0} = \mathbf{0}$. Thus $x \wedge \mathbf{0} = x \wedge (x^* \wedge x) = x \wedge x^* = \mathbf{0} = \mathbf{0} \wedge \mathbf{0}$, i.e. $\mathbf{0} \leq x$ directly by (5). Since $x \wedge \mathbf{0} = \mathbf{0}$, (K3) yields

$$x \wedge \mathbf{1} = x \wedge \mathbf{0}^* = x \wedge x;$$

thus $x \leq \mathbf{1}$ for each $x \in A$. ■

Theorem 5. *Let $\mathcal{P} = (A; \wedge, ^*, \mathbf{0})$ be a pseudocomplemented q-semilattice. Introduce the term operation $x \sqcup y := (x^* \wedge y^*)^*$. Then $\mathcal{B} = (A^*; \wedge, \sqcup, ^*, \mathbf{0}, \mathbf{1})$ is a Boolean algebra.*

Proof. By Theorem 4, $(A^*; \wedge)$ is a semilattice with the least element $\mathbf{0}$. We prove that $(A^*; \sqcup)$ is also a semilattice.

Suppose $a^*, b^*, c^* \in A^*$. Then $a^* \sqcup a^* = (a^{**} \wedge a^{**})^* = (a^{**})^* = a^*$ (by (K1) and (K2)).

Further, $a^* \sqcup (b^* \sqcup c^*) = (a^{**} \wedge (b^{**} \wedge c^{**})^{**})^* = (a^{**} \wedge (b^{**} \wedge c^{**}))^* = ((a^{**} \wedge b^{**}) \wedge c^{**})^* = (a^* \sqcup b^*) \sqcup c^*$; thus $\sqcup$ is associative. Commutativity of $\sqcup$ is evident. Thus $(A^*; \sqcup)$ is a semilattice.

We have to check the absorption laws: $a^* \wedge b^* \leq a^*$, yields $a^{**} \leq (a^* \wedge b^*)^*$; thus

$$a^* \sqcup (a^* \wedge b^*) = (a^{**} \wedge (a^* \wedge b^*)^*)^* = (a^{**} \wedge a^{**})^* = a^{***} = a^*.$$

Analogously, $a^{**} \wedge b^{**} \leq a^{**}$ yields $a^{***} = a^* \leq (a^{**} \wedge b^{**})^*$; thus

$$a^* \wedge (a^* \sqcup b^*) = a^* \wedge (a^{**} \wedge b^{**})^* = a^* \wedge a^* = a^*.$$

Hence, $(A^*; \wedge, \sqcup)$ is a lattice with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$ (see Lemma 6). By Lemma 4, $a^* \wedge a^{**} = \mathbf{0}$. Further,

$$a^* \sqcup a^{**} = (a^{**} \wedge a^{***})^* = (a^{**} \wedge a^*)^* = \mathbf{0}^* = \mathbf{1};$$

thus this lattice is complemented and a^{**} is a complement of $a^* \in A^*$. It remains to prove distributivity. We easily verify

$$(a^* \wedge b^*)^* \wedge (a^* \wedge c^*)^* \wedge (a^* \wedge b^*) = \mathbf{0}, \quad (a^* \wedge b^*)^* \wedge (a^* \wedge c^*)^* \wedge (a^* \wedge c^*) = \mathbf{0}.$$

Applying (K3) we obtain

$$(a^* \wedge b^*)^* \wedge (a^* \wedge c^*)^* \wedge a^* \leq b^{**}, \quad (a^* \wedge b^*)^* \wedge (a^* \wedge c^*)^* \wedge a^* \leq c^{**},$$

whence

$$(a^* \wedge b^*)^* \wedge (a^* \wedge c^*)^* \wedge a^* \leq b^{**} \wedge c^{**} = (b^{**} \wedge c^{**})^{**}.$$

Thus

$$(a^* \wedge b^*)^* \wedge (a^* \wedge c^*)^* \wedge (b^{**} \wedge c^{**})^* \wedge a^* = \mathbf{0},$$

which gives

$$a^* \wedge (b^* \sqcup c^*) = a^* \wedge (b^{**} \wedge c^{**})^* \leq ((a^* \wedge b^*)^* \wedge (a^* \wedge c^*)^*)^* = (a^* \wedge b^*) \sqcup (a^* \wedge c^*).$$

The converse inequality is trivial. Thus the lattice is distributive and hence $\mathcal{B} = (A^*; \wedge, \sqcup, ^*, \mathbf{0}, \mathbf{1})$ is a Boolean algebra. ∎

Now, we are ready to set up a characterization of quasi-implication algebras by means of the pseudocomplemented q-semilattices.

Theorem 6. *Let $\mathcal{P} = (A; \wedge, ^*, \mathbf{0})$ be a pseudocomplemented q-semilattice. Define $x \cdot y := (x \wedge y^*)^*$. Then $\mathcal{A} = (A; \cdot, \mathbf{0})$ is a quasi-implication algebra.*

Proof. By definition of the binary operation $\cdot$, we have immediately $a \cdot \mathbf{0} = (a \wedge \mathbf{0}^*)^* = (a \wedge \mathbf{1})^* = a^*$. The verification of (q1), (q2), (q3) and (q4) is practically the same as in the proof of Theorem 3, only Theorem 4, Lemma 4 and Lemma 5 are used instead of Lemma 3. Hence, the rest of proof is left to the reader as an easy exercise. ∎

The quasi-implication algebra, derived from a pseudocomplemented q-semilattice $\mathcal{P}$ as shown by Theorem 6, will be called "*the induced quasi-implication algebra*" and be denoted by $\mathcal{A}(\mathcal{P})$.

Lemma 7. *Let $\mathcal{P} = (A; \wedge, ^*, \mathbf{0})$ be a pseudocomplemented q-semilattice and let $\mathcal{A}(\mathcal{P})$ be the induced quasi-implication algebra. If $x \leq y$ in $\mathcal{P}$, then $x \cdot y = \mathbf{1}$ in $\mathcal{A}(\mathcal{P})$.*

Proof. Suppose $x \leq y$ in $\mathcal{P}$. Then $x \wedge y = x \wedge x$ and hence $x \cdot y = (x \wedge y^*)^* = ((x \wedge x) \wedge y^*)^* = ((x \wedge y) \wedge y^*)^* = (x \wedge (y \wedge y^*))^* = (x \wedge \mathbf{0})^* = (\mathbf{0} \wedge \mathbf{0})^* = \mathbf{0}^* = \mathbf{1}$. ■

We are going to show that also, conversely, every quasi-implication algebra induces a pseudocomplemented q-semilattice:

Theorem 7. *Let $\mathcal{A} = (A; \cdot, \mathbf{0})$ be a quasi-implication algebra. Define $x \wedge y = (x \cdot (y \cdot \mathbf{0})) \cdot \mathbf{0}$ and $x^* = x \cdot \mathbf{0}$. Then $\mathcal{P} = (A; \wedge, ^*, \mathbf{0})$ is a pseudocomplemented q-semilattice.*

Proof. We check the corresponding axioms.

(P1): By (q3), we derive

$$x \wedge y = (x \cdot (y \cdot \mathbf{0})) \cdot \mathbf{0} = (y \cdot (x \cdot \mathbf{0})) \cdot \mathbf{0} = y \wedge x.$$

(P2): Using of (P1), (q3) and (c) of Theorem 2, we compute $(x \wedge y) \wedge z = z \wedge (x \wedge y) = (z \cdot (((x \cdot (y \cdot \mathbf{0})) \cdot \mathbf{0}) \cdot \mathbf{0})) \cdot \mathbf{0} = (z \cdot (x \cdot (y \cdot \mathbf{0}))) \cdot \mathbf{0} = (x \cdot (z \cdot (y \cdot \mathbf{0}))) \cdot \mathbf{0} = (x \cdot (((z \cdot (y \cdot \mathbf{0})) \cdot \mathbf{0}) \cdot \mathbf{0})) \cdot \mathbf{0} = x \wedge (z \wedge y) = x \wedge (y \wedge z)$.

(P3): By (b) of Theorem 2, $x \cdot (x \cdot \mathbf{0}) = x \cdot \mathbf{0}$ and, by (c) of Theorem 2, $y \wedge (x \wedge x) = (y \cdot (((x \cdot (x \cdot \mathbf{0})) \cdot \mathbf{0}) \cdot \mathbf{0})) \cdot \mathbf{0} = (y \cdot (x \cdot (x \cdot \mathbf{0}))) \cdot \mathbf{0} = (y \cdot (x \cdot \mathbf{0})) \cdot \mathbf{0} = y \wedge x$.

(K2): By (q4) and (q1), we have

$$x^* = x \cdot \mathbf{0} = ((x \cdot \mathbf{0}) \cdot \mathbf{0}) \cdot ((\mathbf{0} \cdot \mathbf{0}) \cdot \mathbf{0}) = ((x \cdot \mathbf{0}) \cdot \mathbf{0}) \cdot \mathbf{0} = x^{***}.$$

(K1): By (b) of Theorem 2, we get $x^* \wedge x^* = ((x \cdot \mathbf{0}) \cdot ((x \cdot \mathbf{0}) \cdot \mathbf{0})) \cdot \mathbf{0} = ((x \cdot \mathbf{0}) \cdot \mathbf{0}) \cdot \mathbf{0} = x^{***} = x^*$.

(K3): Suppose $x \wedge y = \mathbf{0}$. By (P1) also $y \wedge x = \mathbf{0}$ which gives $(y \cdot (x \cdot \mathbf{0})) \cdot \mathbf{0} = \mathbf{0}$

and, by (c) and (d) of Theorem 2, $((y \cdot (x \cdot \mathbf{0})) \cdot \mathbf{0}) \cdot \mathbf{0} = \mathbf{0} \cdot \mathbf{0} = \mathbf{1}$. Applying this, we compute $y \wedge x^* = (y \cdot ((x \cdot \mathbf{0}) \cdot \mathbf{0})) \cdot \mathbf{0} = ((y \cdot (x \cdot \mathbf{0})) \cdot (y \cdot \mathbf{0})) \cdot \mathbf{0} = (\mathbf{1} \cdot (y \cdot \mathbf{0})) \cdot \mathbf{0} = (y \cdot \mathbf{0}) \cdot \mathbf{0} = (y \cdot (y \cdot \mathbf{0})) \cdot \mathbf{0} = y \wedge y$ by (a), (b) and (d) of Theorem 2.

Conversely, suppose $y \wedge x^* = y \wedge y$. Then, by (d) of Theorem 2, (P1), (P2), (P3) and (a) of Lemma 4, $x \wedge y = x \wedge (y \wedge y) = x \wedge (x^* \wedge y) = (x \wedge x^*) \wedge y = \mathbf{0} \wedge y = y \wedge \mathbf{0} = (y \cdot (\mathbf{0} \cdot \mathbf{0})) \cdot \mathbf{0} = (y \cdot \mathbf{1}) \cdot \mathbf{0} = \mathbf{1} \cdot \mathbf{0} = \mathbf{0}$. ■

The pseudocomplemented q-semilattice derived from a quasi-implication algebra $\mathcal{A}$, as shown by Theorem 7, will be called *"the induced pseudocomplemented q-semilattice"* and be denoted by $\mathcal{P}(\mathcal{A})$.

Lemma 8. *Let $\mathcal{A} = (A; \cdot, \mathbf{0})$ be a quasi-implication algebra and $\mathcal{P}(\mathcal{A}) = (A; \wedge, ^*, \mathbf{0})$ the induced pseudocomplemented q-semilattice. If $x, y \in A$ satisfy $x \cdot y = \mathbf{1}$ in $\mathcal{A}$, then $x \leq y$ in $\mathcal{P}(\mathcal{A})$.*

Proof. Suppose $x \cdot y = \mathbf{1}$. Then, by (a), (d) and (b) of Theorem 2, $x \wedge y = (x \cdot (y \cdot \mathbf{0})) \cdot \mathbf{0} = ((x \cdot y) \cdot (x \cdot \mathbf{0})) \cdot \mathbf{0} = (\mathbf{1} \cdot (x \cdot \mathbf{0})) \cdot \mathbf{0} = (x \cdot \mathbf{0}) \cdot \mathbf{0} = (x \cdot (x \cdot \mathbf{0})) \cdot \mathbf{0} = x \wedge x$. By (5), we conclude $x \leq y$. ■

Corollary 2. *Let $\mathcal{A}$ be a quasi-implication algebra. Then $\mathcal{A}(\mathcal{P}(\mathcal{A})) = \mathcal{A}$.*

Proof. Denote by $\odot$ the binary operation in $\mathcal{A}(\mathcal{P}(\mathcal{A}))$. Applying (c) of Theorem 2 two times, we have $x \odot y = (x \wedge y^*)^* = ((x \cdot ((y \cdot \mathbf{0}) \cdot \mathbf{0})) \cdot \mathbf{0}) \cdot \mathbf{0} = x \cdot y$. ■

Remark. The converse deduction is not true in general, i.e. if $\mathcal{P}$ is a pseudocomplemented q-semilattice, then $\mathcal{P}(\mathcal{A}(\mathcal{P}))$ need not coincide with $\mathcal{P}$. Namely, $x \cdot y = (x \wedge y^*)^*$ in $\mathcal{A}(\mathcal{P})$ and hence, for $x \wedge y$ in $\mathcal{P}(\mathcal{A}(\mathcal{P}))$, we have $x \wedge y = (x \wedge y^{**})^{**}$. However, this identity does not hold even if $\mathcal{P}$ is a pseudocomplemented semilattice: Consider a chain $\mathbf{0} < x < y < \mathbf{1}$. Then $x \wedge y = x$, but $(x \wedge y^{**})^{**} = (x \wedge \mathbf{1})^{**} = x^{**} = \mathbf{1} \neq x$.

References

[1] J.C. Abbott, *Semi-boolean algebras*, Mat. Vesnik **4** (1967), 177–198.

[2] R. Balbes, *On free pseudo-complemented and relatively pseudo-complemented semi-lattices*, Fund. Math. **78** (1973), 119–131.

[3] I. Chajda, *Semi-implication algebra*, Tatra Mt. Math. Publ. **5** (1995), 13–24.

[4] I. Chajda, *An extension of relative pseudocomplementation to non-distributive lattices*, Acta Sci. Math. (Szeged), to appear.

[5] A. Diego, *Sur les algèbres de Hilbert*, Gauthier-Villars, Paris 1966 (viii+55pp.).

[6] O. Frink, *Pseudo-complements in semi-lattices*, Duke Math. J. **29** (1962), 505–514.

Received 10 October 2002
Revised 23 January 2003